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Uncertain Controllability and Observability of an Optimal Control Model

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Abstract: The interest of this paper is to examine the controllability and observability of a control system in the configuration state-space of an uncertain optimal control system. The control system is designed based on the realization of capital asset values where a special case of asset management is modelled and optimized. Thus some necessary and sufficient conditions of the controllability and observability of the deterministic systems and the corresponding uncertain systems for the case of the uncertain optimal control system with application in capital asset management are considered.

Keywords: controllability, observability, uncertain systems, optimal control, capital asset management

1. INTRODUCTION

Controllability and observability are important properties in control systems. They represent the ability to move a system around its entire configuration space using certain manipulations. The controllability and observability of a system are mathematical duals that play important roles in control problems such as optimal control. These have played important roles in control theories such as in [1–5]. Recently, researchers such as [6–8] and a host of others have been considering controllability and observability problems in dynamic systems. However, most works done in this area have concentrated on the deterministic and stochastic controllability and observability problems. In this work, uncertain controllability and observability of dynamic systems are carried out by formulating a capital asset management control problem for the uncertain dynamic system such that the uncertain dynamic system is limited to systems involving uncertain processes.

The choice of uncertainty theory over the conventional probability theory exists when the sample size is small to estimate a probability distribution and degree beliefs are ascertained from experts to work in place of frequency since human beings always over-weigh unlikely events. Here, the general controllability and observability for the uncertain system in uncertainty theory are presented based on Klamka and Mahmudov works.

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2. PRELIMINARIES

Uncertainty theory is a branch of mathematics for modelling belief degrees. The theory is based on some concepts which may be referred to [9]. For easy interpretation, some of the concepts are given.

Let Γ be a nonempty set and L be a σ- algebra over Γ such that (Γ, L) is a measurable space. Each element $\Lambda \in L$ is called an event.

Definition 2.1:
A set function M defined on the σ-algebra over L is called an uncertain measure if it satisfies the following axioms:
Axiom 1. *(Normality Axiom): $M\{\Lambda\}$ = 1 for the universal set Γ.*
Axiom 2. *(Duality Axiom): $M\{\Lambda\}$ + $M\{\Lambda^c\}$ = 1 for any event Λ.*
Axiom 3. *(Subadditivity Axiom): For every countable sequence of events, $\Lambda_1, \Lambda_2, \ldots$, we have*

$$M\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} M\{\Lambda_i\}.$$

Axiom 4. *(Product Axiom): Let (Γ_k, L_k, M_k) be uncertainty spaces for $k = 1, 2, \ldots$ The product uncertain measure M is an uncertain measure satisfying*

$$M\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \min_{1\le k\le\infty} M_k\{\Lambda_k\},$$

where Λ_k are arbitrarily chosen events from L_k for $k = 1, 2, \cdots$, respectively; see [9].

Definition 2.2:
Let (Γ, L, M) be an uncertainty space and let T be a totally ordered set. An uncertain process is a function $X_t(\gamma)$ from $T \times (\Gamma, L, M)$ to the set of real numbers such that $\{X_t \in B\}$ is an event for any Borel set B of real numbers at each time t; see [10].

Definition 2.3:
An uncertain process C_σ is said to be a Liu process if
(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
(ii) C_σ has stationary and independent increments,
(iii) every increment $C_{s+\sigma} - C_s$ is a normal uncertain variable with expected value 0 and variance σ^2. The uncertainty distribution of C_σ is

$$\Phi_\sigma(x) = \left[1 + \exp\left(\frac{-\pi x}{\sqrt{3}\sigma}\right)\right]^{-1}, \quad x \in \mathbb{R}, \tag{2.1}$$

and the inverse distribution is

$$\Phi_\sigma^{-1}(y) = \frac{\sigma\sqrt{3}}{\pi} \ln \frac{y}{1-y}, \quad y \in \mathbb{R}; \tag{2.2}$$

see [11].

