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# Models of Uncertain-Random Programming

Georgy Veresnikov[1*], Ludmila Pankova[1], Valeriya Pronina[1]

[1)] *V.A. Trapeznikov Institute of Control Sciences of Russian Academy of Scienses*

*Moscow, Russian*

*E-mail: veresnikov@mail.ru*

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**Abstract**: The article proposes the models of optimization with constraints under conditions of parametric mixed uncertainty – aleatory and epistemic. We model parameters with aleatory uncertainty by random values with probability distribution functions obtained from statistical data. We model parameters with epistemic uncertainty by uncertain values introduced in the uncertainty theory of Liu B. Experts define the uncertainty distribution functions. We model a function of random and uncertain parameters by uncertain-random value, interpreted as epistemic value parameterized by random values. Optimization criteria (deterministic duplicates of objective functions) are combination of different characteristics of random and uncertain values, which allows both to average objective functions and to take into account risks or reliability arising from the variability of random and uncertain values. Using the proposed models of uncertain-random programming, we formalized as a two-criterion optimization problem with constraints and solved the task of preliminary aerodynamic design in the conditions of parametric mixed uncertainty – calculation of aircraft weight parameters. The uncertainty theory makes possible under certain conditions (for sufficiently wide class of functions) to obtain analytical expressions for characteristics of uncertain functions, that significantly reduces computational costs. To calculate weight parameters of aircraft, we use multicriteria genetic algorithm and statistical modeling. We investigate the dependence of the optimization result on the given probability levels for random values and the expert belief degree for epistemic values reflecting the reliability of the obtained solution.

***Keywords***: aleatory uncertainty, epistemic uncertainty, uncertain-random quantity, deterministic duplicate, uncertain-random programming, mixed uncertainty.

## 1. INTRODUCTION

Aleatory (objective) and epistemic (subjective) uncertainties are two types of uncertainty, reflecting nondeterminism. In the context of modeling technical objects and decision making aleatory uncertainty occurs when information about a stochastic parameter is accumulated in statistical data and parameters are modeled by random variables with certain distributions. Epistemic uncertainty arises when information about a parameter is obtained from experts, while the parameter may be either stochastic, but there are no or insufficient statistical data, or deterministic, but its value is unknown to date. The parameters with epistemic uncertainty are modeled by fuzzy, possibilistic, uncertain [1], and others values.

Decision making in the design of technical objects, as a rule, occurs under conditions of mixed uncertainty, when there are parameters both aleatory and epistemic. The existing methods and design tools do not take into account the presence of parameters with epistemic uncertainty. However if we consider the epistemic parameters as random variables, it can lead to errors, which is associated with the nonadditivity of expert judgments (the measure of uncertainty is nonadditive in contrast to the probabilistic measure).

[*] Corresponding author: veresnikov@mail.ru

It is important to note that in construction of optimization models in conditions of uncertainty, transition from nondeterministic objective functions and restrictions to their deterministic duplicates is necessary. Deterministic duplicates of objective functions are their numerical characteristics (mean, variance, quantile, etc.).

Let $f(\bar{\xi},\bar{\varpi})$ be a function of epistemic values $\bar{\xi}$ and aleatory values $\bar{\varpi}$, i.e. a value with mixed uncertainty. There are two known approaches to modeling a value with mixed uncertainty and its characteristics (deterministic duplicates) based on different interpretations of a value with mixed uncertainty [1-4].

The first approach considers $f(\bar{\xi},\bar{\varpi})$ as random value parameterized by epistemic values. We determine the characteristic of function $f(\bar{\xi},\bar{\varpi})$ in two stages. First, we calculate the numerical characteristic of the function as a random quantity for each implementation of epistemic quantities. This characteristic does not depend on random quantities and, as a function of epistemic quantities, is an epistemic quantity. Then we calculate the characteristic of this epistemic quantity, which is the characteristic of the function with mixed uncertainty.

