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On Properties of Coincidence Points of Mappings between (q_1, q_2)-Quasimetric Spaces

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Abstract: In this paper, the properties of coincidence points of mappings acting between (q_1, q_2)-quasimetric spaces are studied. For a pair of mappings, we obtain estimates for the distance from a point to the coincidence points set and intersection of the respective graphs of the mappings. In addition, the stability of coincidence points is studied. A generalization of Lim's lemma is obtained.

Keywords: (q_1, q_2)-quasimetric spaces, coincidence points, set-valued mappings

1. INTRODUCTION AND STATEMENT OF THE PROBLEM

The paper is devoted to the investigation of coincidence points of pairs of set-valued mappings acting between (q_1, q_2)-quasimetric spaces. In order to proceed to the statement of the problem, let us recall the definitions of the concepts in use.

Let X be a nonempty set, numbers $q_0 \geq 1$, $q_1 \geq 1$, $q_2 \geq 1$ be given. A function $\rho_X : X \times X \to \mathbb{R}_+$ is called a (q_1, q_2)-*quasimetric* if

- $\rho_X(x, y) = 0 \Leftrightarrow x = y \; \forall \, x, y \in X$
  (the identity axiom);
- $\rho_X(x, z) \leq q_1\rho_X(x, y) + q_2\rho_X(y, z) \quad \forall x, y, z \in X$
  (the (q_1, q_2)-generalized triangle inequality).

If ρ_X is a (q_1, q_2)-quasimetric, then the space (X, ρ_X) is called a (q_1, q_2)-*quasimetric space*.

The concept of the (q_1, q_2)-quasimetric space was introduced in [1]. If $q_1 = q_2 = 1$ then this concept coincides with the concept of a quasimetric space. If we additionally assume that a quasimetric satisfies the symmetry axiom, i.e. $\rho_X(x, y) \equiv \rho_X(y, x)$, it becomes a metric.

The detailed description of topological properties of (q_1, q_2)-quasimetric spaces was provided in [2]. Recall some basic definitions. A sequence $\{x_i\} \subset X$ is said to *converge* to a point $x \in X$ if $\rho_X(x, x_i) \to 0$ as $i \to \infty$. The point x is called a limit of $\{x_i\}$. A subset of X is said to be *closed* if every limit of every convergent sequence from this set belongs to this set. A sequence $\{x_i\} \subset X$ is said to be a Cauchy sequence if

$$\forall \, \varepsilon > 0 \quad \exists \, N \in \mathbb{N} : \quad \rho_X(x_j, x_i) < \varepsilon \quad \forall \, i > j > N.$$

Let (X, ρ_X) and (Y, ρ_Y) be (q_1, q_2)-quasimetric spaces, $\Phi, \Psi : X \rightrightarrows Y$ be set-valued mappings that map points $x \in X$ to non-empty subsets of Y.

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A point $\xi \in X$ is called a *coincidence point* of the set-valued mappings Φ, Ψ if

$$\Phi(\xi) \cap \Psi(\xi) \neq \emptyset.$$

Below we use the following notation: $\mathrm{Coin}(\Psi, \Phi)$ stands for the set of all coincidence points of mappings Ψ and Φ; $\mathrm{gph}\Psi$ stands for the graph of Ψ, i.e.

$$\mathrm{gph}\Psi = \{(x, y) \in X \times Y : \, y \in \Psi(x)\};$$

$\Gamma(\Psi, \Phi)$ stands for the intersection of the graphs of Ψ and Φ, i.e.

$$\Gamma(\Psi, \Phi) := \{(x, y) \in X \times Y : \, y \in \Psi(x) \cap \Phi(x)\}.$$

It is obvious that

$$\mathrm{Coin}(\Psi, \Phi) \neq \emptyset \;\Leftrightarrow\; \Gamma(\Psi, \Phi) \neq \emptyset.$$

For non-empty sets $U, V \subset X$, denote

$$\mathrm{dist}(U, V) = \inf\{\rho_X(x_1, x_2) : \, x_1 \in U, \, x_2 \in V\},$$

$$h_X^+(U, V) = \sup_{u \in U} \mathrm{dist}(u, V),$$

$$h_X(U, V) := \max\{h_X^+(U, V), h_X^+(V, U)\}.$$

The function h_X^+ is called the *Hausdorff deviation*; the function h_X is called the *Hausdorff* $(\hat{q}_1, \hat{q}_2)$*-quasimetric* (note that $(\hat{q}_1, \hat{q}_2)$ may differ from (q_1, q_2)). For Hausdorff deviation h_X^+, the (q_1, q_2)-generalized triangle inequality holds (see [3, page 25]), i.e.

$$h_X^+(U, W) \leq q_1 h_X^+(U, V) + q_2 h_X^+(V, W) \quad \forall \, U, V, W \subset X.$$

Moreover, the definitions above directly imply that

$$h_X(U, V) \geq h_X^+(U, V) \geq \mathrm{dist}(U, V) \quad \forall \, U, V \subset X.$$

Let us recall now some definitions related to set-valued mappings.

Definition 1.1:
([3, Definition 5.5]) Given a number $\beta \geq 0$, the set-valued mapping $\Phi : X \rightrightarrows Y$ is called β-Lipschitz if

$$h(\Phi(x_1), \Phi(x_2)) \leq \beta \rho_X(x_1, x_2) \quad \forall x_1, x_2 \in X.$$

If $(X, \rho_X) = (Y, \rho_Y)$ and $\beta < 1$ then the β-Lipschitz set-valued mapping Φ is said to be a contraction.

Definition 1.2:
([3, Definition 4.4]) The set-valued mapping $\Phi : X \rightrightarrows Y$ is said to be closed, if for all sequences $\{x_i\} \subset X$, $\{y_i\} \subset Y$ and points $x \in X$, $y \in Y$ such that $x_i \to x$, $y_i \to y$ and $(x_i, y_i) \in \mathrm{gph}(\Phi)$ for all i, we have $(x, y) \in \mathrm{gph}(\Phi)$.

Definition 1.3:
We will say that the graph of the set-valued mapping $\Phi : X \rightrightarrows Y$ is complete, if for all Cauchy sequences $\{x_i\} \subset X$ and $\{y_i\} \subset Y$ such that $\{(x_i, y_i)\} \subset \mathrm{gph}(\Phi)$, there exists a point $(x, y) \in \mathrm{gph}(\Phi)$ such that $x_i \to x$ and $y_i \to y$.

