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# Resource Allocation with a Probabilistic Amount of Resource

Vladimir Burkov*[1], Irina Burkova[1,2], Larisa Rossikhina[2,3]

[1)] *Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia*

*E-mail: vlab17@bk.ru*

[2)] *Academy of Management, the Ministry of Internal Affairs of the Russian Federation, Moscow, Russia*

[3)] *Voronezh Institute of the Federal Penitentiary Service of the Russian Federation, Voronezh, Russia*

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**Abstract***:* The allocation problem of a limited resource under the probabilistic uncertainty over its amount is considered. The Principal has a certain amount of resources to be allocated by him/her among consumers (agents). Each agent submits the request for the resource to the Principal. The Principal allocates the resource in accordance with a specified resource allocation mechanism. In the theory of active systems, the priority-based resource allocation mechanisms were proposed and investigated. With these mechanisms, the resource is allocated proportionally to the values of the agent's priority functions. Three types of the priority-based mechanisms were identified, namely, the mechanism of absolute priorities, the mechanism of straight priorities and the mechanism of reverse priorities. Previously, the priority-based mechanisms were considered under the assumption that the amount of available resource to be allocated by the Principal is known. However, in many real resource allocation problems arising in practice this amount is often unknown. In this paper, the priority-based mechanisms are studied for the case in which the agents know the Principal's resource allocation function. The mechanisms of resource allocation based on the principle of reverse priorities are studied.

***Keywords***: resource allocation; mechanisms of reverse priorities; probabilistic uncertainty; Nash equilibrium.

## 1. INTRODUCTION

The allocation problems of limited resources are widespread in practice [1, 2, 3].

The classical resource allocation scheme is as follows [4]. The Principal has a certain amount of resources to be allocated by him/her among consumers (agents). Each agent submits the request for the resource to the Principal. The Principal allocates the resource in accordance with a specified resource allocation mechanism [5, 6, 7].

In the theory of active systems, the priority-based resource allocation mechanisms were proposed and investigated [8, 9]. With these mechanisms, the resource is allocated proportionally to the values of the agent's priority functions. Three types of the priority-based mechanisms were identified, namely, the mechanism of absolute priorities, the mechanism of straight priorities and the mechanism of reverse priorities.

In absolute priority mechanisms, resource allocation is proportional to the priorities set by the Principal. These mechanisms are non-manipulated. It means that submit a request that

* Corresponding author: vlab17@bk.ru

reflects the true needs of the resource is beneficial for agents. Their disadvantage is the fact that agents do not actually affect resource allocation.

In the mechanisms of direct priority resources are allocated proportionally to the priorities that grow with the request on the principle of "more you ask – more you get". This principle leads to a trend of increasing of the request. It is significant disadvantage of the principle of direct priorities. However, it is still widely used in practice.

In the mechanisms of reverse priority resource allocation is also proportional to functions of priority. However, these functions are decreasing functions of the request value on the principle of "more you ask – less you get". The principle of reverse priorities encourages resource savings. This is its significant advantage. It is experimentally tested in the allocation of water resources [2].

Of the three described mechanisms of priority, the reverse priority principle is the most preferred.

Previously, the priority-based mechanisms were considered under the assumption that the amount of available resource to be allocated by the Principal is known. However, in many real resource allocation problems arising in practice this amount is often unknown [10, 11]. In this paper, the mechanism of reverse priorities is investigated for the case in when agents know the Principal's resource allocation function.

## 2. PROBLEM STATEMENT

Consider an active system consisting of the Principal and agents. The Principal has a resource to be allocated among the agents in accordance with their requests. While submitting the requests, the agents have information about the Principal's allocation function F(R) only, where R is the amount of available resource to be allocated. The goal function of each agent i is an increasing function of the received resource xi. Denote by Si the request for the resource submitted by agent i. The Principal allocates the resource using the mechanism of reverse priorities.

The problem is to determine the Nash equilibrium situation for three cases. In the first case the amount of resource available to the Principal is known to the agents. In the second case, the agents know the distribution function $F(R)$ of the resource available to the Principal. In the third case, the distribution function of the resource available to the Principal is discrete (more precisely, the resource takes two values with corresponding probabilities).

## 3. MECHANISM OF REVERSE PRIORITIES: DETERMINISTIC CASE

Consider the deterministic case in which the amount of resource available to the Principal is known to the agents.

Define the priority function of agent $i$ by

$$\eta_i(S_i) = \frac{A_i}{S_i},\ i = \overline{1,n},$$

where $A_i$ is a parameter that restricts the agent's priority.

