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The "Value at Risk" Principle in Hierarchical Game

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Abstract: A hierarchical game of two persons with random factors is considered. It is assumed that the top-level player has the right of the first move. It is believed that the lower level player at the time of decision making knows exactly the realization of the random factor and the choice of partner. And the top-level player at the time of decision making knows only a probabilistic measure on the set of values of an uncertain factor. The principle of optimality is new: it is believed that a top-level player is ready to neglect some of the "unpleasant" events, the total probability of which is given, but otherwise he is careful. Under these assumptions, the maximum guaranteed result of the top-level player is calculated. The structure of strategies providing such a result is clarified. Two cases were investigated: a game with and without feedback. To solve the problem, an original definition of the maximum guaranteed result is proposed. It is equivalent to the classical definition, but is simpler. Using this technique, solving of the problem reduces to identical transformations of the formulas for predicate calculus. As a result of the solution, the optimal strategy and the set of "unpleasant" cases which are excluded from consideration search task is reduced to calculating multiple maximins on finite-dimensional spaces. In this case, the operation of calculating the expected value with respect to given probabilistic measure is considered to be "elementary". Models of this type can have different interpretations. One can use them for methodological justification of the principle of maximum guaranteed result. One can use them when solving risk management tasks. One can consider them as models for managing the "customer base" of the service company. The proposed method allows to study such models at a qualitative level, and in some cases to obtain quantitative results.

Keywords: informational theory of hierarchical systems, hierarchical games, decision making under risk, maximal guaranteed payoff, risk management

1. INTRODUCTION

A study of hierarchical games with uncertain factors was started in the early seventies of the last century [8,9,15]. Around the same time, similar models were investigated in the theory of active systems [4,5,18] and in the theory of contracts [2,3,16]. In this case, two methods of eliminating uncertainty were mainly studied.

In one of them, it is assumed that the players know only sets of possible values of uncertain factors and they are careful i.e. reckons upon the worst option for themselves. Another considered that the probability measure is defined on the set of uncertain factors, and the players are risk-neutral, that is they are ready to be guided by the expectations of their payoffs. A third, in a sense, intermediate way of eliminating uncertainty is also possible. In the financial engineering it received title principle "value at risk" [6,17]. The development of these ideas can be found in the monograph [1] and other works of the same author.

At a meaningful level, its essence can be explained as follows. I do not think that making decisions in everyday life, people are very often appreciate some probability distributions,

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and even more so calculate mathematic expectations. And on a professional level, I have never had to deal with the customer who formulates the problem in probabilistic terms. But the tendency to the principle of guaranteed results, customers several times clearly formulated. But the principle of maximum guaranteed result has one not a very attractive property. If it is conducted quite consequentially, the possibility cannot be ruled out that the player “does not fall down a brick on his head” or something else unpleasant happens. And constantly focusing on such cases, it is hardly possible to make really effective decisions.

In practice and in theory [7], from this situation, the next way out is used. A part of the completely “fatal” values of the uncertain factor is excluded from consideration, and with respect to the remaining part, the principle of maximal and minimal guaranteed result is used. On what basis are some possibilities excluded from consideration? Probably, it happens because the operating party regards them as “unlikely”.

Of course, there is the temptation to postpone the work for exclusion of unfavorable factors from the level of the model building to the level of its research. For this purpose, it is necessary the operating party accept some probability distribution and said that with a given probability ξ it wants to get a payoff not less than the value of γ. Of course, it is preferably to the value of γ to be larger. Thus, we come to the statement of the problem considered below.

It is unlikely that all this makes sense in the analysis of the problems at the household level. But in the analysis of business decisions risk management problems has recently become very relevant, if not fashionable. Moreover, the risk assessment method discussed above is one of the two most popular (as far as I know, game-theoretic models in which risk is estimated using dispersion also have not yet been investigated). But this method is usually used to investigate problems of centralized decision making under risk. Game-theoretic models of this kind, apparently, have not yet been studied.

Further presentation is constructed as follows. The next section gives a formal statement of the problem. The rest of the article is devoted to its “solution”. By tradition the solution of the problem of calculating of maximal guaranteed result in hierarchical game with a “complicated” structure is regarded as it’s reduction to some “elementary” operations such as computation of maxima and minima on the “simple” sets. We will follow this tradition. In a problem under consideration “non-elementary” operation of selection of a *set* of uncertainties to be excluded from consideration arises already in games not endowed with additional structure. Section 3 is devoted to their study. In the fourth paragraph, the more traditional problem of finding the maximal guaranteed result in a “feedback games” is solved.