Denote by $B_X(x_0, r)$ a closed ball in X centered at $x \in X$ with the radius $r > 0$, i.e.

$$B_X(x_0, r) = \{x \in X : \, \rho_X(x_0, x) \leq r\}.$$

Definition 1.4:
([3, definition 5.4]) Given a number $\alpha > 0$, the set-valued mapping $\Psi : X \rightrightarrows Y$ is called α-covering if

$$\bigcup_{y \in \Psi(x)} B_Y(y, \alpha r) \subseteq \Psi(B_X(x, r)) \quad \forall r \geq 0, \forall x \in X.$$

Let us now recall the coincidence point existence theorem from [3]. Let numbers $\alpha > 0$, $\beta \in [0, \alpha)$ and set-valued mappings $\Psi, \Phi : X \rightrightarrows Y$ be given. Denote

$$M_{\Psi,\Phi}(x, r) := \{y \in \Phi(x) : \mathrm{dist}(\Psi(x), y) < r\}, \quad x \in X, \quad r > 0,$$

$$S(\theta, n) := \frac{1 - \theta^n}{1 - \theta}, \quad \theta \in [0, 1), \quad n = 0, 1, 2 \ldots,$$

$$m_0 := \min\left\{ j \in \mathbb{N} : q_2 \left(\frac{\beta}{\alpha}\right)^j < 1 \right\}.$$

Theorem 1.1:
([3, Theorem 5.7]) Let numbers $\alpha > 0$ and $\beta \in [0, \alpha)$ be given. Assume that

- *the set-valued mapping $\Psi : X \rightrightarrows Y$ is α-covering and its graph is closed;*
- *set-valued mapping $\Phi : X \rightrightarrows Y$ is β-Lipschitz;*
- *at least one of the graphs $\mathrm{gph}(\Psi)$ or $\mathrm{gph}(\Phi)$ is complete.*

Then for all

$$x_0 \in X, \quad r_0 > \mathrm{dist}(\Psi(x_0), \Phi(x_0)), \quad y_1 \in M_{\Psi,\Phi}(x_0, r_0)$$

there exists $\xi \in X$ such that

$$\Psi(\xi) \cap \Phi(\xi) \neq \emptyset,$$

$$\varliminf_{\lambda \to \xi} \rho_X(x_0, \lambda) \leq \frac{q_1^2 \alpha^{m_0 - 1} S(q_2 \frac{\beta}{\alpha}, m_0 - 1) + q_1 (q_2 \beta)^{m_0 - 1}}{\alpha^{m_0} - q_2 \beta^{m_0}} r_0.$$

This assertion not only provides the sufficient conditions for the existence of a coincidence point but also an estimate of the distance from a point $x_0 \in X$ to a coincidence point ξ of the given mappings. A problem to obtain an estimate of distance from the point y_1 to a point $\eta \in \Psi(\xi) \cap \Phi(\xi)$ naturally arises. For the case when (X, ρ_X) and (Y, ρ_Y) are metric spaces, this problem was solved in [4, 5]. The main goal of our paper is to obtain results analogous to those in [4,5] for set-valued mappings acting between (q_1, q_2)-quasimetric spaces. We also discuss a similar problem for fixed points of set-valued mappings and derive propositions on fixed points properties similar to those in [6].

The results of this paper may have applications in the investigation of various nonlinear equations. One of the possible applications of the results is the investigation of nonlinear equations in Banach spaces equipped with an additional bimodule structure over a group ring based on the theory developed in [7]. Note that the results on coincidence points and their analogs (see, for example, [8], [9]) are applied in the study of equations appearing in economic models (see [10]), differential inclusions (see [11], [12], [13]), and other problems.

2. MAIN RESULTS. ESTIMATES OF DISTANCE FROM A POINT TO THE INTERSECTION SET OF TWO GRAPHS

Let $q_1 \geq 1$, $q_2 \geq 1$ be given numbers, (X, ρ_X), (Y, ρ_Y) be (q_1, q_2)-quasimetric spaces.

Given $\alpha > 0$, $\beta \in [0, \alpha)$, denote by $\mathcal{F}_{\alpha,\beta}$ the set of all ordered pairs of set-valued mappings (Ψ, Φ), $\Psi, \Phi : X \rightrightarrows Y$, such that

- the set-valued mapping $\Psi : X \rightrightarrows Y$ is α-covering;
- set-valued mapping $\Phi : X \rightrightarrows Y$ is β-Lipschitz;
- either $\mathrm{gph}(\Psi)$ is complete and $\Phi(x)$ is a closed set for every $x \in X$ or Ψ is closed and $\mathrm{gph}\Phi$ is complete.

Theorem 2.1:
Let numbers $\alpha > 0$, $\beta \in [0, \alpha)$ and an arbitrary ordered pair of set-valued mappings $(\Psi, \Phi) \in \mathcal{F}_{\alpha,\beta}$ be given. Then for all

$$x_0 \in X, \quad r_0 > \mathrm{dist}(\Psi(x_0), \Phi(x_0)), \quad y_1 \in M_{\Psi,\Phi}(x_0, r_0)$$

there exist $\xi \in X$ and $\eta \in Y$ such that

$$\eta \in \Psi(\xi) \cap \Phi(\xi),$$

$$\varliminf_{\lambda \to \xi} \rho_X(x_0, \lambda) \leq \frac{q_1^2 \alpha^{m_0-1} S(q_2 \frac{\beta}{\alpha}, m_0 - 1) + q_1 (q_2\beta)^{m_0-1}}{\alpha^{m_0} - q_2 \beta^{m_0}} r_0, \tag{2.1}$$

$$\varliminf_{\kappa \to \eta} \rho_Y(y_1, \kappa) \leq \beta \frac{q_1^2 \alpha^{m_0-1} S(q_2 \frac{\beta}{\alpha}, m_0 - 1) + q_1 (q_2\beta)^{m_0-1}}{\alpha^{m_0} - q_2 \beta^{m_0}} r_0. \tag{2.2}$$

Before proving Theorem 2.1, let us prove the following lemma.

