Adv Syst Sci Appl 2023; **04**:1–7
Published online at https://ijassa.ipu.ru.

An Exact Solution of the Hunter–Saxton–Calogero Equation by Contact Linearization Method

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Abstract: In this paper we consider a class of generalized nonlinear hyperbolic partial differential equations of the Hunter–Saxton–Calogero type, which arise in the theory of control of liquid crystals and in the control of unsteady gas flows. We found such conditions that the original equation can be reduced to linear one by contact transformations. The general exact multivalued solutions of the Hunter–Saxton–Calogero equation are found. The obtained solutions are visualized.

Keywords: contact transformations, Cartan form, nematic crystals, exact solution, nonlinear partial differential equation, differential forms.

1. INTRODUCTION

Let us consider the generalized nonlinear second-order Hunter–Saxton–Calogero partial differential equation

$$u_{tx} = uu_{xx} + G(u_x), \tag{1.1}$$

where $u(t,x)$ is an unknown function, t and x are the time and the spatial coordinates, respectively.

Such equations with $G(u_x) = \kappa u_x^2$ and $k = \frac{1}{2}$ arise in the theory of nematic liquid crystals. If, initially, all molecules of a liquid crystal are aligned, then some of them will shift slightly and disorientation will spread throughout the crystal. In this case, the function $u(t,x)$ describes the propagation of weak linear orientation waves in the nematic liquid crystal [1].

The equation with $\kappa \neq \frac{1}{2}$ is used in hydrodynamics [2], in the geometry of Einstein–Weyl spaces [3]. The contact equivalence of equation (1.1) and the Euler–Poisson equation was established for $G(u_x) = \kappa u_x^2$ in [4]. Calogero [5], while studying waves in shallow water, found a complex solution of equation (1.1).

In this article we present conditions, under which nonlinear equation (1.1) is equivalent to a linear equation with respect to a pseudo-group of contact transformations. This allows us to construct its exact multivalued solutions. These solutions can be used to control the propagation of orientation waves in a nematic crystal.

This paper continues the series of articles [6–9] on the application of geometric theory of nonlinear differential equations to constructing their exact solutions. We use the methods developed in [10–12].

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2. GEOMETRY OF THE GENERALIZED HUNTER–SAXTON–CALOGERO EQUATION

Let J^1 be the 1-jet space of functions on $\mathbb{R}^2$ with two independent variables t, x and let t, x, u, p_1, p_2 be the canonical coordinates on this space. The Cartan form

$$\varkappa = du - p_1 dt - p_2 dx$$

defines a contact structure on J^1 (the so called Cartan distribution)

$$\mathcal{C} : J^1 \ni \theta \mapsto \mathcal{C}(\theta) = \ker \varkappa_\theta \subset T_\theta J^1.$$

The Cartan distribution $\mathcal{C}$ is generated by the vector fields

$$\frac{\partial}{\partial t} + p_1 \frac{\partial}{\partial u}, \quad \frac{\partial}{\partial x} + p_2 \frac{\partial}{\partial u}, \quad \frac{\partial}{\partial p_1}, \quad \frac{\partial}{\partial p_2}. \tag{2.2}$$

A two-dimensional surface

$$\Gamma_v^1 = \left\{ u = v(t,x), p_1 = \frac{\partial v}{\partial t}, p_2 = \frac{\partial v}{\partial x} \right\} \subset J^1$$

is called a 1-graph of a function $v(t, x)$.

Let $\Omega^2(\mathbb{R}^2)$ be the module of differential 2-forms on $\mathbb{R}^2$. For an arbitrary differential 2-form ω on J^1, we can construct the Lychagin differential operator Δ_ω, which acts by the following rule (see [13]):

$$\Delta_\omega \colon \mathcal{C}^\infty(\mathbb{R}^2) \to \Omega^2(\mathbb{R}^2), \quad \Delta_\omega(v) = \omega|_{\Gamma_v^1}.$$

Here $\omega|_{\Gamma_u^1}$ is a restriction of ω to Γ_v^1. The equation

$$\Delta_\omega(v) = 0 \tag{2.3}$$

is a second-order differential equation of the Monge–Ampere class.

