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Local Normal Forms of Autonomous Quasi-Linear Constrained Differential Systems

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Abstract: The paper presents a study of impasse (singular) points of autonomous quasi-linear constrained differential systems, also called differential-algebraic equations. The interest in such systems is motivated by their applications in various problems of pure and applied mathematics, including control theory, biology, and electric engineering. Local normal forms of such systems in a neighborhood of their impasse points are established.

Keywords: impasse point, direction field, normal form, diffeomorphism, symmetry

1. INTRODUCTION

We consider systems of differential equations in the form

$$A(\xi)\xi' = b(\xi), \;\; \xi' = \frac{d\xi}{dt}, \;\; \xi = (\xi_1, \dots, \xi_n)^* \in \mathbb{R}^n, \tag{1.1}$$

where $A = (a_{ij})$ is an $n \times n$ matrix and $b = (b_1, \dots, b_n)^*$ is a vector function. The components a_{ij}, b_j are assumed to be C^∞-smooth functions on ξ. There are many possible names of systems (1.1): differential systems of Sobolev type, generalized vector fields, descriptor systems, quasi-linear *constrained*[†] differential systems, etc.

The interest in systems (1.1) is motivated by their applications in various problems of pure and applied mathematics (including control theory and electric engineering). For instance, systems (1.1) describe the dynamic of electric circuit in nonlinear RLC-networks (networks consisting of a resistor, an inductor, and a capacitor). See [1]– [14] and the references therein. In a pioneer work [15], systems (1.1) appear as a tool for studying a systems of PDEs of the mixed type, which describes the motion of a body filled with a viscous incompressible fluid. In a recent series of papers [16]– [20], systems (1.1) in dimension $n = 3$ describe geodesic lines in singular metrics.

There exists several different approaches for studying systems (1.1) and, more generally, nonlinear systems of differential equations

$$F(t, \xi, \xi') = 0, \;\; \xi' = \frac{d\xi}{dt}, \;\; \xi = (\xi_1, \dots, \xi_n)^* \in \mathbb{R}^n, \tag{1.2}$$

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[†] The reason for the this term will soon become clear.

which are often called *implicit* differential equations or *differential-algebraic* equations (DAEs). Here $F : \mathbb{R}^{2n+1} \to \mathbb{R}^n$ is assumed to be a C^∞-smooth mapping. There exists several approaches to investigation of systems (1.1), (1.2).

An analytical approach is based on the decoupling procedure. The idea is to rearrange terms within the given system so that it is decomposed in two subsystems of lower dimensions (separated as far as possible), where the first subsystem is equivalent to a standard ODE of of maximum possible dimension and the second one is a DAE having a special form. The highest goal is so-called complete decoupling, where the both subsystems are completely independent, that is, they do not have common unknowns. For linear DAEs with constant coefficients this method is comparable with the Weierstrass–Kronecker normal form of regular matrix pencils. This approach is presented in the book [21].

Another geometric approach goes back to H. Poincaré.[‡] He used it for a single implicit differential equation $F(x, y, y') = 0$, that is, the case $n = 1$. Further development of this approach is given, e.g., in [22]– [34]. In the modern terminology, the main idea is the following.

By J^k denote the space of k-jets of vector functions $\xi(t) : \mathbb{R} \to \mathbb{R}^n$. In the case of systems (1.2), consider a manifold $M_F^{n+1} \subset J^1$, $J^1 \simeq \mathbb{R}^{2n+1}$, defined by the equation $F = 0$. Moreover, DAE (1.2) defines a direction field on M_F^{n+1}, whose integral curves are 1-graphs (Legendrian lifts) of integral curves of (1.2). In other words, we pass from the multivalued vector direction field given by DAE (1.2) in J^0 (i.e., the (t, ξ)-space) to a single-valued direction field on the manifold M_F^{n+1}. The natural projection $\pi : J^1 \to J^0$ sends integral curves of the field defined on M_F^{n+1} to integral curves of DAE (1.2). The restriction of the projection $\pi : M_F^{n+1} \to J^0$ is a mapping of two $(n + 1)$-dimensional manifolds. It has singularities at those points of M_F^{n+1} where the matrix $(\partial F/\partial \xi')$ vanishes. Integral curves of (1.2) have singularities (generically, cusps) at corresponding point of the (t, ξ)-space.