Lemma 2.1:
Let the set-valued mapping $\Psi : X \rightrightarrows Y$ be α-covering, the set-valued mapping $\Phi : X \rightrightarrows Y$ be β-Lipschitz. Then for arbitrary

$$\delta > 0, \quad x_0 \in X, \quad y_1 \in M_{\Psi,\Phi}(x_0, \alpha\delta + \mathrm{dist}(\Psi(x_0), \Phi(x_0)))$$

there exist sequences $\{x_i\} \subset X$ and $\{y_i\} \subset Y$ such that

$$\rho_X(x_0, x_1) \leq \delta + \frac{\mathrm{dist}(\Psi(x_0), \Phi(x_0))}{\alpha}, \tag{2.3}$$

$$\rho_X(x_{i-1}, x_i) \leq \Big(\frac{\beta}{\alpha} + \delta\Big) \rho_X(x_{i-2}, x_{i-1}) \quad \forall i \geq 2, \tag{2.4}$$

$$y_i \in \Psi(x_i) \cap \Phi(x_{i-1}) \quad \forall i \geq 1, \tag{2.5}$$

$$\rho_Y(y_{i-1}, y_i) \leq (\beta + \alpha\delta) \rho_X(x_{i-2}, x_{i-1}) \quad \forall i \geq 2. \tag{2.6}$$

Proof
Let us take an arbitrary $x_0 \in X$ and $\delta > 0$. Set

$$r_0 = \alpha\delta + \mathrm{dist}(\Psi(x_0), \Phi(x_0)).$$

Let us take an arbitrary point $y_1 \in M_{\Psi,\Phi}(x_0, r_0)$. Since the mapping Ψ is α-covering, there exists a point $x_1 \in B_X(x_0, r_0)$ such that $y_1 \in \Psi(x_1)$. Therefore, $y_1 \in \Psi(x_1) \cap \Phi(x_0)$. Let us construct the sought sequences by induction.

If $x_0 = x_1$ then set $x_2 := x_1$, $y_2 := y_1$. Assume that $x_0 \neq x_1$. Set

$$r_1 := (\beta + \alpha\delta) \rho_X(x_0, x_1).$$

Since the mapping Φ is β-Lipschitz, we have $h_Y(\Phi(x_0), \Phi(x_1)) < r_1$. Therefore, there exists a point $y_2 \in \Phi(x_1)$ such that $\rho_Y(y_1, y_2) < r_1$. Since $y_1 \in \Psi(x_1)$, we have

$$y_2 \in \bigcup_{y \in \Psi(x_1)} B_Y(y, r_1).$$

Therefore, since the set-valued mapping Ψ is α-covering, there exists a point $x_2 \in B_X(x_1, r_1)$ such that

$$y_2 \in \Psi(x_2) \quad \text{and} \quad \rho_X(x_1, x_2) \le \frac{r_1}{\alpha}.$$

The sought x_2, y_2 are constructed.

Let us now assume that for a certain j, the sought points x_i, y_i, $i = \overline{1, j}$, are constructed. Let us construct x_{j+1}, y_{j+1}.

If $x_j = x_{j-1}$ then set $x_{j+1} := x_j$, $y_{j+1} := y_j$.

Assume that $x_j \ne x_{j-1}$. Set

$$r_j := (\beta + \alpha\delta)\rho_X(x_{j-1}, x_j).$$

Since the mapping Φ is β-Lipschitz, we have $h_Y(\Phi(x_{j-1}), \Phi(x_j)) < r_j$. Therefore, there exists a point $y_{j+1} \in \Phi(x_j)$ such that $\rho_Y(y_j, y_{j+1}) \le r_j$. Since $y_j \in \Psi(x_j)$, we have

$$y_{j+1} \in \bigcup_{y \in \Psi(x_j)} B_Y(y, r_j).$$

Since the set-valued mapping Ψ is α-covering, there exists a point $x_{j+1} \in B_X(x_j, r_j)$ such that

$$y_{j+1} \in \Psi(x_{j+1}) \quad \text{and} \quad \rho_X(x_j, x_{j+1}) \le \frac{r_j}{\alpha}.$$

The sought x_{j+1}, y_{j+1} are constructed. □

Proof of Theorem 2.1. Without loss of generality, we assume $\alpha = 1$. Take a $\delta > 0$ such that

$$\min\left\{ j \in \mathbb{N} : \ q_2\left(\frac{\beta}{\alpha} + \delta\right)^j < 1 \right\} = m_0, \quad r_0 > \operatorname{dist}(\Psi(x_0), \Phi(x_0)) + \delta.$$

Let us consider the corresponding sequences $\{x_i\}$ and $\{y_i\}$, that were constructed in Lemma 2.1. Let us show that $\{x_i\}$ is a Cauchy sequence.

For integers $i, j \ge 0$, we have

$$\rho_X(x_i, x_{i+j}) \le q_1\rho_X(x_i, x_{i+1}) + q_2\rho_X(x_{i+1}, x_{i+j}) \le$$
$$\le q_1 r_0(\beta+\delta)^i + q_2\big(q_1\rho_X(x_{i+1}, x_{i+2}) + q_2\rho_X(x_{i+2}, x_{i+j})\big) \le$$
$$\le q_1 r_0(\beta+\delta)^i + q_1 q_2 r_0(\beta+\delta)^{i+1} + q_2^2\big(q_1\rho_X(x_{i+2}, x_{i+3}) + q_2\rho_X(x_{i+3}, x_{i+j})\big) \le$$
$$\le \cdots \le q_1 r_0(\beta+\delta)^i\big(1 + q_2(\beta+\delta) + \cdots + q_2^{j-2}(\beta+\delta)^{j-2} + q_2^{j-1}(\beta+\delta)^{j-1}q_1^{-1}\big) =$$
$$= q_1 r_0(\beta+\delta)^i \widetilde{S}(j).$$

Here

$$\widetilde{S}(j) = S(q_2(\beta+\delta), j-1) + q_2^{j-1}(\beta+\delta)^{j-1}q_1^{-1}, \ j \in \mathbb{N}, \qquad \widetilde{S}(0) = 0.$$

Thus, for any non-negative integer i and k, we have

$$\rho_X(x_i, x_{i+k}) \le q_1\rho_X(x_i, x_{i+m_0}) + q_2\rho_X(x_{i+m_0}, x_{i+k}) \le$$
$$\le q_1\rho_X(x_i, x_{i+m_0}) + q_2\big(q_1\rho_X(x_{i+m_0}, x_{i+2m_0}) +$$
$$+ q_2\rho_X(x_{i+2m_0}, x_{i+k})\big) \le q_1\rho_X(x_i, x_{i+m_0}) + q_1 q_2\rho_X(x_{i+m_0}, x_{i+2m_0}) +$$
$$+ q_2^2\big(q_1\rho_X(x_{i+2m}, x_{i+3m_0}) + q_2\rho_X(x_{i+3m_0}, x_{i+k})\big) \le$$