Definition 2.4:
Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} M\{\xi \ge x\}dx - \int_{-\infty}^{0} M\{\xi \le x\}dx$$

provided that at least one of the two integrals is finite; see [9].

Definition 2.5:
An uncertain process X_t is said to have independent increments if

$$X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \cdots, X_{t_k} - X_{t_{k-1}}$$

are independent uncertain variables, where $t_0 < t_1 < \cdots < t_k$. That is, an independent increment process means that its increments are independent uncertain variables whenever the time intervals do not overlap. It is noted that the increments are also independent of the initial state; see [10].

Definition 2.6:
Suppose C_t is a canonical Liu process, and f and g are two functions. Then

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t$$

is called an uncertain differential equation. A solution is a Liu process X_t that satisfies (2.1) and (2.2) identically in t; see [10].

Definition 2.7:
Let X_t be an uncertain process. Then for each $\gamma \in \Gamma$, the function $X_t(\gamma)$is called a sample path of X_t; see [10].

Definition 2.8:
An uncertain process X_t is said to be sample-continuous if almost all sample paths are continuous functions with respect to time t; see [9].

Definition 2.9:
Uncertainty Distribution of Solution. Let α be a number with $0 < \alpha < 1$. An uncertain differential equation

$$dX(t) = f(t, X(t))dt + g(t, X(t))dC(t)$$

is said to have an α-path $X(t)^\alpha$ if it solves the corresponding ordinary differential equation

$$dX(t)^\alpha = f(t, X(t)^\alpha)dt + |g(t, X(t))|\Phi^{-1}(\alpha)dt,$$

where $\Phi^{-1}(\alpha)$ is the inverse uncertainty distribution of a standard normal uncertain variable, that is,

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}, \quad \alpha \in \mathbb{R};$$

see [11].

3. THE ASSET MANAGEMENT MODEL

Asset management problem is mainly based on decision making and the understanding of probable asset degradation and trading-off capital investments, maintenance costs, risks and other uncertainties to optimize decisions made by investors.

However, it is assumed that an individual invests his/her wealth in a capital asset, $A(t)$, of a large business for a time, t, ranging from t_0 to t_n. Supposing he/she starts with a known initial net worth $X_0(t)$. At the time t, what fraction of his/her net worth, ψ, must he/she choose to utilize on the capital asset and what fraction of his net worth, τ, must he/she choose to be incurred on the liability of the business such that the expected present value of the utility of asset, $J(X)$, is maximized.

Table 3.1 represents the definition of the formulated model's parameters.

A dynamic optimization model of the expected present value of assets over a given life cycle based on Uncertainty theory is herein presented following the study of portfolio

Table 3.1. Definition of Parameters to the model.

Parameter	Description
$X(t)$	Net worth at time t (state variable)
$\tau(t)$	Liability ratio (control) at time t, $\tau \in \mathbb{R}$
$\sigma_r(t)$	Diffusion volatility of liability (with variance σ_r^2 per unit time)
$\psi(t)$	Capital asset ratio at time t (control) $\psi \in \mathbb{R}$
$\sigma_b(t)$	Diffusion volatility of asset (with variance σ_b^2 per unit time)
$\kappa(t)$	Capital gain on asset due to inflation at time t
$\sigma_p(t)$	Diffusion volatility on asset price (with variance σ_p^2 per unit time)
$\beta(t)$	Mean rate of return on the asset at time t
$\omega(t)$	Mean interest rate of liability at time t
$C(t)$	Liu canonical process at time t
$\mu(t)$	Consumption level at time t
$j(t)$	Tax ratio at time t
$g(t)$	Depreciation ratio at time t
$h(t)$	Asset supplies ratio at time t
η	subjective discount rate, e.g., $\frac{A}{\eta+1}$ = Present value
λ	degree of relative risk, where $(1-\lambda)$ is the risk aversion
U	Utility function

selection by [12]. It is assumed that the goal of the asset management is to choose the optimal utilization and asset allocation policies for maximizing a value function that discounts exponentially future uncertain values of Hyperbolic Absolute Risk Aversion (HARA) utility function over a given time horizon with the net worth of tangible assets as the state variable.

