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Comparison of Two Dynamic Models of Economic Growth

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Abstract: Two dynamic models of economic growth with the same balance equation are considered. First, we establish the solution of the Harrod-Domar model with time-dependent coefficient of the capital intensity of income growth. (Previously, the only constant coefficients were considered.) Second, we show that in the Solow model with the Cobb-Douglas production function, the capital intensity of income growth depends on time. Comparing these models, we demonstrate the effectiveness of the setting optimal control problems (maximization of the integral discounted utility function) in the extended the Harrod-Domar model.

Keywords: economic growth, production function, Harrod-Domar model, Solow model

1. INTRODUCTION

Models of economic growth became very popular due to their universality. They were applied to various objects in economic structures of many kinds. In mathematical economics, there are two widely acknowledged models of economic dynamics: the Harrod-Domar model and the Solow model, which are presented in scientific and educational literature. See [1] – [12].

In both mentioned models, the total income is the sum of the total investment and the total consumption. Following [13], we establish the exact solution of the Cauchy problem for the differential equation in the Harrod-Domar model of macroeconomic dynamics with the time-dependent coefficient of the capital intensity of income growth (CIIG). Previously, the only constant coefficients of CIIG were considered; see [12]. To confirm economic validity of the assumption that the coefficient of CIIG depends on time, we investigate the exact solution of the Solow model with the Cobb-Douglas production function. See, e.g., [3, 4] and [7] – [10]. Calculation of the income growth capital intensity factor of this solution shows that this coefficient is a time-dependent function.

In the present paper, we also show that the Harrod-Domar model is quite convenient for using the apparatus of the optimal control theory and the calculus of variations. Using these methods, one can find the maximum of the integral discounted utility function of consumption. There are also formulated and investigated several optimal control problems with various constrains that follows from natural economic conditions. Problems of this type are to find the maximum of a functional that expresses the integral discounted utility function in the presence of a differential relation. Consumption and phase constraints are investigated, extremal problems in the Pontryagin and Dubovitsky-Milyutin forms are considered.

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2. THE EXTENDED HARROD-DOMAR MODEL

In the Harrod-Domar model, the differential equation of the macroeconomic dynamics with exogenous dynamics of the consumption of arbitrary character [12, 13] has the form

$$Y(t) = C(t) + BY'(t). \tag{2.1}$$

In this model, time t is continuous. The income $Y(t)$ is equal to the sum of the consumption $C(t)$ and the investment $I(t)$. Usually, $Y(t)$ refers to the gross domestic product, which is identified with the national income. The economy is supposed to be closed, therefore, the net exports are zero and the government expenses are not considered in the model. The main factor of the growth – the speed of income growth – is proportional to the investment. See [6, 12, 13]. That is, $I(t) = BY'(t)$, where B is the coefficient of the capital intensity of income growth (CIIG) and $1/B$ is the limit product of capital at the macroeconomic level.

Previously, the coefficient of CIIG was supposed to be a positive constant [12, 14]:

$$B = \text{const} > 0. \tag{2.2}$$

In the case (2.2), the solution of differential equation (2.1) is given by the formula

$$Y(t) = Y_0 e^{\frac{t-t_0}{B}} - \frac{1}{B}\int_{t_0}^{t} C(\tau) e^{\frac{t-\tau}{B}} d\tau. \tag{2.3}$$

We consider the Cauchy problem: differential equation (2.1) with the initial condition

$$Y(t_0) = Y_0 > 0. \tag{2.4}$$

The main assumption is that

$$B = B(t). \tag{2.5}$$

The solution of this Cauchy problem is given by the formula (see [13]):

$$Y(t) = Y_0 e^{\int_{t_0}^{t} \frac{ds}{B(s)}} - e^{\int_{t_0}^{t} \frac{ds}{B(s)}} \int_{t_0}^{t} \frac{C(\tau)}{B(\tau)} e^{-\int_{t_0}^{t} \frac{ds}{B(s)}} d\tau. \tag{2.6}$$

Obviously, in the case (2.2) formula (2.6) becomes (2.3).

The economic validity of the assumption (2.5) follows from the comparative analysis of the Harrod-Domar model and the Solow model. From the economic viewpoint, it reflects the rate of the technical progress and the rapidly changing of the economic conditions.