The second approach considers $f(\bar{\xi},\bar{\varpi})$ as epistemic value parameterized by random values. We determine the characteristic of function $f(\bar{\xi},\bar{\varpi})$ in two stages. First, we calculate the characteristic of the function as an epistemic quantity for each implementation of random quantities. This characteristic does not depend on epistemic quantities and, as a function of random quantities, is a random quantity. Then we calculate the numerical characteristic of this random quantity, which is the characteristic of the function with mixed uncertainty.

Calculating characteristics of mixed uncertainty values is very expensive, especially in absence of explicit formulas for epistemic and/or random characteristics [2-4]. In this regard, it is relevant to highlight conditions under which there are analytical expressions of characteristics.

Thus, in [4] in framework of the first approach, the authors investigate values with mixed uncertainty, where possibilistic values model epistemic values, while possibilistic-random values have a shift-scale representation: $f(\xi, \omega) = \alpha(\omega) + \sigma(\omega) \cdot \beta(\xi)$, where $\alpha(\omega)$ and $\sigma(\omega)$ are random values, $\beta(\xi)$ is a possibilistic value. The authors obtain formulas for calculating the characteristics of the weighted sum of the possibilistic-random values of the shift-scale form with a certain type distribution for the possibilistic and random values.

Uncertainty theory makes possible for wider class of functions to obtain analytical expressions for characteristics of uncertain functions. That significantly reduces computational costs. In this paper, we model epistemic values by uncertain values introduced in theory of uncertainty [1]. Further, we will call uncertain only such values.

Chance theory [1] introduces uncertain-random value, where uncertain values model epistemic values. The chance is a measure of mixed uncertainty. Uncertain-random value is a real function on uncertain-random space with chance measure and have distribution function of chance measure (Definitions 10-12, Appendix). Characteristics of uncertain-random values, defined as the characteristics of the chance measure distribution function (for example, Definition 13, Appendix), will always be averaged epistemic characteristics, since chance measure is the mathematical expectation of uncertain measure (Definition 1, Appendix). Characteristics are actually determined as in the second approach, where at the first stage the epistemic characteristic is calculated, at the second stage the mathematical expectation is always calculated [1]. Thus, use of chance theory in solving optimization problems leads to models with averaging criteria, i.e. the solution will be effective only “on average”, while risk of unwanted solutions is not considered. When modeling constraints using a chance measure, fulfillment of constraints is required only “on average” and risk of not fulfilling the constraints is not considered [1].

The paper proposes the models of uncertain-random programming, that is, the optimization models with constraints under conditions of parametric mixed uncertainty within the framework of the second approach, which makes it possible to use analytical expressions for sufficiently wide class of functions of uncertain values. The choice of duplicates of objective functions and constraints allows to take into account risk and reliability requirements.

Section 2 describes the models of uncertain-random programming. In Section 3 the task of calculating the weight parameters of an aircraft under conditions of parametric mixed uncertainty is formalized as a multicriteria optimization problem with constraints. We use two proposed models: with averages and with quantiles. The method for solving the task of calculating weight parameters and the results of calculations are given. The appendix contains definitions and statements from the theory of uncertainty and chance theory.

## 2. MODELS OF UNCERTAIN-RANDOM PROGRAMMING

Let $f(\bar{x},\bar{\xi},\bar{\varpi})$ be the objective function, $\bar{x}$ is solution vector, $\xi_i$, $i = 1, \ldots, n$, are uncertain values, $\omega_j$, $j = 1, \ldots, m$, are random values that is, $f(\bar{x},\bar{\xi},\bar{\varpi})$ is uncertain-random value for each $\bar{x}$. We interpret $f(\bar{x},\bar{\xi},\bar{\varpi})$ as the uncertain value parameterized by the random values in the framework of second approach. We determine the characteristic of the function $f(\bar{x},\bar{\xi},\bar{\varpi})$ in two stages. First, we calculate the characteristic of uncertain function $f(\bar{x},\bar{\xi},\bar{\varpi})$, where random values are parameters. Then, we calculate the characteristic of this random value, which is characteristic of the function $f(\bar{x},\bar{\xi},\bar{\varpi})$ with mixed uncertainty.