The risky asset is assumed to earn an uncertain return and an uncertain gain with the mean rate of return and capital gain. Furthermore, we express the change in liability as the sum of liability service with an assumption of uncertainty, consumptions, investment and net foreign supply, less taxation, depreciation and revenue over a period of time; see [13]. Thus, we have

$$J(X) = \max_{\psi} E_C \left[\int_{t_0}^{t_n} \frac{1}{\lambda} e^{-\eta t} (\psi X(t))^{\lambda} dt \right]$$

subject to

$$\begin{aligned} dX(t) = [(\kappa+\beta)\psi - (\omega(\psi-1) + \mu + h - j - g)]X(t)dt \\ +[\psi\sigma_p + \psi\sigma_b - \sigma_r(\psi-1)]X(t)dC(t)^{\cdot} \end{aligned}$$

This model has been solved, characterised, analysed and applied to some real-life situations of sustainable finance; see [14–17].

3.1. Optimality of the Solution

It is important to derive the optimal solutions of the capital asset management problem as it would help in selecting the best available values. The following are utilised in deriving the optimality of the proposed model.

Definition 3.1:
(Principle of Optimality) [19]: For any $(t, x) \in [0, T) \times \mathbb{R}$ *and* $\Delta t > 0$ *with* $t + \Delta t < T$, *we have*

$$J(t, x) = \sup_{D} E \left[\int_t^{t+\Delta t} f(X_s, D, S) ds + J(t + \Delta t, x + \Delta X_t) \right],$$

where $x + \Delta X_t = X_{t+\Delta t}$.

Theorem 3.1:
(Equation of Optimality, [18]) Let $J(t, x)$ *be twice differentiable on* $[0, T) \times \mathbb{R}$, *Then we have*

$$-J_t(t, x) = \sup_D \left[f(x, D, t)) + J_x(t, x)V(x, D)\right],$$

where $J_t(t, x)$ *and* $J_x(t, x)$ *are the partial derivatives of the function* $J(t, x)$ *in* t *and* X *respectively.*

Proof
See ([19], pp. 15-16) □

3.2. *Optimal control of the model*

The above equation of optimality is applied to the uncertain optimal control problem to evaluate the optimal controls analytically.

Applying equation (3.1), we obtain

$$-J_t = \max_\psi \left\{ \frac{1}{\lambda} e^{-\eta t}(\psi X)^\lambda - \psi(\kappa + \beta)XJ_X + (\mu + j + g + h - (\psi - 1)\omega)XJ_X \right\}$$
$$= \max_\psi H$$

where H stands for terms in the braces (condition the optimal ψ satisfies),

$$\frac{\partial H}{\partial \psi} = 0,$$

$$\frac{\partial H}{\partial \psi} = e^{-\eta t}(\psi X)^{\lambda - 1}X - (\kappa + \beta - \omega)XJ_x = 0,$$

$$\psi = \frac{1}{X}\left[(\omega - \kappa - \beta)J_X e^{\eta t}\right]^{\frac{1}{\lambda - 1}}.$$

Hence, by solving the above equations, we obtained the optimal ratio of the net worth in capital assets as

$$\psi^* = \frac{(\mu + j + g + \omega - h)\lambda - \eta}{(1 - \lambda)(\kappa + \beta - \omega)}.$$

However, the optimal liability ratio, τ^* can also be obtained as a control to the system. Since $\tau = \psi - 1$

$$\tau^* = \left[\frac{(\mu + j + g + \omega - h)\lambda - \eta}{(1 - \lambda)(\kappa + \beta - \omega)}\right] - 1$$

or

$$\tau^* = \frac{(\mu + j + g - h)\lambda - (1 - \lambda)(\kappa + \beta) + \omega - \eta}{(1 - \lambda)(\kappa + \beta - \omega)}.$$

3.3. *Solution to the Model*

Here, the analytical and numerical solutions are derived.