2. GAMES WITH RANDOM FACTORS

So, we begin to describe the simulated conflict.

Game with random factors will be hereinafter called a six-tuple $\Gamma = \langle U, V, A, g, h, \wp \rangle$. Here, U, V and A are the sets, g – function which maps the Cartesian product $U \times V$ to the set of real numbers $\mathbb{R}$, $h: U \times V \times A \to \mathbb{R}$, and $\wp$ is probability measure on the set A.

These constructions are interpreted as follows. It is assumed that two participants take part in the game, which we will call the first and second players. A set U is interpreted as the set of controls of the first player, a set V – the set of controls of his partner. We assume that the interests of the first and second players are described by the desire to maximize the functions g and h, respectively. The value of the indefinite factor $\alpha \in A$ is chosen by some third party – Nature. This choice is made randomly in accordance with the distribution $\wp$.

We make the traditional technical assumptions, which significantly simplify the further narration. The sets U, V, and A are assumed to be endowed with topologies and compact ones. The functions g and h are assumed to be continuous. Measure $\wp$ will be considered to be Borel.

Unfortunately, not all results can be obtained in such a general assumptions. Additional conditions on game under consideration, if required, will be indicated when corresponding results will be formulated.

Game Γ describes the possibilities and interests of the players. For the model completion one must also describe the dynamics of decision-making and the attitude of the players to the existing uncertainty. Let's start with the interpretation.

We believe that all the game Γ options are exactly known to the first player. We assume that events take place as follows. At first, the first player chooses his control $u \in U$. Then, the specific value of the indefinite factor $\alpha \in A$ is realized (in accordance with the given probabilistic measure $\wp$). The values of u and α become known to the second player. Thus, for the second player, no uncertainty remains. Consequently, for him all the controls $v \in V$ will be divided into "reasonable", in case of the choice of which he will receive a payoff greater than or equal to a certain number of λ, and "unreasonable", the choice of which promises him payoff smaller then λ (of course, number λ depends on u and α). This principle of behavior is known to the first player. But he is careful and therefore he reckons upon the worst result that can happen when "reasonable" choice of partner will be made. But since the value α is not known to the first player, for him this result is a random variable. The attitude of the first player to this uncertainty is as follows. He agrees to exclude from consideration a certain number of "force majeure" events, the total probability of which does not exceed a given value $1 - \xi$. But other than, he focuses on the worst case for himself and wants to get the maximal guaranteed result.

Formally, the above is described as follows.

Definition 2.1:

Let a real number $\xi \in [0,1]$ be given . Number γ is ξ-guaranteed result of the first player in the game Γ, if there exist a measurable set $B \subset A$, measure $\wp(B)$ of which is greater than or equal to ξ, and such strategy $u \in U$, that for every $\alpha \in B$ there exist a number λ, for which the following conditions are hold true:

1 °. There exists $w \in V$ for which $h(u,w,\alpha) \geq \lambda$;

2 °. For any $v \in V$, either $g(u,v) \geq \gamma$ or $h(u,v,\alpha) < \lambda$.

Supremum of ξ-guaranteed results of the first player is called its maximal ξ-guaranteed result.

Remark 2.1:

It would be possible to refuse the assumption of measurability of set B in this definition, replacing measure $\wp(B)$ with the corresponding outer measure. It will be seen from what follows that, under the assumptions that the measure $\wp$ is Borel, and the function h is measurable, such a modification of the definition do not essentially change anything.

Remark 2.2:

The definition 2.1 is a modification of definition of maximal guaranteed result, proposed by the author in [10]. In more conventional terms the maximal ξ-guaranteed result can be defined by the formula

$$\sup_B \sup_{u \in U} \inf_{\alpha \in B} \min_{v \in BR(u,\alpha)} g(u,v)$$

wherein the outer supremum is taken over all measurable subsets B of the set A, for which $\wp(B) \geq \xi$, and

$$BR(u,\alpha) = \left\{ v \in V : h(u,v,\alpha) = \max_{w \in V} h(u,w,\alpha) \right\}.$$

The proof of the equivalence of these definitions only insignificant differs from the reasoning in [10]. A significant part of this work will, in fact, be done in the next section. For more complex problems Definition 2.1 is more convenient, therefore we will use it.