The resource allocation mechanism has the form

$$x_i(S) = \min\left(S_i, \frac{A_i \cdot R}{S_i \cdot Y}\right), \quad (3.1)$$

where $Y = \sum_j \frac{A_j}{S_j}$.

Assume the goal functions of the agents are increasing in $x_i$.

The deterministic statement of the problem yields the Nash equilibrium

$$Y = \frac{1}{R}\left(\sum_j \sqrt{A_j}\right)^2,$$

$$S_i = \frac{\sqrt{A_i} \cdot R}{\sum_j \sqrt{A_j}}.$$

In this case, $x_i = S_i$ for all $i = \overline{1,n}$.

Now, consider another modification of the priority function given by

$$\eta_i(S_i) = A_i - S_i, \ i = \overline{1,n}.$$

First, analyze the deterministic case:

$$x_i(S) = \min\left(S_i, \frac{A_i - S_i}{Y} R\right),$$

where $Y = \sum_j (A_j - S_j)$.

In the Nash equilibrium,

$$S_i = \frac{A_i - S_i}{Y} R,$$

which gives

$$S_i = \frac{RA_i}{Y + R}$$

and

$$A_i - S_i = \frac{A_i Y}{Y + R}.$$

From the condition

$$Y = \sum_j (a_j - s_j) = \frac{Y}{R + Y} \sum_j A_j$$

it follows that

$$Y^* = A - R,$$

$$S_i^* = \frac{A_i}{\sum_j A_j} R.$$

The Nash equilibrium exists only if the condition A > R is satisfied.

## 4. MECHANISM OF REVERSE PRIORITIES: PROBABILISTIC CASE

Consider the probabilistic case in which the agents choose their requests for the resource using the information about the distribution *F(R)* of the available amount *R* only.

Consider the priority function

$$\eta_i(S_i) = \frac{A_i}{S_i}, \ i = \overline{1,n},$$

and the resource allocation mechanism

$$x_i(S) = \min\left(S_i, \frac{A_i \cdot R}{S_i \cdot Y}\right).$$

Denote by *F(R)* a distribution function of the available amount of the resource and let it be continuously differentiable with respect to *R.*

The expected amount of resource requested by agent *i* is

$$M(S)=\int_0^{R_i}\frac{A_iR}{S_iY}dF(R)+S_i\left[1-F(R_i)\right],$$

where $R_i=\frac{S_i^2\cdot Y}{A_i}$, $i=\overline{1,n}$.

Find the maximum of this value over $S_i$ under the hypothesis of weak contagion (the influence of $S_i$ on Y is small). In other words, while choosing their requests for the resource, the agents neglect the influence of $S_i$ on Y, considering Y to be just a parameter.
Note that

$$\int_0^{R_i}RdF(R)=R_iF(R_i)-\int_0^{R_i}F(R)dR.$$

Calculate

$$\frac{dM}{dS_i}=-\frac{A_i}{S_i^2Y}\left[R_iF(R_i)-\int_0^{R_i}F(R)dR\right]+$$

$$+\frac{A_i}{S_iY}R_iF'(R_i)\frac{dR_i}{dS_i}+1-F(R_i)-S_iF'(R_i)\cdot\frac{dR_i}{dS_i}.$$

A series of trivial transformations finally give

$$\frac{dM}{dS_i}=1-2F(R_i)+\frac{1}{R_i}\int_0^{R_i}F(R)dR.$$

The resulting expression is independent of $A_i$.

Assume the function $M(S)$ is convex and therefore has a maximum point. The equilibrium value $R_i$ is the same for all agents, i.e., $R_i=R^*$ for all $i$.

The maximum point $R^*$ can be determined from the first-order optimality condition

$$2F\left(R^*\right)-\frac{1}{R_i}\int_0^{R^*}F(R)dR=1,$$

where

$$S_i^*=\sqrt{\frac{A_i\cdot R^*}{Y}}.$$

Further, calculate

$$Y^*=\sum_j\frac{A_j}{S_j^*}=\sqrt{\frac{Y}{R^*}}\sum_j\sqrt{A_j},$$

$$Y^*=\sqrt{\frac{\sum_j\sqrt{A_j}}{R^*}}.$$

Finally, the Nash equilibrium is

$$S_i^*=\frac{\sqrt{A_i}}{\sum_j\sqrt{A_i}}\cdot R^*.$$

Now, consider the priority function

$$\eta_i(S_i)=A_i-S_i,\ i=\overline{1,n}$$

and the resource allocation mechanism

$$x_i(S)=\min\left(S_i,\frac{A_i-S_i}{Y}R\right).$$

By analogy with the previous case,

$$M(S_i)=\int_0^{R_i}\frac{(A_i-S_i)}{Y}RdF(R)+S_i(1-F(R_i)),$$

where $R_i=\frac{S_iY}{A_i-S_i}$, $\frac{dR}{dS_i}=\frac{YA_i}{(A_i-S_i)^2}$.