For the analytic solution, the required problem under consideration is

$$J(\psi) = \min_\psi E_C \left[\int_{t_0}^{t_n} \frac{1}{\lambda} e^{-\eta t}(\psi X(t))^\lambda dt\right]$$

subject to

$$dX(t) = [(\kappa+\beta)\psi - (\omega(\psi-1)+\mu+h-j-g)]X(t)dt \\ +[\psi\sigma_p+\psi\sigma_b-\sigma_r(\psi-1)]X(t)dC(t)$$

with α-path equation

$$dX(t)^\alpha = [(\kappa+\beta)\psi - (\omega(\psi-1)+\mu+h-j-g)]X(t)^\alpha dt \\ +|[\psi\sigma_p+\psi\sigma_b-\sigma_r(\psi-1)]X(t)^\alpha|\Phi^{-1}(\alpha)dt.$$

The analytical solution to the constraint is

$$X(t) = X_0 \exp\left([(\kappa+\beta)\psi - (\omega(\psi-1)+\mu+h-j-g)]t \\ +[\psi\sigma_p+\psi\sigma_b-\sigma_r(\psi-1)]C(t)\right)$$

and its inverse uncertainty distribution is

$$\Psi(t)^{-1}(\alpha) = X_0 \exp\Big([(\kappa+\beta)\psi - (\omega(\psi-1)+\mu+h-j-g)]t \\ +\frac{[\psi\sigma_p+\psi\sigma_b-\sigma_r(\psi-1)]t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\Big).$$

Hence,

$$\Psi(t)^{-1}(\alpha) = E(X(t)^\alpha).$$

3.4. Multifactor Model

The multifactor model can be expressed in the following form:

$$J(X) = \max_{\psi} E_C\left[\int_{t_0}^{t_f} \frac{1}{\lambda}e^{-\eta t}(U^\lambda)^T X^{1-\lambda}dt\right] \tag{3.3}$$

subject to

$$dX = FXdt + UPXdt + UQXdC(t), \tag{3.4}$$

where

$$X = \begin{pmatrix} X_{1t} \\ X_{2t} \\ \vdots \\ X_{nt} \end{pmatrix}, \qquad U = \begin{pmatrix} \psi_1 & 0 & \cdots & 0 \\ 0 & \psi_2 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots \\ 0 & 0 & \cdots & \psi_n \end{pmatrix},$$

$$F = \begin{pmatrix} \mu_1+h_1-j_1-g_1-\omega_1 & 0 & \cdots & 0 \\ 0 & \mu_2+h_2-j_2-g_2-\omega_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_n+h_n-j_n-g_n-\omega_m \end{pmatrix},$$

$$P = \begin{pmatrix} \kappa_1+\beta_1-\omega_1 & 0 & \cdots & 0 \\ 0 & \kappa_2+\beta_2-\omega_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \kappa_n+\beta_n-\omega_n \end{pmatrix},$$

$$Q = \begin{pmatrix} \sigma_{1p}+\sigma_{1b}-\sigma_{1r}+1 & 0 & \cdots & 0 \\ 0 & \sigma_{2p}+\sigma_{2b}-\sigma_{2r}+1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{np}+\sigma_{nb}-\sigma_{nr}+1 \end{pmatrix}.$$

See, for example, [18].

Equations (3.3) and (3.4), which are the model of risky capital assets, is an uncertain optimal control system whereby $X(t)$ is the state, U is the control, F is $n \times n$ dimensional constant matrix while P and Q are $n \times m$ dimensional constant matrices, $C(t)$ is the Liu process and $J(X)$ is the objective functional.