3. THE SOLOW MODEL

There are several different ways of the presentation of the Solow macroeconomic model. See, e.g., the original works [3, 4] and the papers [6]– [11]. In the Solow model, the average per capita capital k satisfies a first-order nonlinear differential equation, which follows from the balance equation for funds:

$$\frac{dk}{dt} = -\lambda k + \rho f(k), \tag{3.7}$$

with the initial condition

$$k(0) = k_0 > 0. \tag{3.8}$$

Equation (3.7) with condition (3.8) yield Cauchy problem we shall deal with.

Here $k = K/L$, where K is the capital or funds, L means human resources (or labour) resources, time t is measured in years, ρ is the norm of accumulation (the share of gross investment in the gross domestic product). Following [7], we shall assume that $\rho = \mathrm{const}$, $0 < \rho < 1$. The function

$$f(k) = \frac{F(K, L)}{L} = F(k, 1). \tag{3.9}$$

Here $F(K, L)$ is the neoclassical production function; see [7, 8, 15].

The coefficient $\lambda = \mu + \nu$, where μ is the share of the annual decrease of main production funds. Similarly to [7], we assume that $\mu = \mathrm{const}$. The second term ν means the annual growth rate of the labour force, i.e.

$$\frac{1}{L}\frac{dL}{dt} = \nu. \tag{3.10}$$

Here μ and ν satisfy the restrictions $0 < \mu < 1$, $-1 < \nu < 1$.

The balance equation for funds reads

$$\frac{dK}{dt} = \rho Y - \mu K. \tag{3.11}$$

Then for the average per capita capital we have the following chain of the equalities

$$\frac{dk}{dt} = \frac{d}{dt}\left(\frac{K}{L}\right) = \frac{K_t' L - K L_t'}{L^2} = \frac{K_t'}{L} - K\frac{L_t'}{L^2} = \frac{\rho Y - \mu K}{L} - \nu\frac{K}{L}.$$

This yields equation (3.7).

The transition mode [7, 8] was investigated under assumption that the production function $F(K, L)$ is the Cobb-Douglas function [7, 8, 14]. Indeed, in this case $f(k)$ is the power function with a constant factor; see [7, 8]. Then equation (3.7) is explicitly integrated, and its solution is represented via elementary functions. If ν is constant, equation (3.7) has the stationary solution $k = k_*$, where k_* is the positive root of the equation

$$\rho f(k) - \lambda k = 0. \tag{3.12}$$

Here we assume that k_0 and k_* belong to the interval of average per capita capital under consideration. For $k \neq k_*$, integrating (3.7) with the initial condition (3.8), we obtain the integral equation

$$\int_{k_0}^{k} \frac{ds}{\rho f(s) - \lambda s} = t \tag{3.13}$$

which determines the function $k = k(t)$. Remark that formula (3.13) is correct if the function $(\rho f(s) - \lambda s)^{-1}$ is integrable on the corresponding interval. Sufficient conditions for this can be formulated in several ways. One can impose some conditions on f (as it is done in [11]) or impose conditions on the neoclassical production function $F(K, L)$ and use (3.9).

To satisfy the basic condition of the transient mode in the Solow model [7, 8]

$$k_\infty = \lim_{t\to+\infty} k(t) = k_*, \tag{3.14}$$

equation (3.12) needs to have one positive root k_* on the interval under consideration. In addition, the improper integral

$$\int_{k_0}^{k_*} \frac{ds}{\rho f(s) - \lambda s} \tag{3.15}$$

needs to diverge. This follows from the limit transition in (3.13) as $t \to +\infty$ and (3.14).

Therefore, for the existence of a transition regime in the Solow model one need to assume that function $f(k)$ generated by the neoclassical production function $F(K, L)$ satisfies to conditions mentioned above: the existence of a unique positive root $k_* \neq k_0$ of equation (3.12) in the interval under consideration and the divergence of the integral (3.15).

It is worth observing that if the solution to the Cauchy problem (3.7), (3.8) is known, then all endogenous variables can be found from the equality

$$Y = F(K, L) = C + I. \tag{3.16}$$

Here, the final product Y is used for the non-productive consumption C and the investment I.