Remark 2.3:

All probabilities in this paper can be regarded as subjective, namely, as an evaluation of the operating party (first player) the feasibility of certain events. In the future, no results such as the law of large numbers are used. Therefore, there is no need to take care of any kind of statistical stability. Indeed only readiness of operating party to eliminate uncertainty by the method described in the Definition 2.1 is important.

3. GAME WITHOUT FEEDBACK

Calculation of maximal ξ-guaranteed result, for example, by the formula from the remark 2.2 involves computation of least upper bound on the class of subsets of the set A. Standard methods for calculating of such supremum not exists even for the relatively simple case where the set A is a segment. In this section, we simplify the solution of the problem by replacing the operation of computing such an upper bound with the operation of calculating the expected value. Traditionally such an operation is considered to be "elementary".

Introduce the following notation

$$m(u,\alpha) = \max_{v\in V} h(u,v,\alpha).$$

Let γ be ξ-guaranteed result of the first player in the game Γ. Choose a set $B \subset A$ and strategy $u \in U$, the existence of which is provided for the definition 2.1. Fix an arbitrary $\alpha \in B$.

For strategy $w \in V$, the existence of which postulate the item 1° of definition the inequality $h(u,w,\alpha) \geq \lambda$ is satisfied, the more this inequality must be satisfied for the strategy w^0 determining by the equality $h(u,w^0,\alpha) = m(u,\alpha)$. Consequently the number λ, appearing in Definition 2.1, must satisfy the condition $\lambda \leq m(u,\alpha)$.

Therefore, if the item 2° of definition is satisfied for someone value λ, it moreover holds for $\lambda = m(u,\alpha)$. But with such a value of λ, item 1° also is obviously satisfied: it suffices to choose, for example, $w = w^0$.

Thus, the number of γ is ξ-guaranteed results of the first player in the game Γ, if there is a measurable set $B \subset A$, measure $\wp(B)$ of which is greater than or equal to ξ, and such a strategy $u \in U$, that for every $\alpha \in B$ and any $v \in V$ either $g(u,v) \geq \gamma$ or $h(u,v,\alpha) < m(u,\alpha)$.

Consider the set

$$BR(u,\alpha) = \left\{ v \in V : h(u,v,\alpha) = \max_{w\in V} h(u,w,\alpha) \right\}.$$

If $v \in BR(u,\alpha)$, then $h(u,v,\alpha) = m(u,\alpha)$; therefore , by the second item of Definition 2.1, the inequality $g(u,v) \geq \gamma$ must hold. Conversely, if the last inequality holds for all $v \in BR(u,\alpha)$, then the number γ is ξ-guaranteed result. Indeed, for $w \in BR(u,\alpha)$ and $\lambda = m(u,\alpha)$ the first item of definition is satisfied. For the same values of λ and $v \notin BR(u,\alpha)$ the item 2° is met because of in this case $h(u,v,\alpha) < m(u,\alpha)$, and for $v \in BR(u,\alpha)$ it satisfies as by the assumption $g(u,v) \geq \gamma$.

Thus, the number of γ is ξ-guaranteed result of the first player in the game Γ, if there exist a measurable set $B \subset A$, measure $\wp(B)$ of which is greater than or equal to ξ, and such a strategy $u \in U$, that for every $\alpha \in B$ and any $v \in BR(u,\alpha)$ the inequality $g(u,v) \geq \gamma$ is true. The same condition can be formulated equivalently: the number γ is ξ-guaranteed result of the first player in the game Γ, if there exist a measurable set $B \subset A$, measure $\wp(B)$

of which is greater than or equal to ξ, and such a strategy $u \in U$, that for any $\alpha \in B$, the inequality

$$\min_{v \in BR(u,\alpha)} g(u,v) \geq \gamma \tag{3.1}$$

holds.

Consider the set

$$C(u) = \left\{ \alpha \in A : \min_{v \in BR(u,\alpha)} g(u,v) \geq \gamma \right\}.$$

Since condition (3.1) holds for all $\alpha \in B$, the set B must be contained in the set $C(u)$, and hence , the condition $\wp(C(u)) \geq \wp(B) \geq \xi$ must holds.

Conversely, if the condition $\wp(C(u)) \geq \xi$ is valid for some strategy u, then condition (3.1) will be satisfied for all $\alpha \in B = C(u)$, and hence γ is ξ-guaranteed result.