Mathematical expectation/expected value, quantile (Definition 9, Appendix), variance, probability/ belief degree of non-exceedance of a given value by the function can be chosen as characteristics of random and uncertain values. Using combination of characteristics, you can build different optimization models. The choice of the optimization criterion depends on the specific task, reliability requirements and is the prerogative of the decision maker.

The optimization model with mathematical expectation/expected value criteria, averaging the objective function, gives an effective solution “on average”, while risk or reliability is not taken into account. In robust optimization, when it is necessary to ensure the least variability of the objective function, the variance is used. In tasks of optimal control and design of aircraft under conditions of aleatory uncertainty, quantile and the probability of not exceeding (while minimizing) the objective function of a given threshold value have become widespread, since they are aimed at making optimal decisions based on risk or reliability requirements.

We consider some models of uncertain-random programming and reduce them to mathematical or stochastic programming models.

Let $f(\bar{x},\bar{\xi},\bar{\varpi})$ be the objective function, $\bar{x}$ be the solution vector, $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain values with uncertainty distribution functions $\Phi_1, \Phi_2, \ldots, \Phi_n$ , having inverse distribution functions, $\omega_1, \ldots, \omega_m$ be independent random values with probability distribution functions $\Psi_1, \Psi_2, \ldots, \Psi_m$. Let $f(\bar{x},\bar{\xi},\bar{\varpi})$ be a continuous strictly increasing function with respect to $\xi_1, \ldots, \xi_k$ and a strictly decreasing function with respect to $\xi_{k+1}, \ldots, \xi_n$.

For the model 1 (*EE*), we will select expected value $E^M$ as characteristic at first stage, and mathematical expectation as characteristic at second stage. Then the model 1 is:

$$\min_{\bar{x}} E^P(E^M(f(\bar{x},\bar{\xi},\bar{\varpi}))).$$

Expected value of function $f(\bar{x},\bar{\xi},\bar{\varpi})$ where random values $\bar{\varpi}$ are parameters is (Theorem 2, Appendix):

$$E^M(f(\bar{x},\bar{\xi},\bar{\varpi})) = \int_0^1 f(\bar{x},\Phi_1^{-1}(\alpha),\ldots,\Phi_k^{-1}(\alpha),\Phi_{k+1}^{-1}(1-\alpha),\ldots,\Phi_n^{-1}(1-\alpha),\bar{\varpi})d\alpha \,. \quad (1)$$

Mathematical expectation of random value $E^M(f(\bar{x},\bar{\xi},\bar{\omega}))$ is:

$$E^P(E^M(f(\bar{x},\bar{\xi},\bar{\omega}))) =$$
$$\int_{R^m}\int_0^1 f(\bar{x},\Phi_1^{-1}(\alpha),...,\Phi_k^{-1}(\alpha),\Phi_{k+1}^{-1}(1-\alpha),...,\Phi_n^{-1}(1-\alpha),\bar{\omega})d\alpha d\psi_1...d\psi_m.$$

The model 1 has the form:

$$\min_{\bar{x}} \int_{R^m}\int_0^1 f(\bar{x},\Phi_1^{-1}(\alpha),...,\Phi_k^{-1}(\alpha),\Phi_{k+1}^{-1}(1-\alpha),...,\Phi_n^{-1}(1-\alpha),\bar{\omega})d\alpha d\psi_1...d\psi_m.$$

Thus, we have reduced the model 1 to a deterministic model of mathematical programming.