Calculate

$$\frac{dM}{dS_i}=-\frac{1}{Y}\left[R_iF(R_i)-\int_0^{R_i}F(R)dR\right]+1-F(R_i)=$$

$$1-\left(1+\frac{R_i}{Y}\right)F(R_i)+\frac{1}{Y}\int_0^{R_i}F(R)dR,$$

$$\frac{d^2M}{dR}=-\frac{1}{Y}F(R_i)-\left(1+\frac{R_i}{Y}\right)F'(R_i)+\frac{1}{Y}F(R_i)=$$

$$-\left(1+\frac{R_i}{Y}\right)F'(R_i)<0.$$

Hence, the function $M(S_i)$ is concave, and the maximum point can be determined from the first-order optimality condition (the same for all agents)

$$(R+Y)F(R)-\int_0^R F(x)dx=Y.$$

## 5. MECHANISM OF REVERSE PRIORITIES: DISCRETE CASE

Suppose the available resource $R$ is $Q_1$ with a probability $p_1$ and $Q_2$ with the probability $p_2=1-p_1$.

Adopt the mechanism of reverse priorities with the priority function

$$\eta_i(S_i)=\frac{A_i}{S_i},\ i=\overline{1,n}.$$

Establish how the expected amount of the agent's resource depends his/her request (for simplicity, the agent's number $i$ is omitted).

Calculate $d=\sqrt{\frac{AQ_1}{Y}}$ and $D=\sqrt{\frac{AQ_2}{Y}}$.

Then three situations are possible as follows:

1) $S\le d$. *In this case , with the mechanism of reverse priorities the minimum will be achieved on the request, if* $R=Q_1$ *and if* $R=Q_2$. *Therefore,* $M(S)=S$.

2) $d<S\le D$. In this case, the minimum point $x=\frac{AQ_1}{S\cdot Y}$ will be achieved if $R=Q_1$, and the minimum point $x=S$ will be achieved if $R=Q_2$. The expected amount of the resource will be $M(S)=p_1\frac{AQ_1}{S\cdot Y}+p_2S$.

3) $S\ge D$. In this case, the minimum point $x=\frac{AQ_1}{S\cdot Y}$ will be achieved if $R=Q_1$, and the minimum point $x=\frac{AQ_2}{S\cdot Y}$ will be achieved if $R=Q_2$. The expected amount of the resource will be $M(S)=(p_1Q_1+p_2Q_2)\cdot\frac{A}{Y\cdot S}=\frac{A}{Y\cdot S}R$.

The graph of the function $M(S)$ can be seen in Fig 4.1.

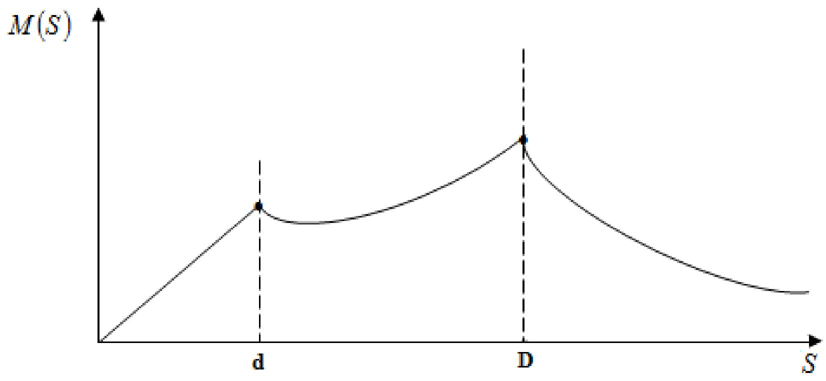


**Fig. 4.1** Graph of function $M(S)$.

On the interval $[d,D]$, $M(S)$ is a convex function of *S*. Therefore, it achieves a maximum at the points d or D. Note that, if $S=d_1$, then $M=d$; with $Q=Q_1$ or $Q=Q_2$, the agent will receive the same amount $x=d$.