4. CONTROLLABILITY AND OBSERVABILITY OF THE UNCERTAIN SYSTEM

Let (Γ, L, M) be a complete uncertainty space with uncertain measure M on Γ and a filtration $\{L(t)|t \in [0,T]\}$ generated by n-dimensional uncertain process $\{C(t) : 0 \leq t \leq T\}$ defined on the uncertainty space (Γ, L, M).

Let $l_2(\Gamma, L(t), \mathbb{R}^n)$ represent the Hilbert space of all $L(t)$-measurable square integrable uncertain variables with all values in $\mathbb{R}^n$. Also, let $l_2^L([0,T], \mathbb{R}^n)$ represent the Hilbert space of all square integrable and $L(t)$-measurable process with the values in $\mathbb{R}^n$. Let $X(t) = X(t+s)$ for $s \in [0,T]$ represent the segment of the trajectory, that is, $X(t) \in l_2^L([0,T], l_2(\Gamma, L(t), \mathbb{R}^n))$.

Let R be a linear operator on the Hilbert space $l_2(\Gamma, L(t), \mathbb{R}^n)$ with domain $D_1(R)$. $R \geq 0$ if $\langle Rc, c\rangle \geq 0$ for all $c \in D_1(R)$, $R > 0$ if $\langle Rc, c\rangle > 0$ for all nonzero $c \in D_1(R)$ and R is coercive $(R - \gamma I \geq 0)$ if there exists a $\gamma > 0$ such that $\langle Rc, c\rangle \geq \gamma \|c\|^2$for all c $\in D_1(R)$.

Now, using the proposed model with the multidimensional constraint of uncertain differential equation

$$dX(t) = FX(t)dt + UPX(t)dt + UQX(t)dC(t) \tag{4.5}$$

for $t \in [0,T]$ with the function initial condition

$$X_0 \in l_2^L([0,T], l_2(\Gamma, L(t), \mathbb{R}^n)),$$

where the state $X(t) \in l_2(\Gamma, L(t), \mathbb{R}^n)$ and control $\psi(t) \in \mathbb{R}^m = U$. U and F are $n \times n$ dimensional constant matrix while P and Q are $n \times m$ dimensional constant matrices.

Thus, suppose the admissible controls $U = l_2^L([0,T], \mathbb{R}^m)$, then for any given initial condition $X_0 \in l_2^L([0,T], l_2(\Gamma, L(t), \mathbb{R}^n))$ and any admissible control $\psi \in U$ for $t \in [0,T]$, there exists a unique solution $X(t; X_0, \psi) \in l_2(\Gamma, L(t), \mathbb{R}^n)$ of the constraint uncertain differential state equation (4.5); see [20].

4.1. Controllability

Definition 4.1:
The uncertain dynamic system (4.5) is said to be relatively exactly controllable on $[0,T]$ *if*

$$R(t)(U) = l_2(\Gamma, L(t), \mathbb{R}^n).$$

This implies that if all the points in $l_2(\Gamma, L(t), \mathbb{R}^n)$ *can exactly be reached at time* T *from any arbitrary initial condition* $X_0 \in l_2^L([0,T], l_2(\Gamma, L(t), \mathbb{R}^n))$.

Definition 4.2:
The uncertain dynamic system (4.5) is said to be relatively approximately controllable on $[0,T]$ *if*

$$\overline{R(t)(U)} = l_2(\Gamma, L(t), \mathbb{R}^n).$$

This implies that if all the points in $l_2(\Gamma, L(t), \mathbb{R}^n)$ can approximately be reached at time T from any arbitrary initial condition $X_0 \in l_2^L([0,T], l_2(\Gamma, L(t), \mathbb{R}^n))$.

The relationship between the controllability concepts for the uncertain dynamic system (4.5) and the controllability of the related deterministic dynamic system below

$$dX(t) = FX(t)dt + UPX(t)dt \quad \texttt{for} \quad t \in [0,T], \tag{4.6}$$

where the admissible control $\psi \in l_2([0,T], \mathbb{R}^m)$.