In the case of systems (1.1), the above construction is essentially simplified: no need to consider the space J^1. Indeed, writing system (1.1) in the pfaffian form

$$\begin{cases} a_{11}d\xi_1 + a_{12}d\xi_2 + \cdots + a_{1n}d\xi_n - b_1dt = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}d\xi_1 + a_{n2}d\xi_2 + \cdots + a_{nn}d\xi_n - b_ndt = 0, \end{cases} \tag{1.3}$$

one can see that system (1.1) defines a direction field in the (t, ξ)-space:

$$\dot{\xi}_1 = \Delta_1(\xi), \dots, \dot{\xi}_n = \Delta_n(\xi), \;\; \dot{t} = \Delta(\xi), \tag{1.4}$$

where Δ is the determinant of the matrix A, and Δ_i is the determinant of the matrix obtained from A by replacing of its ith column with b. Here the dot over a symbol means the differentiation by a new parameter playing the role of time.

A principal difference between system (1.1) and usual autonomous ODEs is that system (1.1) (and consequently, field (1.4)) possesses so-called *degenerate* hypersurface

$$\Gamma = \{\xi : \Delta(\xi) = 0\},$$

which is also called the *criminant* of the system (1.1). Generically, there are no integral curves of (1.1) passing through Γ; see [29]. Therefore, Γ is also called *impasse* hypersurface of the system, and points of Γ are called *impasse* points. It is worth observing that if system (1.1) describes a model of a natural process in real time, the existence of impasse points in the applicability domain often means that the model is not adequate.

[‡]Mémoire sur les courbes définies par les équations différentielles (1885).

The germs of two systems (1.1) are called *C^k-equivalent*, if there exists a local C^k-diffeomorphism of the ξ-space that conjugates the germs of the corresponding direction fields (1.3). Our goal is to simplify field (1.3) (and consequently, the corresponding system (1.1)) in a neighborhood of its impasse points using an appropriate local C^k-diffeomorphism of the ξ-space. Here $k \geq 1$ is integer or ∞ or ω. C^ω is the standard symbol for the class of analytic maps.[§]

The germs of two systems (1.1) are called *orbitally C^k-equivalent*, if there exist a local C^k-diffeomorphism of the ξ-space and C^k-mapping of the independent variable $t \mapsto \tau(t, \xi)$ that conjugate the germs of the corresponding direction fields (1.3).

The classification of systems (1.1) at all impasse points is not observable. According to the general ideology of singularity theory, a number of classes of impasse points that admits relatively simple normal forms is determined. For the C^ω-equivalence, many results of this sort appear in [28], where systems (1.1) are considered in real, complex and even infinite dimensional (Banach) complex spaces. For the C^k-orbital equivalence, many results appear in [32].

In this paper, we present some new results about C^k-normal forms of the germs of systems (1.1) at their impasse points of a special type.

2. MAIN RESULTS

There are several geometric objects naturally connected with system (1.1).

The firts object of thid sort is the criminant Γ. Other important object are the family of linear operators

$$A(\xi) : T_\xi\mathbb{R}^n \to T_\xi\mathbb{R}^n,$$

their images $\operatorname{Im} A(\xi)$ and kernels $\operatorname{Ker} A(\xi)$. The criminant Γ is the locus of points ξ where $\dim \operatorname{Im} A(\xi) < n$ or, equivalently, $\dim \operatorname{Ker} A(\xi) > 0$.

The simplest type of impasse points is so-called *non-singular* impasse points $\xi \in \Gamma$ that satisfy three following conditions:

1. $d\Delta(\xi) \neq 0$, that is, Γ is a regular hypersurface.
2. $\dim \operatorname{Ker} A(\xi) = 1$ and the direction $\operatorname{Ker} A(\xi)$ is transversal to Γ.
3. $b(\xi) \notin \operatorname{Im} A(\xi)$, that is, $\operatorname{Im} A(\xi) \oplus \langle b(\xi) \rangle = \mathbb{R}^n$.

Theorem 2.1:
The germ of system (1.1) *at every its non-singular impasse point is C^∞-equivalent (in the analytic category, C^ω-equivalent) to*

$$\xi_1' = 0, \ldots, \xi_{n-1}' = 0, \;\; \xi_n \, \xi_n' = \pm 1.$$

Moreover, in the category of orbital equivalence, one can reduce ± 1 *to* 1.