The restriction of ω to the surface Γ_v^1 vanishes if and only if the function v is a solution of equation (2.3).

A surface $L \subset J^1\mathbb{R}^2$ is called a multivalued solution of equation (2.3) if $\omega|_L = 0$ and $\varkappa|_L = 0$.

Equation (1.1) belongs to the class of Monge–Ampere equations and, therefore, it can be associated with the differential 2-form

$$\omega = -2G(p_2) dt \wedge dx + dt \wedge dp_1 - dx \wedge dp_2 - 2u dt \wedge dp_2. \tag{2.4}$$

Let us introduce a "non-holonomic symplectic structure" $\Omega \in \Omega^2(\mathcal{C})$:

$$\Omega = d\varkappa|_{\mathcal{C}}$$

Since the Cartan distribution is not completely integrable, this 2-form is defined on vector fields that belong to $\mathcal{C}$ only. Differential form (2.4) is effective, i.e., $\partial_u \rfloor \omega = 0$ and $\omega \wedge \Omega = 0$. Moreover, it is hyperbolic:

$$\omega \wedge \omega + \Omega \wedge \Omega = 0. \tag{2.5}$$

Define the linear operator $A_\omega \colon D(\mathcal{C}) \to D(\mathcal{C})$ as follows:

$$A_\omega X \rfloor \Omega = X \rfloor \omega,$$

where $D(\mathcal{C})$ is a module of vector fields that belong to the Cartan distribution $\mathcal{C}$. The operator A_ω has the following matrix representation in basis (2.2):

$$A_\omega = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2u & -1 & 0 & 0 \\ 0 & -2G(p_2) & 1 & 2u \\ 2G(p_2) & 0 & 0 & -1 \end{pmatrix}.$$

Its square is scalar: $A_\omega^2 = 1$, therefore, its eigenvalues are ± 1. The eigenvectors define two 2-dimensional characteristic distributions

$$C_+ = \left\{ X_+ = \frac{\partial}{\partial t} - u\frac{\partial}{\partial x} + (-p_2 u + p_1)\frac{\partial}{\partial u} + G(p_2)\frac{\partial}{\partial p_2},\ Y_+ = G(p_2)\frac{\partial}{\partial p_1} \right\},$$
$$C_- = \left\{ X_- = -u\frac{\partial}{\partial x} - up_2\frac{\partial}{\partial u} + G(p_2)\frac{\partial}{\partial p_2},\ Y_- = \frac{\partial}{\partial x} + p_2\frac{\partial}{\partial u} + G(p_2)\frac{\partial}{\partial p_1} \right\}.$$

The vector fields $X_\pm, Y_\pm$ form a basis of the module $D(\mathcal{C}_\pm)$. The first derivatives of distributions

$$C_\pm^{(1)} = \{X_\pm,\ Y_\pm, [X_\pm, Y_\pm]\}$$

are 3-dimensional. Therefore, in the 5-dimensional space J^1, they intersect along a 1-dimensional distribution $l = C_+^{(1)} \cap C_-^{(1)}$, which is generated by the vector field

$$Z = G\left(p_2\right)\frac{\partial}{\partial u} + G(p_2)\left(G''\left(p_2\right) - p_2\right)\frac{\partial}{\partial p_1}.$$