Firstly, the deterministic system (4.6) is defined according to [1]. Let

$$Q_j(t) = FQ_{j-1}(t)$$

for $j = 1, 2, 3, \dots$ and $t > 0$, with the initial condition

$$Q(t) = Q_0(0) = J, \quad t = 0,$$

$$Q(t) = Q_0(t) = 0, \quad t \neq 0.$$

For instance, the sequence of the $n \times n$ dimensional matrices $Q_j(t)$ deduced from the determining equation gives:

$$Q_0(0) = P,$$
$$Q_1(0) = FP,$$
$$Q_2(0) = F^2 P.$$

These can be written in a general notation as

$$Q_j(t;T) = \{Q_0(t), Q_1(t), Q_2(t), \dots, Q_{j-1}(t) \quad \texttt{for} t \in [0,T]\}.$$

The following Lemma is given with respect to [1], relating to the controllability of the deterministic system (4.6) in the time interval $[0,T]$:

Lemma 4.1:
The following conditions are equivalent:

1. *the deterministic system (4.6) is relatively controllable on $[0,T]$,*
2. *the relative controllability matrix $B(t)$ is non-singular,*
3. *rank $Q_j(t;T) = k$.*

Hence, the following lemma is formed with respect to [4, 20, 21] which will be useful in the proof of the uncertain controllability and observability.

Lemma 4.2:
For every $c \in l_2(\Gamma, L(t), \mathbb{R}^n)$, $\exists$ a process $q \in l_2^L, \mathbb{R}^{n\times n}$ such that the controllability operator is expressed as:

$$G(t)c = B(t)Ec + \int_0^T B(t)(s)q(s)dC(s).$$

Lemma 4.3:
The uncertain system (4.5) is relatively controllable on $[0,T]$ if and only if one of the following conditions holds true:

1. *$E\langle G(t)c, c\rangle \geq \gamma E\|c\|^2$ for some $\gamma > 0$ and all $c \in l_2(\Gamma, L(t), \mathbb{R}^n)$,*
2. *$R(\lambda_1, G(t))$ converges as $\lambda_1 \to 0^+$ in the uniform operator topology,*

3. *$\lambda_1 R(\lambda_1, G(t))$ converges to the zero operator as $\lambda_1 \to 0^+$ in the uniform operator topology,*
4. *ker$(l(t))^* = \{0\}$ and Im$(l(t))^*$.*

Lemma 4.4:
The uncertain system (4.5) is approximately controllable on $[0,T]$ if and only if one of the following conditions holds true:

1. *$G(t) > 0$,*
2. *$\lambda_1 R(\lambda, G(t))$ converges as $\lambda_1 \to 0^+$ in the strong operator topology,*
3. *$\lambda_1 R(\lambda, G(t))$ converges to the zero operator as $\lambda_1 \to 0^+$ in the weak operator topology,*
4. *ker$(l(t))^* = \{0\}$.*

Theorem 4.1:
The following conditions are equivalent:

1. *The deterministic system (4.6) is relatively controllable on $[0,T]$,*
2. *The uncertain system (4.5) is relatively exactly controllable on $[0,T]$,*
3. *The uncertain system (4.5) is relatively approximately controllable on $[0,T]$.*

Proof
Condition (i) implies condition (ii).
Suppose the deterministic system (4.6) is relatively controllable on $[0,T]$, then the relative controllability matrix $B(t)(s)$ is invertible and strictly positive definite for all $s \in [0,T]$. Hence, for some $\gamma > 0$,

$$\langle B(t)(s)X, X\rangle \geq \gamma \|X\|^2$$

for all $s \in [0,T]$ and $X \in \mathbb{R}^n$.