The proof of Theorem 2.1 can be found in [27, 28, 32].

Now consider a point $\xi_\circ \in \Gamma$ such that the 1st and 2nd conditions hold true, but the 3rd condition fails. From $\dim \operatorname{Ker} A(\xi_\circ) = 1$ (the 2nd condition) and the known identity

$$\dim \operatorname{Ker} A(\xi) + \dim \operatorname{Im} A(\xi) = n$$

it follows that $\dim \operatorname{Im} A(\xi_\circ) = n - 1$, i.e., rank of the matrix $A(\xi_\circ) = n - 1$. Then there exists $i \in \{1, \ldots, n\}$ such that rank of the matrix obtained from $A(\xi_\circ)$ by eliminating the ith column is $n - 1$, and the germs of Δ, $\Delta_1, \ldots, \Delta_n$ belong to the ideal $I = \langle \Delta, \Delta_i \rangle$ in the ring of smooth functions, and the set of singular points (equilibriums) of the corresponding

[§] In speaking about C^ω-equivalence, we assume that the systems (1.1) are also analytic.

field (1.4) is given by two equations: $\Delta = \Delta_i = 0$. (The detailed proof of this fact can be found in [31].)

Therefore, singular point of the field (1.4) are not isolated, they fill a submanifold $W \subset \Gamma$ of codimension 2. At every point $\xi \in W$ in a neighborhood of $\xi_\circ$, the spectrum of the linear part of the field (1.4) is $(\lambda_1, \lambda_2, 0, \dots, 0)$. Here $\lambda_{1,2}$ are complex numbers, generically $\operatorname{Re}\lambda_{1,2} \neq 0$ at almost all points of W.

Lemma 2.1:
Assume that $\operatorname{Re}\lambda_{1,2}(\xi_\circ) \neq 0$. *Then in a neighborhood of* $\xi_\circ$*, the following statements hold true:*

(i) *The set* W *is the center manifold of the field* (1.4).

(ii) *There exist* $i_1, i_2 \in \{1, \dots, n\}$ *such that the ideal* $I = \langle \Delta_{i_1}, \Delta_{i_2} \rangle$.

Proof
The first statement is trivial.

For the second statement, remark that it is possible to select as generators of I any two elements $v, w \in I$ such that dv, dw at the point $\xi_\circ$ are linearly independent. Consider the Jacobi matrices

$$J_0 = \left\| \frac{\partial(\Delta_1, \dots, \Delta_n, \Delta)}{\partial(\xi_1, \dots, \xi_n, t)} \right\|, \quad J = \left\| \frac{\partial(\Delta_1, \dots, \Delta_n)}{\partial(\xi_1, \dots, \xi_n)} \right\|.$$

at the point $\xi_\circ$. Since the functions $\Delta_1, \dots, \Delta_n, \Delta$ do not depend on t, the spectrums of matrices J_0 and J have the form $(\lambda_1, \lambda_2, 0, \dots, 0)$ with the same $\lambda_{1,2}$. Therefore, among the functions $\Delta_1, \dots, \Delta_n$ there exist a couple $v = \Delta_{i_1}$, $w = \Delta_{i_2}$ whose differentials at $\xi_\circ$ are linearly independent. Therefore, v, w are generators of the ideal I. $\square$

Lemma 2.1 shows that using an appropriated renaming of the variables

$$(\xi_1, \dots, \xi_n) \mapsto (x, y, z), \quad z = (z_1, \dots, z_{n-2}),$$

one can write the corresponding field (1.4) in the form

$$\begin{gathered} \dot{x} = v, \quad \dot{y} = w, \quad \dot{z}_i = \alpha_i v + \beta_i w, \quad i = 1, \dots, n-2, \\ \dot{t} = av + bw, \end{gathered} \tag{2.5}$$

Here $a, b, \alpha, \beta, v, w$ are C^∞ (resp., C^ω) functions on x, y, z such that v, w vanish at $\xi_\circ$. The ideal $I = \langle v, w \rangle$, and the center manifold W consisting of singular points is given by the equations $v = w = 0$.