If $S=D$, then $M=\frac{AQ_1}{D\cdot Y}p_1+p_2D$.

The maximum of M is given by

$$\max\left[d;\,p_1\frac{Q_1}{D\cdot Y}+p_2D\right]=\sqrt{\frac{A}{Y}}\max\left[\sqrt{Q_1};\frac{Q_1}{\sqrt{Q_2}}p_1+\sqrt{Q_2}\,p_2\right]$$

If the maximum is achieved on $\sqrt{Q_1}$, the agent will choose the strategy *d*; if on $p_1\frac{Q_1}{\sqrt{Q_2}}+p_2\sqrt{Q_2}$ , then the strategy *D*.

Define $p_1^*$ from the equation

$$\sqrt{Q_1}=\frac{Q_1}{\sqrt{Q_2}}p_1+\sqrt{Q_2}(1-p_2).$$

After simple calculations,

$$p_1^*=\frac{\sqrt{Q_2}}{\sqrt{Q_1}+\sqrt{Q_2}}.$$

So, if $p_1>p^*$, then each agent will choose the strategy *d*; if $p_1<p^*$, then the strategy *D*.

In both cases, in the Nash equilibrium the resources will be allocated as follows:

$$x_i=\frac{\sqrt{A_i}}{B}Q\text{, where } B=\sum_i\sqrt{A_i}.$$

Next, consider the priority function

$$\eta_i(S_i)=A_i-S_i,\ i=\overline{1,n}.$$

By analogy with the previous analysis,

$$x=\min\left(S;\frac{(A-S)Q}{Y}\right)\text{, where } Y=\sum_j(A_j-S_j).$$

Using the equation $S=\frac{(A-S)Q}{Y}$, find

$$d = \frac{AQ_1}{Y + Q_1} \text{ and } D = \frac{AQ_2}{Y + Q_2}.$$

If $S \le d$, then $M[x] = d$.
If $d < S \le D$, then

$$M[x] = p_1 \frac{(A - S)Q_1}{Y} + p_2 S = R\frac{A}{Y} - S\left(\frac{p_1 Q_1}{Y} - p_2\right).$$

If $S \ge D$, then $M[x] = \frac{(A - S)}{Y}(p_1 Q_1 + p_2 Q_2)$.

Obviously, the maximum of $M[x]$ is achieved either at the point *d* or at the point *D*.
Hence,

$$M[x] = \left[d; p_1 \frac{(A - D)Q_1}{Y} + p_2 D\right] =$$

$$= A\max\left[\frac{Q_1}{Q_1 + Y}; \frac{Q_1}{Q_2 + Y} p_1 + \frac{Q_2}{Q_2 + Y} p_2\right].$$

The boundary value $p_1^*$ can be defined from the equation

$$\frac{Q_1}{Q_1 + Y} = \frac{Q_1}{Q_2 + Y} p_1 + \frac{Q_2}{Q_2 + Y} p_2.$$

Trivial calculations yield

$$p_1^* = \frac{Y}{Q_1 + Y}.$$

In contrast to the previous case, the value $p_1^*$ depends on Y.
If $S_j = d_j$, then

$$Y = \sum_j (A_j - d_j) = B,$$

$$Y = B - Q_1,\ p_1^* = 1 - \frac{Q_1}{B}.$$

If $S_i = D_i$, $i = \overline{1, n}$, then

$$p_1^* = 1 - \frac{Q_2}{B}.$$

Again, three cases are possible as follows.

1) $p > 1 - \frac{Q_1}{B}$. In this case, all agents will choose the strategy $d_i$, $i = \overline{1, n}$.

2) $p < 1 - \frac{Q_2}{B}$. In this case, all agent will choose the strategy $D_i$, $i = \overline{1, n}$.

3) $1 - \frac{Q_2}{B} < p < 1 - \frac{Q_1}{B}$.

This situation has uncertainty. If all agents choose the strategy $d_i$, then $p_1^* = 1 - \frac{Q_1}{B}$; therefore, $p < p_1^*$ and they will benefit more from the strategy $D_i$.

If they choose the strategy $D_i$, then $p_1^* = 1 - \frac{Q_2}{B}$; therefore, $p > p_1^*$ and the agent will benefit more from the strategy $d_i$. The situation is difficult to predict.

 

## 6. CONCLUSIONS

In this paper, a study of the mechanism of inverse priorities for resource allocation in the deterministic and probabilistic case has been presented. For different priority functions the Nash equilibrium has been obtained under the assumption that the goal functions of the agents are monotonically increasing in their amounts of resource.

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