Let define the function $\theta(x)$ by the condition

$$\theta(x) = \begin{cases} 1, \text{ if } x \geq 0, \\ 0, \text{ if } x < 0. \end{cases}$$

The measure of the set $C(u)$ is equal to

$$\mathrm{M}\theta\left(\min_{v \in BR(u,\alpha)} g(u,v) - \gamma \right),$$

where the symbol M designate operator of calculating mathematical expectations with respect to measure $\wp$.

This immediately yields the following result.

Theorem 3.1:

In order for the number γ to be ξ-the guaranteed result of the first player in the game Γ, it is necessary and sufficient that either

$$\max_{u \in U} \mathrm{M}\theta\left(\min_{v \in BR(u,\alpha)} g(u,v) - \gamma \right) \geq \xi, \tag{3.2}$$

or

$$\sup_{u \in U} \mathrm{M}\theta\left(\min_{v \in BR(u,\alpha)} g(u,v) - \gamma \right) > \xi,$$

if the upper bound in the last formula is not reached.

Remark 3.1:

In the papers [11] and [12] in the same manner was formulated conditions characterizing maximum guaranteed result for games with undefined interval uncertainty and games with risk-neutral first player. In earlier papers [14] and [15], although with some additional assumptions explicit formulas were obtained for the maximum guaranteed results in these problems. In the problem considered in this paper, such an alternative is not visible even for simpler analogue problem of optimization corresponding the game Γ, wherein a set V consists of one point.

Remark 3.2:

Formula (3.2) could be a starting point in an attempt to give a more traditional definition of maximal ξ-guaranteed result. But such a definition in this case would require clarification. And apparently, this explanation inevitably would be similar to the definition 2.1. In addition, quite a "classical" form of this definition can't be given, because the number γ is part of the argument of function θ. Thus, in this case it is not very convenient to *follow the traditions. In the next section, the advantage of the new definition will become even more obvious.*

In order not to be distracted by the technical details in the proof of theorem 3.1 was left a gap. To fill it, it is necessary to prove the following statement.

Lemma 3.1:

For every $u \in U$, the set $C(u)$ is measurable.

Proof. Fix $u \in U$. It is sufficient to prove that the function $\varphi(\alpha) = \min_{v \in BR(u,\alpha)} g(u,v)$ is measurable. To do this, it sufficient to prove that the function $-\varphi(\alpha) = 0 - \varphi(\alpha)$ is measurable. Therefore, it suffices to prove that for any γ the set

$$C^0(u) = \{\alpha \in A : -\varphi(\alpha) < -\gamma\} = \left\{\alpha \in A : \min_{v \in BR(u,\alpha)} g(u,v) > \gamma\right\}$$

is measurable (for convenience, this part of the proof uses only the facts explicitly stated in [13]).

Since the measure $\wp$ is assumed to be Borel, it is sufficient to prove that the set $C^0(u)$ is open. Suppose the contrary.

Then there exist $\alpha \in C^0(u)$ and a sequence $\alpha_1, \alpha_2 ...$ such that $\lim_{k \to \infty} \alpha_k = \alpha$ and $\alpha_k \notin C^0(u)$ for all $k = 1,2, ...$

The set $BR(u,\alpha_k)$ is defined by the condition of equality type, so it is closed due to the continuity of function h. Since a set V is assumed to be compact the set $BR(u,\alpha_k)$ also will be compact. Hence, at some point $v_k \in BR(u,\alpha_k)$ the minimum of the function $g(u,v)$ on the set $BR(u,\alpha_k)$ is achieved. Since by assumption $\alpha_k \notin C^0(u)$, the inequality $g(u,v_k) \le \gamma$ holds.

Since the set V is compact it is possible without loss of generality to assume that the sequence $v_1, v_2, ...$ converges to an element $v \in V$.

Fix an arbitrary $w \in V$. Since $v_k \in BR(u,\alpha_k)$, the inequality $h(u,v_k,\alpha_k) \ge h(u,w,\alpha_k)$ holds. Since the function h is continuous, going to the limit in this inequality, we obtain $h(u,v,\alpha) \ge h(u,w,\alpha)$. Since w is arbitrary, it follows that $v \in BR(u,\alpha)$. And going to the limit in the inequality $g(u,v_k) \le \gamma$, we get $g(u,v) \le \gamma$, and even more so $\min_{v \in BR(u,\alpha)} g(u,v) \le \gamma$ which contradicts the condition $\alpha \in C_0(u)$.

The obtained contradiction proves the lemma. □

Thus, theorem 3.1 is completely proved.