$$\le q_1\rho_X(x_i,x_{i+m_0})+q_1q_2\rho_X(x_{i+m_0},x_{i+2m_0})+q_1q_2^2\rho_X(x_{i+2m_0},x_{i+3m_0})+{q_2}^3\rho_X(x_{i+3m_0},x_{i+k})\le$$
$$\le\cdots\le q_1\rho_X(x_i,x_{i+m_0})+q_1q_2\rho_X(x_{i+m_0},x_{i+2m_0})+$$
$$+\cdots+q_1q_2^{p-1}\rho_X(x_{i+(p-1)m_0},x_{i+pm_0})+q_2^p\rho_X(x_{i+pm_0},x_{i+k})\le$$
$$\le q_1^2r_0\beta^i\widetilde{S}(m_0)\big(1+q_2(\beta+\delta)^{m_0}+q_2^2(\beta+\delta)^{2m_0}+\cdots+q_2^{p-1}(\beta+\delta)^{(p-1)m_0}\big)+$$
$$+q_2^pq_1r_0(\beta+\delta)^{i+pm_0}\widetilde{S}(k-pm_0)=$$
$$=q_1^2r_0(\beta+\delta)^i\widetilde{S}(m_0)S(q_2(\beta+\delta)^{m_0},p)+q_2^p(\beta+\delta)^{i+pm_0}q_1r_0\widetilde{S}(k-pm_0),$$

where p is the integer part of k/m_0. Since, $q_2(\beta+\delta)^{m_0}<1$, we have

$$\begin{gathered}\rho_X(x_i,x_{i+k})\le\\ \le q_1^2r_0(\beta+\delta)^i\big(\widetilde{S}(m_0)S(q_2(\beta+\delta)^{m_0},p)+q_2^pr_0(\beta+\delta)^{pm_0}q_1^{-1}\widetilde{S}(k-pm_0)\big)\le\\ \le q_1^2r_0(\beta+\delta)^i\Big(\frac{\widetilde{S}(m_0)}{1-q_2(\beta+\delta)^{m_0}}+q_1^{-1}\widetilde{S}(k-pm_0)\Big).\end{gathered}\tag{2.7}$$

Since, $0\le k-pm_0\le m_0$, the value $q_1^{-1}S(k-pm_0)$ is uniformly bounded for all k. Therefore, $\{x_i\}$ is a Cauchy sequence. Let us show that $\{y_i\}$ is also a Cauchy sequence.

According to the lemma, $\rho_Y(y_{i+1},y_{i+j+1})\le(\beta+\delta)\rho_X(x_i,x_{i+j})$ for every i and j. Therefore, repeating the arguments above, we get

$$\begin{gathered}\rho_X(y_{i+1},y_{i+k+1})\le(\beta+\delta)[q_1^2r_0(\beta+\delta)^i\big(\widetilde{S}(m_0)S(q_2(\beta+\delta)^{m_0},p)+\\ +q_2^p(\beta+\delta)^{pm_0}q_1^{-1}\widetilde{S}(k-pm_0)\big)]\le\\ \le q_1^2r_0(\beta+\delta)^{i+1}\Big(\tfrac{\widetilde{S}(m_0)}{1-q_2(\beta+\delta)^{m_0}}+q_1^{-1}\widetilde{S}(k-pm_0)\Big).\end{gathered}\tag{2.8}$$

Therefore, $\{y_i\}$ is a Cauchy sequence.

Consider now two cases. At first, assume that the $\mathrm{gph}(\Psi)$ is complete and each value of Φ is a closed set. Then the Cauchy sequences $\{x_i\}$, $\{y_i\}$ converge to points $\xi\in X$, $\eta\in\Psi(\xi)$ respectively as $(x_i,y_i)\in\mathrm{gph}(\Psi)$. We have

$$h_Y(\Phi(x_i),\Phi(\xi))=h_Y(\Phi(\xi),\Phi(x_i))\le\beta\rho_X(\xi,x_i),$$

and thus $h_Y^+(y_{i+1},\Phi(\xi))\le\beta\rho_X(\xi,x_i)$. Therefore,

$$h_Y^+(\eta,\Phi(\xi))\le q_1\rho_Y(\eta,y_{i+1})+q_2h_Y^+(y_{i+1},\Phi(\xi))\le q_1\rho_Y(\eta,y_{i+1})+q_2\beta\rho_X(\xi,x_i).$$

Since $\{y_{i+1}\}$ tends to η and $\{x_i\}$ tends to ξ, we have $h_Y^+(\eta,\Phi(\xi))=0$. This equality and the closedness of $\Phi(\xi)$ imply $\eta\in\Phi(\xi)$.

Assume now that $\mathrm{gph}(\Phi)$ is complete and Ψ is closed. Then the Cauchy sequences $\{x_i\}$, $\{y_i\}$ converge to some points $\xi\in X$, $\eta\in\Phi(\xi)$ respectively, since $(x_i,y_{i+1})\in\mathrm{gph}(\Phi)$. The set $\mathrm{gph}(\Psi)$ is closed, therefore $\eta\in\Psi(\xi)$.

So, it is proved that

$$\eta\in\Phi(\xi)\cap\Psi(\xi).$$

Passing to the limit in (2.7) as $k\to+\infty$ and putting $i=0$ we obtain

$$\varlimsup_{\zeta\to\xi}\rho_X(x_0,\zeta)\le\frac{q_1^2\alpha^{m_0}S\big(q_2\frac{\beta+\alpha\delta}{\alpha},m_0-1\big)+q_1q_2^{m_0-1}(\beta+\alpha\delta)^{m_0-1}}{\alpha^{m_0}-q_2(\beta+\alpha\delta)^{m_0}}\Big(\delta+\frac{\mathrm{dist}(\Psi(x_0),\Phi(x_0))}{\alpha}\Big)$$

Hence, as the choice of δ is arbitrary, it implies (2.1). Analogously passing to the limit as $k \to \infty$ in (2.8) and substituting $i = 0$ by virtue of the choice of δ we obtain (2.2). □

Let us now obtain an estimate of the distance from a point $(x, y) \in X \times Y$ to the set $\Gamma(\Psi, \Phi)$. Put

$$K(m_0) := \frac{q_1^2 \alpha^{m_0 - 1} S(q_2 \frac{\beta}{\alpha}, m_0 - 1) + q_1 (q_2 \beta)^{m_0 - 1}}{\alpha^{m_0} - q_2 \beta^{m_0}}.$$

For vectors $z = (x, y) \in X \times Y$, $A = (A_X, A_Y) \in \mathbb{R}^2$ and a subset $\Gamma \subset X \times Y$, we write

$$D(z, \Gamma) \le A$$

if

$$\forall\, \varepsilon > 0 \quad \exists (\xi, \eta) \in \Gamma : \quad \varliminf_{\lambda \to \xi} \rho_X(x, \lambda) \le A_X + \varepsilon, \quad \varliminf_{\kappa \to \eta} \rho_Y(y, \kappa) \le A_Y + \varepsilon.$$

Theorem 2.2:
Let $\alpha > 0$ and $\beta \in [0, \alpha)$ be given. If $(\Psi, \Phi) \in \mathcal{F}_{\alpha,\beta}$ then the set $\Gamma(\Psi, \Phi)$ is non-empty and, moreover, for arbitrary $x \in X$, $y \in Y$, the inequality

$$D\big((x, y), \Gamma(\Psi, \Phi)\big) \le A(x, y, y_\phi) \quad \forall\, y_\phi \in \Phi(x), \tag{2.9}$$

holds. Here

$$A(x, y, y_\phi) := \big(K(m_0)\mathrm{dist}(\Psi(x), y_\phi),\ q_1 \rho_Y(y, y_\phi) + q_2 \beta K(m_0) \mathrm{dist}(\Psi(x), y_\phi)\big).$$

Proof
Theorem 2.1 implies that $\Gamma(\Psi, \Phi) \neq \emptyset$.