Now let us discuss the Solow model with the Cobb-Douglas production function [7,8,14]:

$$F(K, L) = AK^{\alpha}L^{1-\alpha}, \quad A = \mathrm{const} > 0, \quad 0 < \alpha < 1. \tag{3.17}$$

Substituting (3.17) in formula (3.9), we obtain

$$f(k) = \frac{F(K, L)}{L} = F(k, 1) = Ak^{\alpha}. \tag{3.18}$$

Taking into account (3.18), one can bring equation (3.7) to the Bernoulli form:

$$\frac{dk}{dt} = -\lambda k + \rho A k^{\alpha}. \tag{3.19}$$

Solving equation (3.19) with the initial condition (3.8) and the additional assumption

$$\lambda = \mathrm{const}, \tag{3.20}$$

we get

$$k(t) = e^{-\lambda t}\left[\frac{\rho A}{\lambda}e^{\lambda(1-\alpha)t} - \frac{\rho A}{\lambda} + {k_0}^{1-\alpha}\right]^{\frac{1}{1-\alpha}}. \tag{3.21}$$

Let us find the capital intensity of the income growth for (3.21), that is, for the average per capita capital in the Solow economic growth model with the Cobb-Douglas production function (3.17) and the additional condition (3.20).

Using the basic premise of the Harrod-Domar model, the definition of the Solow model accumulation norm and the balance formula (3.16), we have

$$I(t) = BY'(t) = \rho Y, \tag{3.22}$$

that is,

$$Y'(t) = \frac{\rho}{B(t)}Y(t). \tag{3.23}$$

Integrating (3.23), we get

$$Y(t) = e^{\rho \int_{t_0}^{t} \frac{ds}{B(s)}},$$

where the exponent is the indicator of the income growth.

Theorem 3.1:
Under the above assumptions, the capital intensity of the income growth is given by the following formula:

$$B(t) = \frac{I(t)}{Y'(t)} = \frac{\rho F(K, L)}{\frac{d}{dt}\left[F(K, L)\right]} = \frac{\rho k}{(\nu - \alpha\nu - \alpha\mu)k + \alpha\rho A k^{\alpha}}. \tag{3.24}$$

Proof
From (3.22) and (3.16), we have

$$B(t) = \frac{I(t)}{Y'(t)} = \frac{\rho Y(t)}{Y'(t)} = \frac{\rho F(K,L)}{\frac{d}{dt}\left[F(K,L)\right]}.$$

Using formula (3.17) and multiplying the numerator and the denominator of the right hand side by $A^{-1}L^{\alpha}K^{1-\alpha}$, one can write the latter equality in the form

$$B(t) = \frac{\rho KL}{\alpha L\frac{dK}{dt} + (1-\alpha)K\frac{dL}{dt}}.$$

Applying the formulas (3.10) and (3.11) to the right hand side of this equality, we obtain

$$B(t) = \frac{\rho KL}{\alpha L(\rho Y - \mu K) + \nu(1-\alpha)KL}.$$

Finally, taking into account (3.16), (3.17) and multiplying the numerator and denominator of the right hand side by L^{-2}, we have

$$B(t) = \frac{\rho(K/L)}{\alpha\rho A(K/L)^{\alpha} + (\nu - \alpha\mu - \alpha\nu)(K/L)}.$$

To complete the proof, recall the definition of the average per capita capital: $k = K/L$. $\square$

Remark 3.1:
Formula (3.24) *justifies the assumption that the capital intensity of the income growth depends on time.*

Remark 3.2:
Equation (3.19) *can be considered without assumption* (3.20)*, i.e.,* $\lambda = \lambda(t)$ *is an arbitrary integrable function. This makes sense, because the annual growth rate of the employment (growth rate of the labour force* (3.10)*) is not constant.*

Remark 3.3:
In the case $\lambda = \lambda(t)$*, the solution of Bernoulli equation* (3.19) *with the initial condition* (3.8) *is given by the formula*

$$k(t) = e^{-\int_0^t \lambda(\tau)d\tau}\left[\rho A(1-\alpha)\int_0^t e^{(1-\alpha)\int_0^s \lambda(\tau)d\tau}ds + {k_0}^{1-\alpha}\right]^{\frac{1}{1-\alpha}}.$$

4. OPTIMAL CONSUMPTION IN THE EXTENDED HARROD-DOMAR MODEL

In the extended Harrod-Domar model, formula (2.6) determines the income through the consumption. A natural question: how to find the consumption? From the mathematical viewpoint, this question can be formulated as an optimal control problem.