For the model 2 (*QE*), we use expected value as characteristic at first stage, and quantile of the random variable as characteristic at second stage. Then the model 2 has the form:

$$\min_{\bar{x}} r,$$
where
$$P(E^M(f(\bar{x},\bar{\xi},\bar{\omega})) \le r) \ge \beta,\ 0.5 \le \beta \le 1,$$

or (according to the formula (1)):

$$\min_{\bar{x}} r,$$
where
$$P((\int_0^1 f(\bar{x},\Phi_1^{-1}(\alpha),...,\Phi_k^{-1}(\alpha),\Phi_{k+1}^{-1}(1-\alpha),...,\Phi_n^{-1}(1-\alpha),\bar{\omega})d\alpha) \le r) \ge \beta,\ 0 \le \beta \le 1.$$

Thus, we reduce the model 2 to stochastic model with quantile criterion.

For the model 3 (*EQ*), we use quantile as characteristic at first stage, and mathematical expectation as characteristic at second stage. Then the model 3 has the form:

$$\min_{\bar{x}} E^P(\inf_{\alpha}(f(\bar{x},\bar{\xi},\bar{\omega}))),$$

where in accordance with Theorem 2 (d) of the Appendix:

$$inf_{\alpha}(f(\bar{x},\bar{\xi},\bar{\omega})) = f(\bar{x},\Phi_1^{-1}(\alpha),...,\Phi_k^{-1}(\alpha),\Phi_{k+1}^{-1}(1-\alpha),...,\Phi_n^{-1}(1-\alpha),\bar{\omega}). \quad (2)$$

Thus, we have reduced the model 3 to a deterministic model of mathematical programming.

For the model 4 (*QQ*), we use quantile of uncertain value as characteristic at first stage, and quantile of random value as characteristic at second stage. Then the model 4 has the form:

$$\min_{\bar{x}} r,$$
where
$$P\ (inf_{\alpha}(\ f(\bar{x},\bar{\xi},\bar{\omega})) \le r\} \ge \beta,\ 0 \le \beta \le 1.$$

Thus, we reduce the model 4 to stochastic model with quantile criterion (formula (2)).

Consider representation of duplicate for constraint $g(\bar{x},\bar{\xi},\bar{\omega}) \le 0$, where $g$ is an uncertain-random value, and $\bar{x}$ is the solution vector, $\xi_1, \xi_2, ..., \xi_n$ are independent uncertain values with uncertainty distribution functions $\Phi_1, \Phi_2, ..., \Phi_n$, having inverse functions, $\omega_1, ..., \omega_m$ are independent random values with probability distribution functions $\Psi_1, \Psi_2, ..., \Psi_m$.

 

In the case of hard constraint, the constraint must be satisfied for any implementation of random and uncertain values, and duplicate value has form:

$$\max_{\bar{\xi},\bar{\varpi}} g(\bar{x},\bar{\xi},\bar{\varpi}) \le 0 \ .$$

In the case of soft constraint, the constraint performs with guaranteed belief degree and probability.

We will build duplicate of soft constraint. We interpret $g(\bar{x},\bar{\xi},\bar{\varpi})$ as uncertain value parameterized by random values. Then in the first stage, where random variables are parameters, the duplicate of constraint $g(\bar{x},\bar{\xi},\bar{\varpi}) \le 0$ has the form: $M(g(\bar{x},\bar{\xi},\bar{\varpi}) \le 0) \ge \alpha$ , where $M$ is the degree of expert confidence (see Appendix), $\alpha$ is given by the designer. If $g(\bar{x},\bar{\xi},\bar{\varpi})$ is continuous strictly increasing function with respect to $\xi_1$, …, $\xi_k$ and a strictly decreasing function with respect to $\xi_{k+1}$,…, $\xi_n$, then the inequality $M(g(\bar{x},\bar{\xi},\bar{\varpi}) \le 0) \ge \alpha$ is equivalent to the inequality (theorem 2, Appendix):

$$g(\bar{x},\Phi_1^{-1}(\alpha),...,\Phi_k^{-1}(\alpha),\Phi_{k+1}^{-1}(1-\alpha),...,\Phi_n^{-1}(1-\alpha),\bar{\varpi}) \le 0 \, .$$

At the second stage, we replace this inequality containing random values by the probability that this inequality is satisfied:

$$P(g(\bar{x},\Phi_1^{-1}(\alpha),...,\Phi_k^{-1}(\alpha),\Phi_{k+1}^{-1}(1-\alpha),...,\Phi_n^{-1}(1-\alpha),\bar{\varpi}) \le 0) \ge \beta,$$

where $\beta$ is the confidence level of probability specified by the designer.