Let us take an arbitrary $x \in X$, $y \in Y$, $y_\phi \in \Phi(x)$, $\varepsilon > 0$. Set $r := \mathrm{dist}(\Psi(x), y_\phi)$. The definition of $\mathrm{dist}(\cdot, \cdot)$ implies

$$\mathrm{dist}(\Psi(x), \Phi(x)) \le r.$$

Therefore, $y_\phi \in M_{\Psi,\Phi}(x, r + \varepsilon)$. It follows from Theorem 2.1 that there exist $\xi \in X$ and $\eta \in Y$ such that $(\xi, \eta) \in \Gamma(\Psi, \Phi)$ and

$$\varliminf_{\lambda \to \xi} \rho_X(x, \lambda) \le K(m_0)(r + \varepsilon), \quad \varliminf_{\kappa \to \eta} \rho_Y(y_\phi, \kappa) \le \beta K(m_0)(r + \varepsilon).$$

Since $\rho_Y(y, \kappa) \le q_1 \rho_Y(y, y_\phi) + q_2 \rho_Y(y_\phi, \kappa)$, we have

$$\varliminf_{\kappa \to \eta} \rho_Y(y, \kappa) \le q_1 \rho_Y(y, y_\phi) + q_2 \varliminf_{\kappa \to \eta}(y_\phi, \kappa) \le q_1 \rho_Y(y, y_\phi) + q_2 \beta K(m_0)(r + \varepsilon).$$

The arbitrariness of $\varepsilon > 0$ implies that (2.9) holds. □

Theorem 2.3:
Let $\alpha > 0$ and $\beta \in [0, \alpha)$ be given. If $(\Psi, \Phi) \in \mathcal{F}_{\alpha,\beta}$ then the set $\Gamma(\Psi, \Phi)$ is non-empty and, moreover, for arbitrary $x \in X, y \in Y$, the inequality

$$D((x, y), \Gamma(\Psi, \Phi)) \le A(x, y) \tag{2.10}$$

holds. Here

$$A(x, y) := \big(K(m_0)(r_1 + r_2),\ q_1 r_2 + q_2 \beta K(m_0)(q_1 r_1 + q_2 r_2)\big),$$

$$r_1 := \mathrm{dist}(\Psi(x), y), \quad r_2 := \mathrm{dist}(y, \Phi(x)).$$

Proof
Take arbitrary points $x \in X$, $y \in Y$, $y_\psi \in \Psi(x)$, $y_\phi \in \Phi(x)$ and a number $\varepsilon > 0$ such that

$$\rho_Y(y_\psi, y) < \mathrm{dist}(\Psi(x), y) + \frac{\varepsilon}{2}, \quad \rho_Y(y, y_\phi) < \mathrm{dist}(y, \Phi(x)) + \frac{\varepsilon}{2}.$$

Set

$$\varepsilon_1 := \frac{\varepsilon(q_1 + q_2)}{2}, \quad r := q_1 r_1 + q_2 r_2.$$

Let us show that $y_\phi \in M_{\Psi,\Phi}(x, r + \varepsilon_1)$.

Indeed, by the assumption $y_\phi \in \Phi(x)$. Moreover,

$$\rho_Y(y_\psi, y_\phi) < q_1 r_1 + q_2 r_2 + \frac{\varepsilon(q_1 + q_2)}{2} \leq r + \varepsilon_1$$

and

$$\mathrm{dist}(\Psi(x), \Phi(x)) \leq \mathrm{dist}(\Psi(x), y_\phi) \leq \rho_Y(y_\psi, y_\phi).$$

Hence, $\mathrm{dist}(\Psi(x), y) < r + \varepsilon_1$. Therefore, $y_\phi \in M_{\Psi,\Phi}(x, r + \varepsilon_1)$.

Theorem 2.1 implies that there exist $\xi \in X$ and $\eta \in Y$ such that $(\xi, \eta) \in \Gamma(\Psi, \Phi)$ and

$$\underline{\lim}_{\lambda \to \xi} \rho_X(x, \lambda) \leq K(m_0)(r + \varepsilon_1), \quad \underline{\lim}_{\kappa \to \eta} \rho_Y(y_\phi, \kappa) \leq K(m_0)\beta(r + \varepsilon_1). \tag{2.11}$$

Since

$$\rho_Y(y, \kappa) \leq q_1 \rho_Y(y, y_\phi) + q_2(y_\phi, \kappa)$$

the inequality (2.11) implies

$$\underline{\lim}_{\kappa \to \eta} \rho_Y(y, \kappa) \leq q_1 \rho_Y(y, y_\phi) + q_2 \underline{\lim}_{\kappa \to \eta} (y_\phi, \kappa) \leq q_1 r_2 + q_2 K(m_0)\beta(r + \varepsilon_1).$$

Moreover, it follows from (2.11) that

$$\underline{\lim}_{\lambda \to \xi} \rho_X(x, \lambda) \leq K(m_0)(r + \varepsilon_1).$$

The arbitrariness of $\varepsilon > 0$ implies that the inequality (2.10) holds. □

Remark 2.1:
In the case when (X, ρ_X) and (Y, ρ_Y) are metric spaces, Theorem 2.1 coincides with [4, Theorem 1] and Theorem 2.2 coincides with [4, Theorem 3].

3. COROLLARIES. ESTIMATES OF DISTANCE BETWEEN INTERSECTIONS OF GRAPHS AND SETS OF COINCIDENCE POINTS

Let $q_1 \geq 1$, $q_2 \geq 1$ be given numbers, (X, ρ_X), (Y, ρ_Y) be (q_1, q_2)-quasimetric spaces.

Let us define one more function which characterizes a distance between subsets of (q_1, q_2)-quasimetric spaces. For $U, V \subset X$ set

$$e^+(U, V) := \sup_{v \in V} \mathrm{dist}(U, v).$$

Even though the definitions of e^+ and h^+ look quite similar, these functions are actually different as ρ_X is not necessarily symmetric.