We consider this problem is the most general form, as the problem of maximizing the integral discounted utility of the consumption [17, 18]:

$$\int_{t_0}^{t_1} u(C(t))\exp(-\delta t)dt \Rightarrow \max, \tag{4.25}$$

where $\delta > 0$ is the discount factor. Here we use the analogy with problems of optimal consumption management in the household economy, see [19] – [23]. Following [19] – [21], we suppose that the utility of consumption is represented by the function $u(C)$, which reflects constant aversion to risk by Arrow-Pratt:

$$a = -\frac{u''(C)C}{u'(C)} \geq 0. \tag{4.26}$$

The economic meaning of (4.26) is clear from the notation

$$g(C) = u'(C). \tag{4.27}$$

Then $g(C)$ is the limit utility of consumption. Further, taking into account (4.26) and (4.27), we have the equalities:

$$g'(C) = u''(C), \quad E_C(g) = \frac{g'(C)}{g(C)/C} = \frac{u''(C)C}{u'(C)} = -a.$$

Here, $E_C(g)$ is the elasticity of g with respect to C. Further, we assume that $g(C)$ is a monotonically decreasing function, therefore, $a \geq 0$. Conversely, the assumption that $g(C)$ is increasing, is associated with the risk. It is natural to call the condition that $g(C)$ does not increase the *disgust to risk*.

Lemma 4.1:
The derivative of the utility function u describing the constant aversion to risk by Arrow-Pratt (4.26) *is given by the formula*

$$u'(C) = g(C) = \frac{\gamma}{C^a} = \gamma C^{-a}, \quad \gamma = \mathrm{const} > 0. \tag{4.28}$$

Proof
Considering (4.26) as a differential equation with the unknown function *u*, we get

$$\frac{a}{C} = -\frac{u''(C)}{u'(C)}.$$

Integrating this equation, we obtain

$$a\int \frac{dC}{C} = \alpha \ln|C| = -\int \frac{u''(C)}{u'(C)}dC = -\ln|u'(C)| + \mathrm{const} = \ln\left|\frac{C_0}{u'(C)}\right|.$$

Since logarithm is a monotonic function, this yields

$$|C|^a = \left|\frac{C_0}{u'(C)}\right|.$$

Since the consumption is positive, one can put $|C_0| = \gamma > 0$. Taking into account (4.27), the last equality implies (4.28). □

Corollary 4.1:
The utility function u satisfying the condition of Lemma 4.1 has the form

$$u(C) = \begin{cases} \dfrac{\gamma C^{1-a}}{1-a} + \chi, & a \neq 1; \\ \gamma \ln C + \chi, & a = 1; \end{cases} \qquad \gamma = \mathrm{const} > 0, \quad \chi = \mathrm{const}. \tag{4.29}$$

Proof
Integrating equation (4.28), we obtain (4.29). □

Theorem 4.1:
The consumption function

$$C(t) = (u')^{-1}\left[\frac{1}{B(t)}C_1 \exp\left\{\delta t - \int_{t_0}^{t}\frac{d\tau}{B(\tau)}\right\}\right] \tag{4.30}$$

gives the maximum in the variation problem (4.25) *with fixed boundaries.*

Proof
Substituting the expression of the consumption from (2.1) into (4.25), we obtain the functional

$$J(Y) = \int_{t_0}^{t_1} u(Y - B(t)Y')\exp(-\delta t)dt, \tag{4.31}$$

whose increment

$$J(Y+h) - J(Y) = \Delta J(Y,h)$$

has the form

$$\Delta J(Y,h) = \int_{t_0}^{t_1}[u(Y+h-B(t)(Y'+h')) - u(Y-B(t)Y')]\exp(-\delta t)dt. \tag{4.32}$$

From the equality $Y - B(t)Y' = C(t)$, it follows that

$$u(Y+h-B(t)(Y'+h')) - u(Y-B(t)Y') = u(C(t)+h-B(t)h') - u(C(t)),$$

$$\Delta J(Y,h) = \int_{t_0}^{t_1}[u(C(t)+h-B(t)h') - u(C(t))]e^{-\delta t}dt. \tag{4.33}$$

Substituting the Taylor expansion

$$u(C(t)+h-B(t)h') = \\ u(C(t)) + u'(C(t))[h-B(t)h'] + \frac{1}{2}u''(C(t))[h-B(t)h']^2 + R(t),$$

where $R(t) = o[h-B(t)h']^2$ as $h - B(t)h' \to 0$, into (4.33), we obtain

$$\Delta J(Y,h) = \int_{t_0}^{t_1} u'(C(t))[h-B(t)h']e^{-\delta t}dt + \\ \frac{1}{2}\int_{t_0}^{t_1} u''(C(t))[h-B(t)h']^2 e^{-\delta t}dt + \int_{t_0}^{t_1} R(t)e^{-\delta t}dt. \tag{4.34}$$