Remark 3.3:

The assumptions formulated in the previous section about the topological and metric structure of the game Γ can be changed, and, perhaps, simply weakened. The question of the weakest assumptions under which Lemma 3.1 and further results of this type remain true, is of some interest, but beyond the scope of this article. The most interesting model examples certainly satisfy the conditions formulated above, so we omit further discussion.

Remark 3.4:

The proof of Lemma 3.1 is quite standard, but rather long. For this reason, further, the proof of similar results is omitted.

A question of search optimal strategy of the first player in this game is quite meaningful. Given the results obtained, the answer to it is not difficult to find.

First of all, we note that the supremum in the definition of maximal ξ-guaranteed result can't be achieved. Therefore, there may not exist a strategy allowing one to obtain such a result with probability ξ. This effect is understandable because similar fact takes place already in the game with no uncertainty (i.e., with single-point set A).

If the number γ is a ξ-guaranteed result, then any solution of the inequality

$$\mathrm{M}\theta\left(\min_{v\in BR(u,\alpha)} g(u,v)-\gamma\right)\geq \xi,$$

if the upper bound in formula (3.2) is reached, or the inequality

$$\mathrm{M}\theta\left(\min_{v\in BR(u,\alpha)} g(u,v)-\gamma\right)> \xi,$$

otherwise, is the desired strategy. In both cases, the inequalities have solutions, since by assumption γ is ξ-guaranteed result.

One of sets B "suitable" for this strategy can be defined by the condition $B = C(u)$.

Of course, such selection of the set B is not the only possible. However, in the general case, the same applies to the choice of the optimal strategy u.

4. GAME WITH FEEDBACK

Consider another game $\Gamma_* = \langle U_*, V_*, A, g_*, h_*, \wp\rangle$, in a certain way related to the game Γ.

Denote by $\Phi(X,Y)$ the class of all functions from the set X into the set Y. Let $U_* = \Phi(V\times A,U)$, $V_* = V\times A$, and the functions g_* and h_* are determined by the conditions

$$g_*(u_*,v_*) = g(u_*(v,\beta),v),\ h_*(u_*,v_*,\alpha) = h(u_*(v,\beta),v,\alpha),$$

where $v_* = (v,\beta)$. The set A and the measure $\wp$ on it are the same as in the game Γ.

These constructions can be interpreted as follows. Players choose their "physical" controls from the sets U and V. But by the time of the selection of its control $u\in U$ the first player receives reliable information about control $v\in V$ selected by his partner. In addition, the second player can transmit to the first player information on the realized value of the uncertain factor. However, this information does not have to be reliable, that is, the second player has the right to select some message $\beta\in A$, which he will transmit to the partner. His physical control $u_*(v,\beta)$ the first player chooses on the basis of all the information received, and both players' payoffs depend only on the physical choices made by them, and do not depend on information they exchanged.

Game Γ_* has the same structure as the game Γ, so the question can be put of finding the maximal ξ-guaranteed result in this game. We will deal with this task.

When analyzing this problem, using the English language is already quite inconvenient. Therefore, we turn to the language of predicate calculus.

For the game Γ_* the definition of ξ-guaranteed result γ will look as follows:

$$\begin{gathered}\exists B\exists u_*\in\Phi(V\times A,U)\forall\alpha\in B\exists\lambda:\wp(B)\geq\xi\ \&\\ \&\left[\exists w\in V\exists \upsilon\in A: h(u_*(w,\upsilon),w,\alpha)\geq\lambda\right]\&\\ \&\left[\forall v\in V\forall\beta\in A\ g(u_*(v,\beta),v)\geq\gamma\vee h(u_*(v,\beta),v,\alpha)<\lambda\right].\end{gathered}\tag{4.1}$$

This formula is not "elementary", because it contains one existential quantifier refers to a class of all measurable subsets B of the set A, and another – to the class of all functions u from the set $V\times A$ to the set U. It can be simplified, but one has to make the following assumption.

Hypothesis 4.1:

The function h is such that there exists such control $u^p\in U$, that for any $v\in V$ and any $\alpha\in A$ the equality $h(u^p,v,\alpha)=\min_{u\in U} h(u,v,\alpha)$ holds.

In fact, it assumes the existence of a universal (independent of α) strategy of punishing the second player by first.