Let us describe some properties of the function e^+.

Proposition 3.1:
For arbitrary sets $U,\ V,\ W \subset X$, the following inequalities hold

$$e^+(U,W) \le q_1 e^+(U,V) + q_2 e^+(V,W), \tag{3.12}$$

$$\mathrm{dist}(U,W) \le q_1 e^+(U,V) + q_2 e^+(V,W), \tag{3.13}$$

Proof
Let us prove (3.12). Take arbitrary sets $U,\ V,\ W \subset X$, a point $w \in W$ and a number $\varepsilon > 0$. According to the definition of e^+ there exists $v \in V$ and such that $\rho_X(v,w) \le e^+(V,W)$. Moreover, there exists $u \in U$ such that $\rho_X(u,v) \le e^+(U,V)$. We have,

$$\mathrm{dist}(U,w) \le \rho_X(u,w) \le q_1\rho_X(u,v) + q_2\rho_X(v,w) \le q_1 e^+(U,V) + q_2 e^+(V,W).$$

Due to the arbitrariness of $w \in W$, the above inequality implies (3.12). Inequality (3.13) follows from (3.12) as $\mathrm{dist}(U,W) \le e^+(U,W)$. □

We will also use the following inequality which was proved in ([3, Property 5.1]):

$$\mathrm{dist}(U,W) \le q_1 \mathrm{dist}(U,V) + q_2 h^+(V,W). \tag{3.14}$$

Theorem 3.1:
Let $\alpha > 0$ and $\beta \in [0,\alpha)$ be given. If $(\Psi,\Phi) \in \mathcal{F}_{\alpha,\beta}$ then for arbitrary $(\tilde{\xi},\tilde{\eta}) \in \Gamma(\tilde{\Psi},\tilde{\Phi})$ and $\varepsilon > 0$ there exists $(\xi,\eta) \in \Gamma(\Psi,\Phi)$ such that

$$\varliminf_{\lambda\to\xi} \rho_X(\tilde{\xi},\lambda) \le K(m_0)\big(q_1 e^+(\Psi(\tilde{\xi}),\tilde{\Psi}(\tilde{\xi})) + q_2 h^+(\tilde{\Phi}(\tilde{\xi}),\Phi(\tilde{\xi}))\big)+\varepsilon,$$

$$\varliminf_{\kappa\to\eta} \rho_Y(\tilde{\eta},\kappa) \le q_1 q_2 h^+(\tilde{\Phi}(\tilde{\xi}),\Phi(\tilde{\xi})) + q_2\beta K(m_0)\big(q_1^2 e^+(\Psi(\tilde{\xi}),\tilde{\Psi}(\tilde{\xi})) + q_2^2 h^+(\tilde{\Phi}(\tilde{\xi}),\Phi(\tilde{\xi}))\big) + \varepsilon.$$

Proof
Fix an arbitrary pair $(\tilde{\xi},\tilde{\eta}) \in \Gamma(\tilde{\Psi},\tilde{\Phi})$ and take an arbitrary $\varepsilon > 0$. Applying Theorem 2.3 to the mappings Φ and Ψ we obtain that there exists $(\xi,\eta) \in \Gamma(\Psi,\Phi)$ such that

$$\varliminf_{\lambda\to\xi} \rho_X(\tilde{\xi},\lambda) \le K(m_0)(r_1 + r_2) + \varepsilon, \tag{3.15}$$

$$\varliminf_{\kappa\to\eta} \rho_Y(\tilde{\eta},\kappa) \le q_1 r_2 + q_2\beta K(m_0)(q_1 r_1 + q_2 r_2) + \varepsilon. \tag{3.16}$$

Here $r_1 := \mathrm{dist}(\Psi(\tilde{\xi}),\tilde{\eta})$, $r_2 := \mathrm{dist}(\tilde{\eta},\Phi(\tilde{\xi}))$.

It follows from (3.13) that $\mathrm{dist}(\Psi(\tilde{\xi}),\tilde{\eta}) \le q_1 e^+(\Psi(\tilde{\xi}),\tilde{\Psi}(\tilde{\xi})) + q_2 e^+(\tilde{\Psi}(\tilde{\xi}),\tilde{\eta})$. Since $\tilde{\eta} \in \tilde{\Psi}(\tilde{\xi})$, we have

$$r_1 = \mathrm{dist}(\Psi(\tilde{\xi}),\tilde{\eta}) \le q_1 e^+(\Psi(\tilde{\xi}),\tilde{\Psi}(\tilde{\xi})).$$

Since $\tilde{\eta} \in \tilde{\Phi}(\tilde{\xi})$, we have $\mathrm{dist}(\tilde{\eta},\tilde{\Phi}(\tilde{\xi})) = 0$. So, it follows from (3.14) that

$$r_2 = \mathrm{dist}(\tilde{\eta},\Phi(\tilde{\xi})) \le q_2 h^+(\tilde{\Phi}(\tilde{\xi}),\Phi(\tilde{\xi})).$$

Hence, the desired inequalities follow from (3.15) and (3.16). □

Let us introduce one more notation. Given arbitrary sets $\widetilde{\Gamma},\Gamma \subset X \times Y$ and a vector $A = (A_X, A_Y) \in \mathbb{R}^2$, the notation

$$H^+(\widetilde{\Gamma},\Gamma) \le A$$

means that for arbitrary $(x, y) \in \widetilde{\Gamma}$ we have $D((x, y), \Gamma) \leq A$.

Let arbitrary set-value mappings $\Psi, \Phi, \tilde{\Phi}, \tilde{\Psi} : X \rightrightarrows Y$ and numbers $\alpha > 0$, $\beta \in [0, \alpha)$ be given. Set

$$A_X(x) := K(m_0)\big(q_1 e^+(\Psi(x), \tilde{\Psi}(x)) + q_2 h^+(\tilde{\Phi}(x), \Phi(x))\big),$$

$$A_Y(x) := q_1 q_2 h^+(\tilde{\Phi}(x), \Phi(x)) + q_2 \beta K(m_0)\big(q_1^2 e^+(\Psi(x), \tilde{\Psi}(x)) + q_2^2 h^+(\tilde{\Phi}(x), \Phi(x))\big),$$

$$A(x) := (A_X(x), A_Y(x)), \quad x \in X.$$

Given a set $\Xi \subset X$, denote

$$\sup_{\xi \in \Xi} A(\xi) = \{\Lambda \in \mathbb{R}^2 : \ \Lambda \geq A(\xi) \quad \forall \xi \in \Xi\}.$$

Here the inequality is understood in the coordinate-wise sense.