Then, integrating by parts, we have

$$-\int_{t_0}^{t_1} u'(C(t))B(t)h'e^{-\delta t}dt = -\int_{t_0}^{t_1} u'(C(t))B(t)e^{-\delta t}dh = \\ -u'(C(t))B(t)e^{-\delta t}h\Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} h\frac{d}{dt}[u'(C(t))B(t)e^{-\delta t}]dt. \tag{4.35}$$

Since we consider the variation problem (4.25) with fixed boundaries, in addition to the initial condition (2.4), i.e., the boundary condition on the left edge, we have the boundary condition on the right edge:

$$Y(t_1) = Y_1 > 0. \tag{4.36}$$

From (2.4) and (4.36), it follows that for the function $h = h(t)$ the equalities

$$h(t_0) = h(t_1) = 0. \tag{4.37}$$

hold true. From (4.37), it follows that the first term in the right hand part of (4.35) is zero. Therefore, the equality (4.35) reads

$$-\int_{t_0}^{t_1} u'(C(t))B(t)h'e^{-\delta t}dt = \int_{t_0}^{t_1} h\frac{d}{dt}[u'(C(t))B(t)e^{-\delta t}]dt. \tag{4.38}$$

From (4.38) and (4.34) we conclude that

$$\Delta J(Y,h) = \int_{t_0}^{t_1} h\Big(u'(C(t))e^{-\delta t} + \frac{d}{dt}[u'(C(t))B(t)e^{-\delta t}]\Big)dt+ \\ \frac{1}{2}\int_{t_0}^{t_1} u''(C(t))[h - B(t)h']^2e^{-\delta t}dt + \int_{t_0}^{t_1} R(t)e^{-\delta t}dt. \tag{4.39}$$

The sign of the left hand side of (4.39) coincides with the sign of the first term in the right hand side of (4.39). Indeed, replacing h with βh, where $\beta = \mathrm{const}$, we get

$$\Delta J(Y,\beta h) = J(Y+\beta h) - J(Y) = \\ = \beta\int_{t_0}^{t_1} h\Big(u'(C(t))e^{-\delta t} + \frac{d}{dt}[u'(C(t))B(t)e^{-\delta t}]\Big)dt+ \\ \frac{1}{2}\beta^2\int_{t_0}^{t_1} u''(C(t))[h - B(t)h']^2e^{-\delta t}dt + \int_{t_0}^{t_1} \tilde{R}(t)e^{-\delta t}dt, \tag{4.40}$$

where $\tilde{R}(t) = o(\beta^2[h - B(t)h']^2)$ as $\beta(h - B(t)h') \to 0$. Passing to the limit $\beta \to 0$, one can see that the first term in the right hand side of (4.40) tends to zero with the first order of smallness, while the second term has the second order of smallness and the third term has the order greater than two. This proves the statement.

Now let us prove that the first term in the right hand side of (4.39) is zero. Suppose the contrary. Then, replacing β with $-\beta$, we change the sign of the first term in the right hand side of (4.40), while the sign of the left hand side of (4.40) remains the same. This contradiction shows that

$$\int_{t_0}^{t_1} h\Big(u'(C(t))e^{-\delta t} + \frac{d}{dt}[u'(C(t))B(t)e^{-\delta t}]\Big)dt = 0.$$

By the fundamental lemma of the calculus of variations, this yields the Euler equation

$$u'(C(t))e^{-\delta t} + \frac{d}{dt}[u'(C(t))B(t)e^{-\delta t}] = 0. \tag{4.41}$$

Taking into account (4.41), one can simplify (4.39) as follows:

$$\Delta J(Y,h) = \frac{1}{2}\int_{t_0}^{t_1} u''(C(t))[h - B(t)h']^2e^{-\delta t}dt + \int_{t_0}^{t_1} R(t)e^{-\delta t}dt. \tag{4.42}$$

Let us check that the solution of the Euler equation (4.41) with the boundary conditions (2.4) and (4.36), and consequently, (4.37), give the maximum of the functional (4.25), or equivalently, (4.31). First, we remark that the sign of the left hand side of (4.42) coincides with the sign of the first term in the right hand side of (4.42). Indeed, replacing in (4.42) h with βh, where $\beta = \text{const}$, we have

$$\Delta J(Y, \beta h) = \frac{1}{2}\beta^2 \int_{t_0}^{t_1} u''(C(t))[h - B(t)h']^2 e^{-\delta t} dt + \int_{t_0}^{t_1} \tilde{R}(t) e^{-\delta t} dt. \tag{4.43}$$

Here $\tilde{R}(t) = o(\beta^2 [h - B(t)h']^2)$ as $\beta(h - B(t)h') \to 0$.