Now we can start converting the formula (4.1). As in the previous section let’s start with the specification of the value λ. Put

$$H(\gamma)=\{(u,v)\in U\times V : g(u,v)\geq\gamma\},$$

$$l(\alpha,\gamma)=\max_{(u,v)\in H(\gamma)} h(u,v,\alpha).$$

In meaningful terms, $H(\gamma)$ is the set of “acceptable” outcomes for the first player. The number $l(\alpha,\gamma)$ characterizes the maximum payoff that the second player can get, provided that the first one somehow ensures that he gets an “acceptable” result (of course, this maximum payoff depends on the indefinite factor α).

According to the condition (4.1), there exist a set B and a function ω_* such that

$$\begin{gathered}\forall\alpha\in B\exists\lambda:\wp(B)\geq\xi\,\&\left[\exists w\in V\exists\upsilon\in A: h(\omega_*(w,\upsilon),w,\alpha)\geq\lambda\right]\&\\ \&\left[\forall v\in V\forall\beta\in A\ g(\omega_*(v,\beta),v)\geq\gamma\vee h(\omega_*(v,\beta),v,\alpha)<\lambda\right],\end{gathered}\tag{4.2}$$

Fix such a set and a function. Then there exists a function u_*, for which the condition is satisfied:

$$\begin{gathered}\forall\alpha\in B\ \wp(B)\geq\xi\,\&\\ \&\left[\exists w\in V\exists\upsilon\in A: h(u_*(w,\upsilon),w,\alpha)\geq l(\alpha,\gamma)\right]\&\\ \&\left[\forall v\in V\forall\beta\in A\ g(u_*(v,\beta),v)\geq\gamma\vee h(u_*(v,\beta),v,\alpha)<l(\alpha,\gamma)\right].\end{gathered}\tag{4.3}$$

Let’s prove it. Let the condition (4.2) be satisfied. Then the set $H(\gamma)$ is not empty. Indeed, let’s fix any $\alpha\in B$. Then by virtue of condition (4.2) we have $h(\omega_*(w,\upsilon),w,\alpha)\geq\lambda$. Then, due to the same condition $g(\omega_*(w,\upsilon),w)\geq\gamma$ and therefore $(\omega_*(w,\upsilon),w)\in H(\gamma)$.

For each $\alpha\in A$ let’s fix a pair $(u^\alpha,v^\alpha)\in H(\gamma)$ such that $h(u^\alpha,v^\alpha,\alpha)=l(\alpha,\gamma)$. Put

$$u_*(v,\beta)=\begin{cases}u^\alpha,\ \text{if }\ v=v^\alpha\ \text{ and }\ \beta=\alpha,\\ \omega_*(v,\beta)\ \text{in other cases.}\end{cases}$$

Then for any $\alpha\in A$ a condition

$$\exists w\in V\exists\upsilon\in A: h(u_*(w,\upsilon),w,\alpha)\geq l(\alpha,\gamma)$$

is satisfied (one can take $w=v^\alpha$ and $\upsilon=\alpha$). In addition, from the inequality $h(\omega_*(w,\upsilon),w,\alpha)\geq\lambda$ it follows that $\lambda\leq l(\alpha,\gamma)$, so condition (4.2) implies

$$g(\omega_*(v,\beta),v)\geq\gamma\vee h(\omega_*(v,\beta),v,\alpha)<l(\alpha,\gamma).\tag{4.4}$$

If $\omega_*(v,\beta)\neq u_*(v,\beta)$, then by construction inequality $g(u_*(v,\beta),v)\geq\gamma$ is true, hence the condition

$$g(u_*(v,\beta),v)\geq\gamma\vee h(u_*(v,\beta),v,\alpha)<l(\alpha,\gamma)\tag{4.5}$$

is satisfied. Otherwise, conditions (4.4) and (4.5) are equivalent.

Thus, it is proved that condition (4.2) follows condition (4.3). The reverse implication is obvious. Therefore, conditions (4.2) and (4.3) are equivalent.

Exactly the same “modification” of the strategy of the first player proves that the condition

$$\begin{gathered}\exists B\exists u_*\in\Phi(V\times A,U)\forall\alpha\in B\ \wp(B)\geq\xi\,\&\\ \&\left[\exists w\in V\exists\upsilon\in A: h(u_*(w,\upsilon),w,\alpha)\geq l(\alpha,\gamma)\right]\&\\ \&\left[\forall v\in V\forall\beta\in A\ g(u_*(v,\beta),v)\geq\gamma\vee h(u_*(v,\beta),v,\alpha)<l(\alpha,\gamma)\right]\end{gathered}$$

is equivalent to a simpler condition

$$\exists B \exists u_* \in \Phi(V \times A, U) \forall \alpha \in B \ \wp(B) \geq \xi \ \&$$
$$\& \left[\forall v \in V \forall \beta \in A \ g(u_*(v,\beta),v) \geq \gamma \vee h(u_*(v,\beta),v,\alpha) < l(\alpha,\gamma) \right].$$

The relevant reasoning is practically the same as the above, so we omit them.