Since all components of the field (2.5) do not depend on t, one can consider the projection of (2.5) to the ξ-space:

$$\dot{x} = v, \quad \dot{y} = w, \quad \dot{z}_i = \alpha_i v + \beta_i w, \quad i = 1, \dots, n-2. \tag{2.6}$$

The classification of fields (2.6) much simpler than the classification of fields with isolated singular points with the same spectrum, see [17, 18, 31]. It is explained by the fact that the non-trivial dynamics is connected with the center manifold of the field. In the case of (2.6), the center manifolds consists of singular points of the fields, therefore the non-trivial dynamics is identically zero.

Lemma 2.2:
Assume that $\operatorname{Re}\lambda_{1,2}(\xi_\circ) \neq 0$ *and there are no resonances*

$$p_1\lambda_1(\xi_\circ) + p_2\lambda_2(\xi_\circ) = 0, \quad p_{1,2} \in \mathbb{Z}_+, \quad p_1 + p_2 \geq 1. \tag{2.7}$$

Then for any integer $k \geq 1$*, the germ of* (2.6) *is* C^k*-equivalent to the germ*

$$\dot{x} = \tilde{v}(x, y, z), \quad \dot{y} = \tilde{w}(x, y, z), \quad \dot{z}_i = 0, \quad i = 1, \dots, n-2, \tag{2.8}$$

at the origin, with some $\tilde v, \tilde w \in \langle x, y\rangle$*. Moreover, if* $\operatorname{Re}\lambda_{1,2}(\xi_\circ)$ *have the same sign, the above equivalence holds true with* $k = \infty$*.*

Without loss of generality, further we assume that $|\lambda_1(\xi_\circ)| \le |\lambda_2(\xi_\circ)|$. Put

$$\lambda_\circ = \lambda_1(\xi_\circ) : \lambda_2(\xi_\circ), \ \ \mu_\circ = \lambda_2(\xi_\circ) : \lambda_1(\xi_\circ).$$

Lemma 2.3:
Assume that the conditions of Lemma 2.2 hold true. Then the germ of the field (2.6) *is orbitally* C^k*-equivalent to the germ* (2.8)*, where* $\tilde v, \tilde w$ *have the form*
(a) $\tilde v = x$, $\tilde w = \lambda(z)y$, *if* $\lambda_{1,2}(\xi_\circ)$ *are real and* $\mu_\circ \notin \mathbb{N} \cup \mathbb{Q}_-$*,*
(b) $\tilde v = \lambda(z)x$, $\tilde w = y + \alpha(z)x^{\mu_\circ}$*, if* $\mu_\circ \in \mathbb{N} \setminus \{1\}$*,*
(c) $\tilde v = \alpha(z)x + \beta(z)y$, $\tilde w = -\beta(z)x + \alpha(z)y$*, if* $\lambda_{1,2}(\xi_\circ)$ *are complex.*

Note that $\lambda(0) = \lambda_\circ$, $\mu(0) = \mu_\circ$ in the cases (a), (b). In the case (c), $\alpha(0), \beta(0)$ are respectively the real and imaginary parts of $\lambda_{1,2}(\xi_\circ)$.
Lemmas 2.2, 2.3 can be found in [17, 18, 31].

In the cases (a), (b), (c) enumerated in Lemma 2.3, consider three families of local diffeomorphisms $(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ given by the following changes of the variables x, y:

$$\text{(a)} \quad x \mapsto x\varphi, \ \ y \mapsto y\varphi^{\lambda(z)}, \tag{2.9}$$

$$\text{(b)} \quad x \mapsto x\varphi, \quad \begin{cases} y \mapsto y\varphi^{\mu(z)} + \alpha(z)\mu(z)x^{\mu_\circ}\dfrac{\varphi^{\mu(z)} - \varphi^{\mu_\circ}}{\mu(z) - \mu_\circ}, & z \ne 0, \\ y \mapsto y\varphi^{\mu(z)} + \alpha(z)\mu(z)x^{\mu_\circ}\varphi^{\mu_\circ}\ln\varphi, & z = 0, \end{cases} \tag{2.10}$$

with all smooth strictly positive functions $\varphi = \varphi(x, y, z)$, and

$$\text{(c)} \quad \begin{aligned} x &\mapsto e^{\alpha(z)\varphi}(x\cos\beta(z)\varphi + y\sin\beta(z)\varphi), \\ y &\mapsto e^{\alpha(z)\varphi}(y\cos\beta(z)\varphi - x\sin\beta(z)\varphi), \end{aligned} \tag{2.11}$$

with all smooth functions $\varphi = \varphi(x, y, z)$.