At any point $a \in J^1$, the tangent space $T_a J^1$ can be decomposed into a direct sum

$$T_a J^1 = C_+(a) \oplus l(a) \oplus C_-(a).$$

Denote the distributions C_+, l, and C_- as P_1, P_2, and P_3, respectively. Let D_j be the module of vector fields from the distribution P_j and let $\mathbf{P}_j \colon D(J^1) \to D_j$ be projectors. Define the tensors $q_{j,k}^s \in \Omega^2(J^1) \otimes D(J^1)$ (see [12]):

$$q_{j,k}^s(X, Y) := -\mathbf{P}_s[\mathbf{P}_j X, \mathbf{P}_k Y],$$

where $j, k, s = 1, 2, 3;\ s \neq j, k$, and skew contraction of two decomposable tensors $\alpha \otimes X, \beta \otimes Y \in \Omega^2(J^1) \otimes D(J^1)$:

$$\langle \alpha \otimes X, \beta \otimes Y \rangle = (Y \rfloor \alpha) \wedge (X \rfloor \beta).$$

This definition is extended to the remaining tensors by linearity. Tensor invariants of equation (1.1) have the form:

$$q_{2,3}^1 = (p_2 dt \wedge dx + dt \wedge du) \otimes \left(G(p_2)^2 \frac{\partial}{\partial p_1} - G\left(p_2\right)\frac{\partial}{\partial x} + G\left(p_2\right) p_2 \frac{\partial}{\partial u} \right),$$
$$q_{1,2}^3 = (p_2 dt \wedge dx - dt \wedge du + p_1 dt \wedge dp_2 + p_2 dx \wedge dp_2 - du \wedge dp_2)$$
$$\otimes \left(-\left(G''\left(p_2\right) - 2\right)\left(G\left(p_2\right)\right)^2 \frac{\partial}{\partial p_1} \right),$$
$$q_{1,1}^2 = \left(G\left(p_2\right) dt \wedge dx + u dt \wedge dp_2 + dx \wedge dp_2\right) \otimes \left(-G\left(p_2\right)\frac{\partial}{\partial u} + \left(G'\left(p_2\right) - p_2\right) G(p_2)\frac{\partial}{\partial p_1} \right),$$
$$q_{3,3}^2 = \left(\left(G'\left(p_2\right) p_2 - p_2^2 - G\left(p_2\right)\right) dt \wedge dx + \left(-G'\left(p_2\right) + p_2\right) dt \wedge du + dt \wedge dp_1 - u dt \wedge dp_2\right)$$
$$\otimes \left(G\left(p_2\right)\frac{\partial}{\partial u} - \left(G'\left(p_2\right) - p_2\right) G\left(p_2\right)\frac{\partial}{\partial p_1} \right).$$

The invariant Laplace forms for equation (1.1) are

$$\lambda_+ = \left\langle q^2_{1,1}, q^1_{2,3} \right\rangle = -dt \wedge dp_2, \qquad \lambda_- = \left\langle q^2_{3,3}, q^3_{1,2} \right\rangle = -(G''(p_2) - 2)dt \wedge dp_2.$$

Equation (1.1) satisfies the conditions of contact linearization

$$\lambda_- = 0,\ \lambda_+ \wedge \lambda_+ = 0,\ d\lambda_+ = 0$$

if and only if the function $G(p_2)$ has the form

$$G(p_2) = p_2^2 + 2k_1 p_2 + k_0,$$

where k_0, k_1 are arbitrary constants. Then equation (1.1) has the form

$$u_{tx} - uu_{xx} - 2k_1 u_x - u_x^2 - k_0 = 0. \tag{2.6}$$

Let us construct a linearizing contact transformation. Equation (2.6) corresponds to the differential 2-form

$$\omega = -2(u_x^2 + 2k_1 u_x + k_0)dt \wedge dx + dt \wedge du_t - 2udt \wedge du_x - dx \wedge du_x. \tag{2.7}$$