Change the order of the generality quantifiers in the last formula:

$$\exists B \exists u_* \in \Phi(V \times A, U) \forall v \in V \forall \beta \in A \ \wp(B) \geq \xi \ \&$$
$$\& \left[g(u_*(v,\beta),v) \geq \gamma \vee \forall \alpha \in B \ h(u_*(v,\beta),v,\alpha) < l(\alpha,\gamma) \right].$$

Now, one can use the structure of the set of strategies of the first player to change the order of quantifiers of existence and generality:

$$\exists B \forall v \in V \forall \beta \in A \exists u \in U : \wp(B) \geq \xi \ \& \left[g(u,v) \geq \gamma \vee \forall \alpha \in B \ h(u,v,\alpha) < l(\alpha,\gamma) \right].$$

The variable β in square brackets has “disappeared”, so this formula can be further simplified:

$$\exists B \forall v \in V \exists u \in U : \wp(B) \geq \xi \ \& \left[g(u,v) \geq \gamma \vee \forall \alpha \in B \ h(u,v,\alpha) < l(\alpha,\gamma) \right].$$

Denote

$$E(\gamma) = \left\{ v \in V : \max_{u \in U} g(u,v) < \gamma \right\}.$$

Then the previous condition can be rewritten in an equivalent form:

$$\exists B \forall v \in E(\gamma) \exists u \in U : \wp(B) \geq \xi \ \& \ \forall \alpha \in B \ h(u,v,\alpha) < l(\alpha,\gamma),$$

or

$$\exists B \forall v \in E(\gamma) \wp(B) \geq \xi \ \& \ \exists u \in U : \forall \alpha \in B \ h(u,v,\alpha) < l(\alpha,\gamma).$$

Now let us use hypothesis 4.1 to change the order of the generality and existence quantifiers:

$$\exists B \forall v \in E(\gamma) \wp(B) \geq \xi \ \& \ \forall \alpha \in B \exists u \in U : h(u,v,\alpha) < l(\alpha,\gamma).$$

Once again changing the order of the generality quantifiers, we get

$$\exists B \wp(B) \geq \xi \ \& \ \forall \alpha \in B \forall v \in E(\gamma) \exists u \in U : h(u,v,\alpha) < l(\alpha,\gamma).$$

Let

$$\vartheta(x) = \begin{cases} 1, \text{ if } \ x > 0, \\ 0, \text{ if } \ x \leq 0. \end{cases}$$

Replacing the quantifiers of generality and existence with the operators of maximum, minimum and expectation, how was it done in the previous section, we get the following result.

Theorem 4.1:

Let hypothesis 4.1 be fulfilled. Then in order for the number γ to be a ξ-guaranteed result, it is necessary that

$$\mathrm{M}\theta\left(\inf_{v \in E(\gamma)} \max_{u \in U} \left(l(\alpha,\gamma) - h(u,v,\alpha) \right) \right) \geq \xi,$$

and it is sufficient that

$$\mathrm{M}\vartheta\left(\inf_{v \in E(\gamma)} \max_{u \in U} \left(l(\alpha,\gamma) - h(u,v,\alpha) \right) \right) > \xi. \tag{4.6}$$

Remark 4.1:

There are games Γ *for which the least upper bound of the numbers* γ *satisfying the necessary condition in the theorem differs from the least upper bound of the numbers* γ *satisfying the sufficient condition. For such games, the results obtained do not give a definitive answer to the question, what is the maximum* ξ*-guaranteed result? But it does not make sense to refine the obtained necessary and sufficient conditions in this case, since it is clear that for such games the problem of calculating the maximal* ξ*-guaranteed result is not stable with respect to small changes in the parameters of the game* Γ*. Therefore, additional study this problem is required, which is beyond the scope of this article. However, such games are in a sense "exceptional".*

Remark 4.2:

The acceptance of hypothesis 4.1 from a formal point of view seems rather restrictive, since it is essentially assumed that all functions from some parametric family have saddle points. But in many meaningful models, its use seems justified. For example, if the first player chooses a price, he can choose it as the minimum possible, if he allocates a resource to a partner, he can allocate it "at a minimum" under all conditions. A. F. Kononenko generally believed that in economic models, hypothesis 4.1 is always fulfilled. I do not quite share this view, for the principle of proportion of the severity of the punishment to the severity of the offence must be considered. But in this case it is impossible to abandon this hypothesis. In this sense, the problem considered in this paper is more complicated than the problem with a risk-neutral player, where, as shown in [12], a similar hypothesis can be abandoned by introducing a "gauge" additive to the payoff function of the second player.