Thus, we reduced constraint with uncertain-random value to probabilistic constraint.

Reducing the model of uncertain-random programming to a deterministic model of mathematical programming greatly simplifies the solution. Reducing model of uncertain-random programming to model of stochastic programming simplifies solution, but in general case they remain quite complicated. This is due to complexity of finding the analytical form of probabilistic criterion and constraint, and, in absence of one, complexity of numerical solution methods. In particular cases, deterministic equivalents of probabilistic criteria and constraints can be found [5, 6].

## 3. CALCULATION OF AIRCRAFT WEIGHT PARAMETERS

### *3.1. Models*

We formalize the one of the tasks of calculating weight parameters of passenger aircraft as optimization problem with constraints under conditions of mixed uncertainty based on the method of calculating the weight report of passenger aircraft [7]. In the original form, the functions for calculating the basic weight parameters of aircraft at the preliminary design stage are:

$$M_{fuel} = M_0 - M_{airframe} - M_{ch} - M_{pp} - M_{ec} - M_{rae} - M_{payload} - M_{pe}$$

$$M_0 = \gamma_a (V_a - V_{pc}) + M_{payload} + M_{pe}$$

$$M_{airframe} = (1+K_m) q_s A_{wet}$$

$$q_s = k_{qs0} + (k_{qs1} + \lg V_a) \lg V_a$$

$$M_{ch} = 0.3 M_{airframe} K_{eq}$$

$$M_{pp} = \gamma_{eng} T (k_{eng\,0} + k_{eng\,1} \frac{V_a}{A_{wet}})$$

where $M_0$ is takeoff mass; $M_{airframe}$ is airframe mass; $M_{ch}$ is equipment mass for control and hydraulics; $M_{pp}$ is power plant mass; $M_{fuel}$ is fuel mass; $M_{ec}$ is weight of crew and equipment; $M_{rae}$ is mass of board radioelectronic equipment; $M_{payload}$ is commercial load; $M_{pe}$ is passenger equipment; $\gamma_a$ is the average density of aircraft; $V_a$ is aircraft volume; $V_{pc}$ is volume of the passenger compartment; $K_m$ is composites utilization coefficient; $q_s$ is specific weight per square meter of airframe; $A_{wet}$ is area of washed surface of aircraft; $k_{qs0}$ is statistical coefficient; $k_{qs1}$ is statistical coefficient; $K_{eq}$ is coefficient taking into account production technology; $\gamma_{eng}$ is technological coefficient, which shows the ratio of weight of engine to maximum thrust at $H = 0$, $V = 0$ ($H$, $V$ are height and speed, respectively); $T$ is required thrust at $H = 0$, $V = 0$; $k_{eng0}$ and $k_{eng1}$ are statistical coefficients.

To calculate the optimal design parameters we select optimization criteria that affect the fulfillment of specified technical requirements: operating costs and flight range. We minimize takeoff weight to reduce fuel consumption, as the main part of operating costs consists of fuel costs. On the other hand, we maximize the fuel supply, which increases the range of the aircraft.