We apply the partial Legendre transform to this 2-form:

$$\Phi\colon\ (t, x, u, p_1, p_2) \mapsto (t,\ -p_2, -xp_2 + u,\ p_1, x)\,.$$

Applying this transformation to differential form (2.7), we obtain a new form

$$\omega_1 = \Phi^*(\omega) = (2xp_2 - 2u)dt \wedge dx + dt \wedge dp_1 + \left(2x^2 + 4k_1 x + 2k_0\right) dt \wedge dp_2 - dx \wedge dp_2,$$

which corresponds to the linear equation

$$u_{tx} + (x^2 + k_1 x + k_0)u_{xx} + xu_x - u = 0. \tag{2.8}$$

Equation (2.8) can be solved by cascade integration method:

$$u(t,x) = e^{k_1 t}\left(\int_{t_0}^{t} F_1(\tau)e^{-k_1\tau}\cosh\left((\tau - t)\sqrt{k_1^2 - k_0} - \operatorname{arctanh}\left(\frac{x + k_1}{\sqrt{k_1^2 - k_0}}\right)\right)d\tau + \right.$$
$$\left. + F_2\left(-t - \frac{\operatorname{arctanh}\left(\frac{x+k_1}{\sqrt{k_1^2 - k_0}}\right)}{\sqrt{k_1^2 - k_0}}\right)\right)\sqrt{\frac{2k_1 x + x^2 + k_0}{k_0 - k_1^2}}, \tag{2.9}$$

where F_1, F_2 are arbitrary functions.

Note that the Legendre transformation maps the multivalued solutions of equation (2.6) to the solutions of equation (2.8). But the inverse Legendre transformation maps classical solutions (2.9) to multivalued ones.

Apply the inverse transformation

$$\Phi^{-1}\colon (t, x, u, p_1, p_2) \mapsto (t, p_2, -xp_2 + u, p_1, -x)$$

to (2.9). Let us choose t and p_2 as parameters β, α, respectively. Then we get general multivalued solution of equation (2.6):

$$L : \begin{cases} t =\beta, \\ x = -\dfrac{1}{\sqrt{-(\alpha^2+2\alpha k_1+k_0)\gamma}} \left(e^{k_1\beta} \left((\alpha+k_1) \left(\displaystyle\int_{\beta_0}^{\beta} F_1(\tau)e^{-k_1\tau}\cosh(\psi)d\tau \right) \right.\right. \\ \qquad \left.\left. +(\alpha+k_1)F_2(\eta) + \gamma \left(\displaystyle\int_{\beta_0}^{\beta} F_1(\tau)e^{-k_1\tau}\sinh(\psi)\,d\tau + F_2'(\eta) \right) \right) \right), \\ u = -\dfrac{1}{\sqrt{-(\alpha^2+2\alpha k_1+k_0)\gamma}} \left(\left((\alpha k_1+k_0) \left(\displaystyle\int_{\beta_0}^{\beta} F_1(\tau)e^{-k_1\tau}\cosh(\psi)\,d\tau \right) \right.\right. \\ \qquad \left.\left. +(\alpha k_1+k_0)F_2(\eta) - \alpha \left(\gamma \left(\displaystyle\int_{\beta_0}^{\beta} F_1(\tau)e^{-k_1\tau}\sinh(\psi)\,d\tau \right) + F_2'(\eta) \right) \right) e^{k_1\beta} \right), \\ u_t = e^{k_1\beta}\sqrt{\dfrac{\alpha^2+2\alpha k_1+k_0}{-\gamma}} \left(-\gamma \left(\displaystyle\int_{\beta_0}^{\beta} F_1(\tau)e^{-k_1\tau}\sinh(\psi)\,d\tau \right) \right. \\ \qquad \left. +k_1 \left(\displaystyle\int_{\beta_0}^{\beta} F_1(\tau)e^{-k_1\tau}\cosh(\psi)\,d\tau \right) - F_2'(\eta) + k_1F_2 \right) + F_1(\beta), \\ u_x =\alpha, \end{cases}$$

where F_1, F_2 are arbitrary functions, α, β are parameters, $\gamma = \sqrt{k_1^2 - k_0}$,

$$\eta = \left(-\frac{\beta\gamma + \operatorname{artanh}\left(\dfrac{\alpha+k_1}{\gamma}\right)}{\gamma} \right), \ \psi = \left(-\operatorname{arctanh}\left(\frac{\alpha+k_1}{\gamma}\right) + (\tau-\beta)\gamma \right).$$

To show that L is indeed a multivalued solution, it is enough to check that the restriction of the 2-form ω to it vanishes.