Lemma 2.4:
The families of diffeomorphisms (2.9)*,* (2.10)*,* (2.11) *consist of symmetries of the direction field* (2.8) *in the cases (a), (b), (c), respectively.*

In other words, every diffeomorphism from the families (2.9), (2.10), (2.11) sends vector field (2.8) into a parallel vector field¶ in the cases (a), (b), (c), respectively.
It is worth observing that families (2.9), (2.10), (2.11) are subgroups of the whole group of symmetries of the direction field (2.8) in the cases (a), (b), (c), respectively, but non of them coincide with the whole group. For instance, the family (2.9) does not contain the linear changes

$$x \mapsto c_1 x, \ \ y \mapsto c_2 y$$

with arbitrary constants $c_{1,2} \ne 0$.
The proof of Lemma 2.4 is by direct calculations and omitted.
Now we can get the main result of the paper.

Theorem 2.2:
Let $\xi_\circ \in \Gamma$ *be an impasse point of the system* (1.1) *satisfying the above conditions. In addition, in the cases (a), (b), we assume that the eigenvectors with* $\lambda_{1,2}(\xi_\circ)$ *are transversal to the*

¶Two vector fields are parallel, if one of them is obtained from another by multiplication by a non-vanishing smooth function. In other words, two vector fields are parallel, if the corresponding direction fields coincide.

criminant Γ. *Then for any integer* $k \geq 1$, *the germ of system* (1.1) *is* C^k*-equivalent to the germ*

$$(x+y)\psi x' = \tilde{v}, \;\; (x+y)\psi y' = \tilde{w}, \;\; z_1' = 0, \, \ldots, \, z_{n-2}' = 0, \tag{2.12}$$

at the origin, where $\tilde{v}, \tilde{w}$ *are defined in Lemma 2.3 and* $\psi = \psi(x, y, z)$ *is a* C^∞*-smooth function,* $\psi(0) \neq 0$. *Under the same conditions, the germ of system* (1.1) *is orbitally* C^k*-equivalent to* (2.12) *with the same* $\tilde{v}, \tilde{w}$ *and* $\psi \equiv 1$. *Moreover, if* $\operatorname{Re} \lambda_{1,2}(\xi_\circ)$ *have the same sign, the above equivalences hold true with* $k = \infty$.

Proof

By Lemma 2.3, there exist a C^k-smooth local diffeomorphism

$$f: \;\; (\xi_1, \ldots, \xi_n) \mapsto (x, y, z), \;\; z = (z_1, \ldots, z_{n-2}),$$

that sends the point $\xi_\circ$ to 0 and brings the germ of the direction field (2.6) at $\xi_\circ$ to the orbital normal form (2.8). Then f transforms the corresponding direction field (2.5) into the form

$$\begin{gathered} \dot{x} = \tilde{v}, \;\; \dot{y} = \tilde{w}, \;\; \dot{z}_i = 0, \;\; i = 1, \ldots, n-2, \\ \dot{t} = a\tilde{v} + b\tilde{w}, \end{gathered} \tag{2.13}$$

where $\tilde{v}, \tilde{w}$ are defined in 2.3 and a, b are smooth functions on the variables x, y, z.

The criminant of the system (1.1) corresponding to (2.13) is given by the equation $a\tilde{v} + b\tilde{w}$. The condition that the eigenvectors with $\lambda_{1,2}(\xi_\circ)$ are transversal to the criminant yields $a(0) \neq 0$, $b(0) \neq 0$. Let us show that there exists a diffeomorphism of the families (2.9), (2.10), (2.11) in the cases (a), (b), (c), respectively, that convert the criminant Γ into a hyperplane, for instance, the hyperplane $x + y = 0$.