Theorem 3.2:
Let $\alpha > 0$ and $\beta \in [0, \alpha)$ be given. If $(\Psi, \Phi) \in \mathcal{F}_{\alpha,\beta}$ then

$$H^+(\Gamma(\tilde{\Psi}, \tilde{\Phi}), \Gamma(\Psi, \Phi)) \leq \Lambda \quad \forall \Lambda \in \sup_{\tilde{\xi} \in \mathrm{Coin}(\tilde{\Psi}, \tilde{\Phi})} A(\tilde{\xi}). \tag{3.17}$$

If, in addition, ρ_X is lower semicontinuous with respect to the second argument, then

$$h^+(\mathrm{Coin}(\tilde{\Psi}, \tilde{\Phi}), \mathrm{Coin}(\Psi, \Phi)) \leq \sup_{\tilde{\xi} \in \mathrm{Coin}(\tilde{\Psi}, \tilde{\Phi})} A_X(\tilde{\xi}) \tag{3.18}$$

Proof
Take arbitrary $\Lambda \in \mathbb{R}^2$ such that $\Lambda \geq A(\xi)$ for all $\tilde{\xi} \in \mathrm{Coin}(\tilde{\Psi}, \tilde{\Phi})$. Take arbitrary $(\tilde{\xi}, \tilde{\eta}) \in \Gamma(\tilde{\Psi}, \tilde{\Phi})$. Theorem 3.1 implies that $D((\tilde{\xi}, \tilde{\eta}), \Gamma(\Psi, \Phi)) \leq \Lambda$. Hence, (3.17) is proved.

Let us prove (3.18). Assume now that ρ_X is lower semicontinuous with respect to the second argument. Theorem 2.3 implies that for every pair $(\tilde{\xi}, \tilde{\eta}) \in \Gamma(\tilde{\Psi}, \tilde{\Phi})$ and every $\varepsilon > 0$ there exists a point $\xi \in \mathrm{Coin}(\Psi, \Phi)$ such that

$$\rho_X(\tilde{\xi}, \xi) \leq K(m_0)(\mathrm{dist}(\Psi(\tilde{\xi}), \tilde{\eta}) + \mathrm{dist}(\tilde{\eta}, \Phi(\tilde{\xi}))) + \varepsilon, \tag{3.19}$$

It follows from (3.13) that $\mathrm{dist}(\Psi(\tilde{\xi}), \tilde{\eta}) \leq q_1 e^+(\Psi(\tilde{\xi}), \tilde{\Psi}(\tilde{\xi})) + q_2 e^+(\tilde{\Psi}(\tilde{\xi}), \tilde{\eta})$. Since $\tilde{\eta} \in \tilde{\Psi}(\tilde{\xi})$, we have

$$\mathrm{dist}(\Psi(\tilde{\xi}), \tilde{\eta}) \leq q_1 e^+(\Psi(\tilde{\xi}), \tilde{\Psi}(\tilde{\xi})).$$

Since $\tilde{\eta} \in \tilde{\Phi}(\tilde{\xi})$, we have $\mathrm{dist}(\tilde{\eta}, \tilde{\Phi}(\tilde{\xi})) = 0$. Thus, (3.14) implies

$$\mathrm{dist}(\tilde{\eta}, \Phi(\tilde{\xi})) \leq q_2 h^+(\tilde{\Phi}(\tilde{\xi}), \Phi(\tilde{\xi})).$$

Substituting these estimates into (3.19) we obtain

$$\rho_X(\tilde{\xi}, \xi) \leq A_X(\tilde{\xi}) + \varepsilon.$$

Hence,

$$h^+(\mathrm{Coin}(\tilde{\Psi}, \tilde{\Phi}), \mathrm{Coin}(\Psi, \Phi)) =$$

$$= \sup_{\tilde{\xi} \in \mathrm{Coin}(\tilde{\Psi}, \tilde{\Phi})} \mathrm{dist}(\tilde{\xi}, \mathrm{Coin}(\Psi, \Phi)) \leq \sup_{\tilde{\xi} \in \mathrm{Coin}(\tilde{\Psi}, \tilde{\Phi})} A_X(\tilde{\xi}).$$

Inequality (3.18) is proved. □

Corollary 3.1:
Let assumptions of Theorem 3.1 hold and ρ_X be lower semi-continuous with respect to the second argument. Then for arbitrary $(\tilde{\xi}, \tilde{\eta}) \in \Gamma(\tilde{\Psi}, \tilde{\Phi})$ and $\varepsilon > 0$ there exists $(\xi, \eta) \in \Gamma(\Psi, \Phi)$ such that

$$\rho_X(\tilde{\xi}, \xi) \leq K(m_0)(q_1 e^+(\Psi(\tilde{\xi}), \tilde{\Psi}(\tilde{\xi})) + q_2 h^+(\tilde{\Phi}(\tilde{\xi}), \Phi(\tilde{\xi}))) + \varepsilon, \tag{3.20}$$

$$\rho_Y(\tilde{\eta}, \eta) \leq q_1 q_2 h^+(\tilde{\Phi}(\tilde{\xi}), \Phi(\tilde{\xi})) + q_2 \beta K(m_0)(q_1^2 e^+(\Psi(\tilde{\xi}), \tilde{\Psi}(\tilde{\xi})) + q_2^2 h^+(\tilde{\Phi}(\tilde{\xi}), \Phi(\tilde{\xi}))) + \varepsilon.$$

Recall Lim's lemma (see [14]). Let X be a complete metric space, $\beta \in [0, 1)$, $\Phi, \tilde{\Phi} : X \rightrightarrows X$ be β-contractive set-valued mappings such that $\Phi(x), \tilde{\Phi}(x)$ are closed for every x.

$$h(\mathrm{Fix}(\Phi), \mathrm{Fix}(\tilde{\Phi})) \leq \frac{1}{1-\beta} \sup_{x \in X} h(\Phi(x), \tilde{\Phi}(x)).$$

Here $\mathrm{Fix}(\Phi)$ is the set of fixed points of the mapping Φ.

Let us now derive a generalization of Lim's lemma for coincidence points of mappings between (q_1, q_2)-quasimetric spaces.