3. VISUALIZATION

Let us consider an example of visualization for constructed solution. Let $k_0 = 2,\ k_1 = 0$. Choose the functions $F_1(\tau) = -\tau,\ F_2(\eta) = -\eta$. Then we have

$$\begin{cases} t =\beta, \\ x =\dfrac{\beta + (\arctan(\alpha)\alpha + 1 - \beta\alpha)\sqrt{\alpha^2+1} + \alpha^2\beta}{\alpha^2+1}, \\ u =\dfrac{(\beta + \alpha - \arctan(\alpha))\sqrt{\alpha^2+1} - \alpha^2 - 1}{\alpha^2+1}. \end{cases}$$

The graph of this solution is shown in Fig. 3.1. Solution graphs for other F_1 and F_2 are presented in Fig. 3.2 and Fig. 3.3.

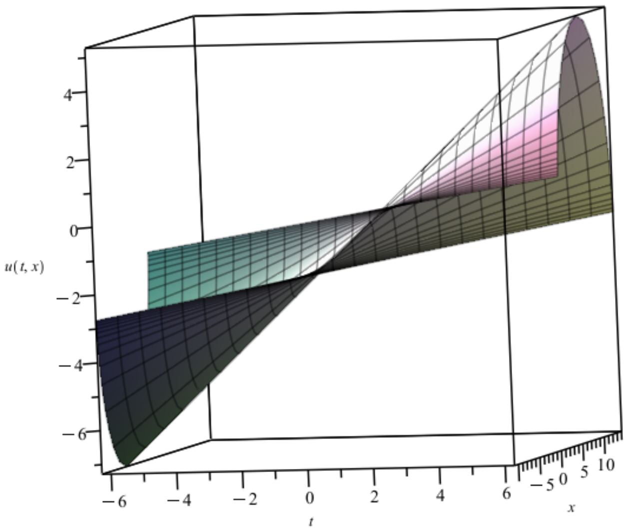

Fig. 3.1. Solution of equation (2.6) with $F_1(\tau) = -\tau,\ F_2(\eta) = -\eta$.

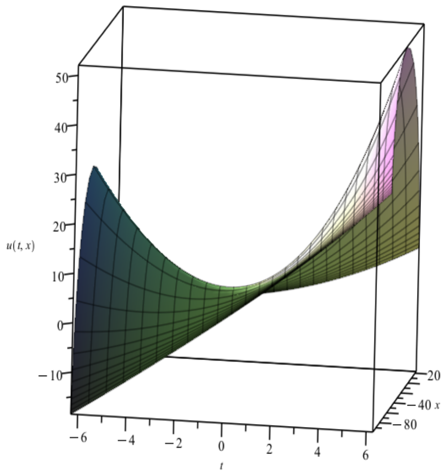

Fig. 3.2. Solution with $F_1(\tau) = \tau^2,\ F_2(\eta) = \eta^2$

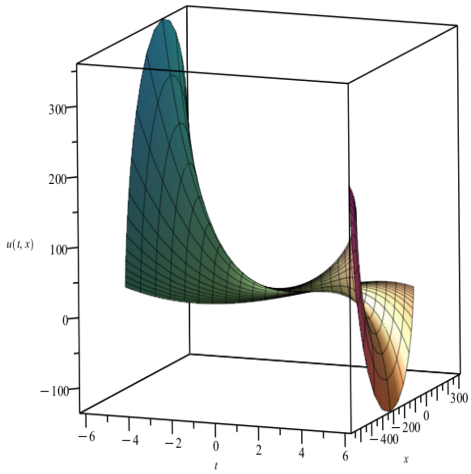

Fig. 3.3. Solution with $F_1(\tau) = \tau^3,\ F_2(\eta) = \eta^3$

ACKNOWLEDGEMENTS

This work is supported by the Russian Science Foundation (grant 23-21-00390).

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