Analysis of the proof shows that hypothesis 4.1 can be replaced by the following assumption.

Hypothesis 4.2:

There exist a control $u \in U$ *such that the inequality* $\max_{v\in V} h(u,v,\alpha) < l(\alpha,\gamma)$ *holds for all* $\alpha \in A$.

Since the inequality in hypothesis 4.2 is strict, it cannot be argued that it is weaker than hypothesis 4.1. However, it is quite possible to expect that there are quite a lot of meaningful models in which hypothesis 4.2 holds and hypothesis 4.1 does not. Hypothesis 4.1 is accepted as the main one, since it is easier to interpret.

If the sufficient condition (4.6) of theorem 4.1 is satisfied for some number γ, then the results obtained above allow us to construct one of the strategies that allow us to obtain the result γ with probability ξ.

As the set B, which appears in the definition of the maximum guaranteed result, we can take the set

$$B = \left\{\alpha \in A : \inf_{v\in E(\gamma)} \max_{u\in U} \big(l(\alpha,\gamma) - h(u,v,\alpha)\big) > 0\right\}.$$

For any α from the set B thus chosen let's choose an arbitrary pair (u^α,v^α) from the set $H(\gamma)$ satisfying the condition $h(u^\alpha,v^\alpha) = l(\alpha,\gamma)$. Put

$$u_*(v,\alpha) = \begin{cases} u^\alpha, \text{ if } \alpha \in B \text{ и } v = v^\alpha, \\ u^p \text{ in all other cases.} \end{cases}$$

It is directly verified that the so-defined strategy u_* is the desired one.

The interpretation of these constructions is standard. The second player is asked to select control v^α, if the value of the undefined factor $\alpha \in B$ has been realized, and to report the true information about this factor. In this case, the first player promises to use the "encouraging" control u^α. Otherwise, the second player faces punishment. Controls u^α and v^α are chosen so that the message of reliable information is really beneficial to the second player. Cases $\alpha \notin B$ are excluded from consideration by the first player. Therefore, in particular, for such values α, the set $H(\gamma)$ can be empty. In these cases, the strategy of punishment is used for greater reliability.

5. CONCLUSION

In addition to the “methodological” interpretation described in the introduction, the studied model has another, perhaps more interesting one. The value $1-\xi$ in this model can be naturally considered as a measure of risk. Thus, the model explicitly describes both the “yield”, estimated by the value of the payoff $g(u,v)$, and the risk. This seems important enough.

It is quite natural to assume that the value ξ is the control of the operating party (the first player), along with u. Here we can assume that the order of decision-making is as follows. First, the first player fixes the value ξ and the strategy u (or u_*, respectively), then the value of the uncertain factor α is realized, then the second player chooses his control. In this case, it does not matter whether the second player receives information about the selected value ξ, since his payoff does not depend on him.

However, the model is not fully formed, because in this case it is natural to assume the presence of two criteria: the risk measure $1-\xi$ and the corresponding ξ-guaranteed result. It follows directly from the definition that, as ξ increases, the corresponding ξ-guaranteed result does not increase. The choice of balance between the two criteria is left to the operating party. But if the researcher of the operation has an effective way of calculating the ξ-guaranteed result, it will be a serious help in solving this problem.

The described method of risk accounting, of course, is not the only one possible. But already on the basis of the studied model it is possible to construct other meaningful problem statements.

For example, one can assume that the operating party selects a number of values $\xi^1,\xi^2,\ldots,\xi^n$. For each strategy $u \in U$ it is possible to find the payoff of the first player $\gamma^i(u)$ which with a probability of ξ^i is provided with the choice of strategy u under the rational actions of a partner. And then a multi-criteria problem is solved with the criteria $\gamma^1(u),\gamma(u)^2,\ldots,\gamma(u)^n$. To these criteria, one can add the expected value of a guaranteed payoff of the first player.

Thus, we get a fairly wide range of statements, each of which can be “tried on” for the simulated situation. Apparently, the key step in the study of the corresponding problems is made in this article.

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