In order to prove the relative exact controllability of the uncertain system (4.5) on $[0,T]$, the relationship between the controllability operator $G(t)$ and the controllability matrix $B(t)$ given in Lemma 4.2 as $E\langle G(t)c, c\rangle$ to be expressed in terms of $\langle G(t)Ec, Ec\rangle$.

Firstly,

$$\begin{aligned} E\langle G(t)c, c\rangle &= E\left\langle B(t)Ec + \int_0^T B(t)(s)q(s)dC(s), Ec + \int_0^T q(s)dC(s)\right\rangle \\ &= \langle B(t), Ec, Ec\rangle + E\int_0^T \langle B(t)(s)q(s), q(s)\rangle\, ds \\ &\geq \gamma\left(\|Ec\|^2 + E\int_0^T \|q(s)\|^2\, ds\right) = \gamma E\|c\|^2 . \end{aligned}$$

Thus, in the view of the controllability operator,

$$G(t) \geq \gamma I,$$

which implies that the relative controllability operator $G(t)$ is strictly positive definite and, the inverse operator $G(t)^{-1}$ is bounded, [1]. Hence, the uncertain relative exact controllability of uncertain dynamic system (4.5) on $[0,T]$ is proved from the relative controllability of deterministic system (4.6) on $[0,T]$.

Condition (ii) implies condition (iii).
Since the state space for the uncertain dynamic system (4.5) is finite-dimensional, which implies that the exact and approximate controllability coincide; see [1]. Hence, it is easy to conclude that the uncertain system (4.5) is relatively approximately controllable on $[0,T]$.

Condition (iii) implies condition (i).
Suppose the uncertain dynamic system (4.5) is uncertainly relatively approximately

controllable on $[0, T]$ and its controllability operator is positive definite, that is, $G(t) > 0$, then applying the resolvent operator $\lambda_1 R(\lambda_1, G(t))$, where $\lambda > 0$, [20], gives

$$E \left\|\lambda_1 R(\lambda_1, G(t))c\right\|^2 \to 0.$$

This implies

$$E \left\|\lambda_1 R(\lambda_1, G(t))c\right\|^2 = \left\|\lambda_1 R(\lambda_1, B(t))Ec\right\|^2 + E\int_0^T \left\|\Lambda_1 R(\lambda_1, B(t)(s))q(s)\right\|^2 ds \to 0.$$

Therefore,

$$E\int_0^T \left\|\Lambda_1 R(\lambda_1, B(t)(s))q(s)\right\|^2 ds \to 0$$

for all $q(s) \in l_2^L[0, T], \mathbb{R}^{n\times n}$, and consequently there exists a subsequence λ_n such that for every $c \in l_2(\Gamma, L(t), \mathbb{R}^n)$ we have

$$\left\|\lambda_n R(\lambda_k, B(t)(s))c\right\| \to 0$$

almost everywhere on $[0, T]$.

Thus, the property holds for all $0 \leq s < T$ because of the continuity of $R(\lambda, B(t)(s))$. This implies that the deterministic system (4.6) is relatively approximately controllable on $[0, T]$. However, considering the state space for the deterministic system (4.6) is finite-dimensional, that is, exact and approximate controllabilities coincides, [1]. Therefore, it is concluded that the deterministic system (4.6) is relatively controllable on $[0, T]$. □

4.2. Observability

Here, the dual concepts of observability for the uncertain dynamic system (4.5) are treated.