Passing to the limit $\beta \to 0$, one can see that the first term tends to zero with the second order of smallness, while the second term the order greater than two. Therefore, the sign of the left side of (4.43) coincides with the sign of the first term in the right hand side of (4.43). It remains to use the inequality $u''(C(t)) \leq 0$, which follows from (4.26).

Taking into account (4.27), we can write equation (4.41) in the form

$$g(C(t))e^{-\delta t} + \frac{d}{dt}[g(C(t))B(t)e^{-\delta t}] = 0. \tag{4.44}$$

The change of variables

$$w = g(C(t))B(t)e^{-\delta t}, \tag{4.45}$$

i.e., $g(C(t))e^{-\delta t} = w/B(t)$, transforms equation (4.44) into

$$\frac{w}{B(t)} + \frac{dw}{dt} = 0.$$

Integrating the latter equation, we obtain

$$w = C_1 \exp\left\{ - \int_{t_0}^{t} \frac{d\tau}{B(\tau)} \right\}, \quad C_1 = \text{const}.$$

Substituting the obtained equality in (4.45), after obvious transformations we obtain

$$g(C(t)) = \frac{1}{B(t)} C_1 \exp\left\{ \delta t - \int_{t_0}^{t} \frac{d\tau}{B(\tau)} \right\}, \quad C_1 = \text{const} > 0. \tag{4.46}$$

Finally, recall that $g(C)$ monotonically decreases. Therefore, it is invertible, and

$$C(t) = g^{-1}\left[\frac{1}{B(t)} C_1 \exp\left\{ \delta t - \int_{t_0}^{t} \frac{d\tau}{B(\tau)} \right\} \right], \quad C_1 = \text{const} > 0. \tag{4.47}$$

Taking into account (4.27), from (4.47) it follows (4.30). The proof is complete. □

Remark 4.1:
The consumption function (4.30) *can be also written in the form*

$$C(t) = \left[\frac{\gamma B(t)}{C_1} \right]^{\frac{1}{a}} \exp\left\{ \frac{1}{a} \left[\int_{t_0}^{t} \frac{d\tau}{B(\tau)} - \delta t \right] \right\}. \tag{4.48}$$

Proof
From (4.28) and (4.46) we have the equality

$$[C(t)]^a = \frac{\gamma B(t)}{C_1} \exp\left\{ \int_{t_0}^{t} \frac{d\tau}{B(\tau)} - \delta t \right\}, \tag{4.49}$$

which can be resolves by C and it gives (4.48). □

Remark 4.2:
The consumption function (4.30) *or, equivalently,* (4.48) *gives the maximum in the variation problem* (4.25) *with fixed boundary conditions* (2.4) *and* (4.36). *Formula* (4.49) *expresses the consumption under the condition* (4.26)*, which means that the utility function satisfies the constant risk aversion according to Arrow-Pratt.*

It is very convenient to formulate the above problems using the terminology from the control theory. The problem of maximization of the functional (4.25) under constraints (2.1), (2.4), (4.36), and the additional restriction

$$0 < \underline{C} \leq C(t) \leq \overline{C} < +\infty \tag{4.50}$$

is called the Pontryagin problem. In the inequality (4.50), the lower bound $\underline{C}$ means the total subsistence minimum and the upper bound $\overline{C}$ is the total subsistence maximum.

In turn, the Pontryagin problem can be also formulated with an additional phase constraint, for example, $Y(t) \geq \text{const} \geq 0$. Such a problem is often called the Dubovitsky-Milyutin problem. See [19] – [21].

5. CONCLUSION

We presented the comparative analysis of two models of economic dynamics: the model of the economic growth by Harrod-Domar and the model by Solow. There were several earlier models of the economic growth, but they are not widely acknowledged; see, e.g., [24]. At present, more advanced models are gaining popularity, however, they are based on the models discussed in the present paper; see [25] – [27].

The comparative analysis confirms the economic viability of the assumption that the CIIG depends on time. Comparing these models and using the analogy with household economies [19] – [23], we demonstrate the efficiency of the control theory approach in the extended Harrod-Domar model. The obtained results shaw that despite significant differences between these models, their comparative analysis is substantial.

It is worth observing that using the approach [28] – [31] based on the theory of covering mappings, the both considered models can be generalized to the market of many goods with various production functions.

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