Consider that case (a). Then $\tilde{v} = x$, $\tilde{w} = \lambda(z)y$. By Lemma 2.4, any diffeomorphism of the family (2.9) transforms the direction field (2.13) into

$$\begin{aligned} &\dot{x} = x, \;\; \dot{y} = \lambda(z)y, \;\; \dot{z}_i = 0, \;\; i = 1, \ldots, n-2, \\ &\dot{t} = ax\varphi + b\lambda(z)y\varphi^{\lambda(z)} = a\varphi\,(x + a^{-1}b\lambda(z)\varphi^{\lambda(z)-1}y). \end{aligned} \tag{2.14}$$

Without loss of generality, assume that $a^{-1}(0)b(0)\lambda(0) > 0$. Otherwise one can make the change variables $y \mapsto -y$, which preserves the first $n-1$ components of the field (2.14) and brings its last component to the desirable form.

Thus, it suffices to establish the existence of a positive smooth function φ satisfying the equation

$$a^{-1}(x\varphi, y\varphi^{\lambda(z)}, z)\, b(x\varphi, y\varphi^{\lambda(z)}, z)\, \lambda(z)\, \varphi^{\lambda(z)-1} = 1. \tag{2.15}$$

The derivative of the left-hand side of (2.15) by φ is not equal to zero. By the implicit function theorem, a function φ with the desirable properties exists in a neighborhood of the origin.

The diffeomorphism (2.9) with φ founded above preserves the first $n-1$ components of the direction field (2.14) and transforms its last component into

$$\dot{t} = \psi(x, y, z)\,(x + y),$$

where ψ is a smooth function non-vanishing at the origin. The corresponding system (1.1) can be written in the form

$$(x+y)\psi x' = x, \;\; (x+y)\psi y' = \lambda(z)y, \;\; z_1' = 0, \, \ldots, \, z_{n-2}' = 0. \tag{2.16}$$

Finally, if we deal with the orbital equivalence, one can make the change of the independent variable: $t \mapsto \tau$, where the new and old variables are connected with the differential equation

$$\frac{dt}{d\tau} = \psi(x, y, z).$$

The geometric sense this equation is that we make a parameterization integral curves of system (2.16), which depends on a point on every curve, so that the function ψ in (2.16) becomes identically 1.

The case (b) is similar to (a) and omitted.

Consider that case (c). It is easy to see that $(\alpha-\beta)a-(\alpha+\beta)b$, $(\alpha+\beta)a+(\alpha-\beta)b$ do not simultaneously vanish at the origin. Without loss of generality, we assume that the latter functions does not vanish at the origin.

By Lemma 2.4, any diffeomorphism of the family (2.11) transforms the direction field (2.13) with

$$\tilde{v}=\alpha(z)x+\beta(z)y, \quad \tilde{w}=-\beta(z)x+\alpha(z)y$$

into the direction field whose first $n-1$ components coincides with those in (2.13) and the nth component is

$$\dot{t}=xe^{\alpha\varphi}((\alpha a-\beta b)\cos\beta\varphi-(\alpha b+\beta a)\sin\beta\varphi)+ \\ ye^{\alpha\varphi}((\alpha a-\beta b)\sin\beta\varphi+(\alpha b+\beta a)\cos\beta\varphi).$$

Thus, it suffices to establish the existence of a smooth function φ such that

$$(\alpha a-\beta b)\cos\beta\varphi-(\alpha b+\beta a)\sin\beta\varphi=(\alpha a-\beta b)\sin\beta\varphi+(\alpha b+\beta a)\cos\beta\varphi, \tag{2.17}$$

where α,β are given functions on z, while a,b depend on φ:

$$a=a(e^{\alpha\varphi}(x\cos\beta\varphi+y\sin\beta\varphi),e^{\alpha\varphi}(-x\sin\beta\varphi+y\cos\beta\varphi),z),$$

$$b=b(e^{\alpha\varphi}(x\cos\beta\varphi+y\sin\beta\varphi),e^{\alpha\varphi}(-x\sin\beta\varphi+y\cos\beta\varphi),z).$$

Equation (2.17) can be transformed to the equivalent form

$$\tan(\beta\varphi)=\frac{(\alpha-\beta)a-(\alpha+\beta)b}{(\alpha+\beta)a+(\alpha-\beta)b} \tag{2.18}$$

The existence of a function φ satisfying equation (2.18) follows from the implicit function theorem. The remaining part of the proof in the case (c) repeats the reasonings carried out in the case (a). □

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