Corollary 3.2:
Let $\alpha > 0$ and $\beta \in [0, \alpha)$ be given. Assume that ρ_X is lower semicontinuous with respect to the second argument. If $(\Psi, \Phi), (\tilde{\Psi}, \tilde{\Phi}) \in \mathcal{F}_{\alpha,\beta}$ then

$$h^+(\mathrm{Coin}(\tilde{\Psi}, \tilde{\Phi}), \mathrm{Coin}(\Psi, \Phi)) \leq K(m_0) \sup_{x \in X} \big(q_1 e^+(\Psi(x), \tilde{\Psi}(x)) + q_2 h^+(\tilde{\Phi}(x), \Phi(x))\big) \tag{3.21}$$

Proof
It follows from Theorem 3.2 that

$$h^+(\mathrm{Coin}(\tilde{\Psi}, \tilde{\Phi}), \mathrm{Coin}(\Psi, \Phi)) \leq \sup_{\tilde{\xi} \in \mathrm{Coin}(\tilde{\Psi}, \tilde{\Phi})} A_X(\tilde{\xi}) \leq \sup_{x \in X} A_X(x) =$$

$$= K(m_0) \sup_{x \in X} \big(q_1 e^+(\Psi(x), \tilde{\Psi}(x)) + q_2 h^+(\tilde{\Phi}(x), \Phi(x))\big).$$

Hence, (3.21) holds.

□

Let us now derive a generalization of Lim's lemma for fixed points of self-mappings of (q_1, q_2)-quasimetric spaces.

Corollary 3.3:
Assume that (X, ρ_X) is a complete (q_1, q_2)-quasimetric space and ρ_X is lower semicontinuous with respect to the second argument. Given a number $\beta \in [0, 1)$, assume that mappings $\Phi, \tilde{\Phi} : X \to X$ are β-contractions and closed. Then

$$h_+(\mathrm{Fix}(\tilde{\Phi}), \mathrm{Fix}(\Phi)) \leq q_2 \frac{q_1^2 S(q_2\beta, n_0 - 1) + q_1 (q_2\beta)^{n_0 - 1}}{1 - q_2 \beta^{n_0}} \sup_{x \in X} h^+(\tilde{\Phi}(x), \Phi(x)).$$

Here $n_0 := \min\{j \in \mathbb{N} : q_2 \beta^j < 1\}$.

Proof
Set $\Psi(x) := \{x\}$, $\tilde{\Psi}(x) := \{x\}$, $\alpha := 1$. Then $(\Psi, \Phi), (\tilde{\Psi}, \tilde{\Phi}) \in \mathcal{F}_{\alpha,\beta}$, $m_0 = n_0$, $e^+(\Psi(x), \tilde{\Psi}(x)) = 0$, and

$$K(m_0) = \frac{q_1^2 S(q_2\beta, n_0 - 1) + q_1 (q_2\beta)^{n_0 - 1}}{1 - q_2 \beta^{n_0}}.$$

Hence, applying Corollary 3.2 we obtain the desired inequality. □

Corollary 3.4:
Let (X, ρ_X) be a complete (q_1, q_2)-quasimetric space, ρ_X be lower semicontinuous in the second argument, (Σ, ρ_Σ) be a (q_1, q_2)-quasimetric space, $\Phi : X \times \Sigma \to X$ be given. Given numbers $\beta \in [0, 1)$ and $l \geq 0$, assume that mapping $\Phi(\cdot, \sigma)$ is a β-contraction and closed for every $\sigma \in \Sigma$, $\Phi(x, \cdot)$ is l-Lipschitz.

Then the set-valued mapping $\sigma \mapsto \mathrm{Fix}(\Phi(\cdot, \sigma))$ is Lipschitz.

Proof
Take arbitrary $\sigma, \tilde{\sigma} \in \Sigma$. It follows from Corollary 3.3 that

$$h_+\big(\mathrm{Fix}(\Phi(\cdot, \tilde{\sigma})), \mathrm{Fix}(\Phi(\cdot, \sigma))\big) \leq c \sup_{x \in X} h^+(\Phi(x, \tilde{\sigma}), \Phi(x, \sigma)),$$

where

$$c = q_2 \frac{q_1^2 S(q_2\beta, n_0 - 1) + q_1(q_2\beta)^{n_0 - 1}}{1 - q_2\beta^{n_0}}.$$

Hence, $h\big(\mathrm{Fix}(\Phi(\cdot, \tilde{\sigma})), \mathrm{Fix}(\Phi(\cdot, \sigma))\big) \leq c \sup_{x \in X} h(\Phi(x, \tilde{\sigma}), \Phi(x, \sigma))$. Moreover, since $\Phi(x, \cdot)$ is l-Lipschitz, we have

$$\sup_{x \in X} h(\Phi(x, \tilde{\sigma}), \Phi(x, \sigma)) \leq l \rho_\Sigma(\tilde{\sigma}, \sigma).$$

Thus, $h\big(\mathrm{Fix}(\Phi(\cdot, \tilde{\sigma})), \mathrm{Fix}(\Phi(\cdot, \sigma))\big) \leq lc\rho_\Sigma(\tilde{\sigma}, \sigma)$, which completes the proof. □

Let us now discuss the question of the stability of coincidence points. Given a sequence of pairs of set-valued mappings (Ψ_n, Φ_n), $\Psi_n, \Phi_n : X \rightrightarrows Y$, which tend in a certain sense to a pair of set-valued mappings (Ψ, Φ), $\Psi_n, \Phi_n : X \rightrightarrows Y$, and a point $\xi \in \mathrm{Coin}(\Psi, \Phi)$. Our goal is to derive conditions for the existence of points $\xi_n \in \mathrm{Coin}(\Psi_n, \Phi_n)$ such that $\xi_n \to \xi$.

Corollary 3.5:
Assume that ρ_X is lower semicontinuous with respect to the second argument, $\{(\Psi_n, \Phi_n)\} \subset \mathcal{F}_{\alpha,\beta}$ for every n, and there exists a point $\xi \in \mathrm{Coin}(\Psi, \Phi)$ such that

$$e_+(\Psi_n(\xi), \Psi(\xi)) \to 0, \quad h^+(\Phi(\xi), \Phi_n(\xi)) \to 0.$$

Then there exists a sequence $\{\xi_n\}$, such that

$$\xi_n \in \mathrm{Coin}(\Psi_n, \Phi_n) \quad \forall\, n, \quad \xi_n \to \xi \quad \textit{as} \quad n \to \infty.$$

Proof
Take an arbitrary point $\eta \in \Psi(\xi) \cap \Phi(\xi)$. Corollary 3.1 implies that for every n there exists a point $\xi_n \in \mathrm{Coin}(\Psi_n, \Phi_n)$ such that

$$\rho_X(\xi, \xi_n) \leq K(m_0)\big(q_1 e^+(\Psi_n(\xi), \Psi(\xi)) + q_2 h^+(\Phi(\xi), \Phi_n(\xi))\big) + 2^{-n}.$$

Since $e_+(\Psi_n(\xi), \Psi(\xi)) \to 0$ and $h^+(\Phi(\xi), \Phi_n(\xi)) \to 0$, we have $\rho_X(\xi, \xi_n) \to 0$. Therefore, $\xi_n \to \xi$. □

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