Suppose F and J generate C_0-semigroup S_1 and S_2 on the Hilbert space $l_2(\Gamma, L(t), \mathbb{R}^n)$, that is, F and J generate a continuous representation of semigroup S. Then the dual of the uncertain system (4.5) is:

$$dX(t) = [F^* + J^* + W_1^* + W_2^*]X(t)dt + UQX(t)dC(t).$$

Thus, the following concepts are described:

- The observability map of the uncertain system (4.5) on $[0, T]$ is expressed as the linear operator $O(t) : l_2(\Gamma, L(t), \mathbb{R}^n) \to l_2^L([0, T])$ which is defined by

$$O(t)c = (W_1 + W_2)(S_1 + S_2)(T - s)E\{c|L\}.$$

- The observability Gramian of the uncertain system (4.5) on $[0, T]$ is defined by:

$$\Theta = (O(t))^* O(t).$$

Definition 4.3:
The uncertain system (4.5) is said to be relatively observable on $[0, T]$ if the operator $O(t)$ is injective and its inverse bounded on the range $O(t)$. The means that the initial state can be uniquely and continuously constructed from outputs in $l_2^L(\Gamma, L(t), \mathbb{R}^n)$.

Definition 4.4:
The uncertain system (4.5) is said to be approximately observable on $[0, T]$ if

$$O(t) = \{0\},$$

that is, the initial state uniquely depends on the knowledge of the result in $l_2^L(\Gamma, L(t), \mathbb{R}^n)$.

Theorem 4.2:
For the uncertain dynamic system (4.6), the following duality results hold true:

1. *The uncertain system (4.5) is relatively observable on* $[0, T]$ *if and only if the dual system (4.2) is relatively controllable on* $[0, T]$.
2. *The uncertain system (4.5) is approximately controllable on* $[0, T]$ *if and only if the dual system is approximately controllable on* $[0, T]$.

Proof
Since F and J generate a C_o-semigroup $S_1(t)$ and $S_2(t)$ respectively on $l_2^L(\Gamma, L(t), \mathbb{R}^n)$, then F^* and J^* generate the C_o-semigroup $S_1^*(t)$ and $S_2^*(t)$ respectively. Also, since $O(t) \in l_2^L([0, T], l_2(\Gamma, L(t), \mathbb{R}^n))$,

$$(O(t))^* q = \int_0^T (S_1^* + S_2^*)(T - s)(W_1^* + W_2^*) q(s) ds.$$

Thus, the range of $O(t)$ implies that of the controllability operator of the dual system (4.2). Let the controllability of the dual system (4.2) be represented by $N(t)$, then

$$(O(t))^* = N(t), \qquad (N(t))^* = O(t).$$

1. Suppose the deterministic system (4.5) is relatively observable, there exists an inverse $(O(t))^{-1}$ on the range of $O(t)$. Then,

$$\left\|(O(t))^{-1} q\right\| \leq l \|q\|$$

for all $q \in l_2^L([0, T], \mathbb{R}^{n \times n})$ and $l > 0$. Hence,

$$\|c\| = \left\|(O(t))^{-1} O(t) c\right\| \leq l \left\|O^T c\right\| = l \|(N(t))^* c\| .$$

Therefore, the relative controllability of the uncertain system (4.5) follows from Lemma 4.3 as thus. Suppose that the uncertain system is relatively controllable, then $(N(t))^*$ is injective and has a closed range. This implies that from $(N_0)^* = O(t)$, $O(t)$ is injective and has a closed range. Thus, by the Closed Graph Theorem, the inverse of $O(t)$ is bounded on the range of $O(t)$.
2. However, by definition, the deterministic system (4.6) is approximately observable if and only if

$$\texttt{ker} O(t) = \texttt{ker}(N(t))^* = \{0\}.$$

Therefore by Lemma 4.4, $\texttt{ker}(N(t))^* = \{0\}$ if and only if the uncertain system (4.5) is approximately controllable. Hence the proof of the equivalence.

□

5. CONCLUSION

The necessary and sufficient conditions for the uncertain controllability and observability of a finite-dimensional uncertain dynamic control system have been established and proved. Thus, the effectiveness of each input and output in the general operation of the control system can be determined. The application of this work is to measure the controllability and observability degree of the system by using certain admissible input factors to determine how the system can move around its configuration space. Subsequently, we can observe the analytic and numerical solutions to the multifactor system.

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