In this regard, the task of weight calculation is formalized and presented in form of following optimization model:

$$\begin{cases} \min M_0, \max M_{fuel}, \\ 400 \le \gamma_a \le 500, \\ M_0 - 0.8 M_{fuel} \le M_{landing} \end{cases} \tag{1}$$

Based on the study of technical requirements, the designer choices the design parameters. The design parameters are $A_{wet}$, $V_a$, $M_{payload}$, $M_{pe}$, $M_{rae}$. The available information about the parameters determines the separation of the parameters into random and uncertain. Statistics provide information on random parameters. Experts provide information on uncertain parameters. The parameters with epistemic uncertainty are $K_m$, $K_{eq}$, $V_{pc}$, $M_{ec}$, $\gamma_a$. The parameters with aleatory uncertainty are $k_{qs0}$, $k_{qs1}$, $k_{eng0}$, $k_{eng1}$, $\gamma_{eng}$, $T$.

The expression $M_0 - 0.8 M_{fuel} \le M_{landing}$ determines the constraint on landing weight of aircraft, associated with length of run during landing.

Let $\Phi^{-1}_{\gamma_a}$, $\Phi^{-1}_{V_{pc}}$, $\Phi^{-1}_{M_{ec}}$, $\Phi^{-1}_{K_{eq}}$, $\Phi^{-1}_{K_m}$ be inverse uncertainty distribution functions of uncertainty for parameters $\gamma_a$, $V_{pc}$, $M_{ec}$, $K_{eq}$, $K_m$; $\Psi_{k_{qs0}}$, $\Psi_{k_{qs1}}$, $\Psi_{k_{eng0}}$, $\Psi_{k_{eng1}}$, $\Psi_{\gamma_{eng}}$, $\Psi_T$ be probability distribution functions of uncertainty for parameters $k_{qs0}$, $k_{qs1}$, $k_{eng0}$, $k_{eng1}$, $\gamma_{eng}$, $T$.

To solve this problem, as well as to compare the results and study the models, we construct two two-criterion models based on the single-criterion models proposed in section 2. In model *A*, for each criterion, select the "averaging" model (*EE*). In model *B*, for each criterion we choose the "quantile" model (*QQ*).

**Model A:**

$$\begin{cases} \min\ E[M_0],\ \max E[M_{fuel}], \\ M(\gamma_a \le 500) \ge \alpha_{\gamma_a}^{\le}, \\ M(\gamma_a \ge 400) \ge \alpha_{\gamma_a}^{\ge}, \\ P(M(M_0 - 0.8M_{fuel} \le M_{landing}) \ge \alpha_{M_{landing}}) \ge P_{M_{landing}},\ P_{M_{landing}} \in [0,1], \end{cases}$$

where $M$ is measure of uncertainty (expert belief degree), $\alpha_{\gamma_a}^{\le}$, $\alpha_{\gamma_a}^{\ge}$, $\alpha_{M_{landing}}$ are levels of belief degrees that the inequalities in question are satisfied, $P_{M_{landing}}$ is level of probability that belief degree in the implementation of inequality $M_0 - 0.8M_{fuel} \le M_{landing}$ will be greater than $\alpha_{M_{landing}}$.

To solve this task, first the designer sets $\alpha_{\gamma_a}^{\le}$, $\alpha_{\gamma_a}^{\ge}$, $\alpha_{M_{landing}}$, $P_{M_{landing}}$. Then the multi-criteria optimization algorithm is applied. For each combination of design parameters varied in process of optimization, we calculate expected values of objective functions according to the formulas:

$$E[M_0] = \int_0^1 (\Phi_{\gamma_a}^{-1}(\alpha)(V_a - \Phi_{V_{pc}}^{-1}(1-\alpha)) + M_{payload} + M_{pe})d\alpha,$$

$$E[M_{fuel}] = \int_{R^m} (\int_0^1 (M_0' - M_{airframe} - M_{ch} - M_{pp} - \Phi_{M_{ec}}^{-1}(1-\alpha) -$$

$$- M_{rae} - M_{payload} - M_{pe})d\alpha)d\psi_{k_{qs0}} d\psi_{k_{qs1}} d\psi_{k_{eng0}} d\psi_{k_{eng1}} d\psi_{\gamma_{eng}} d\psi_T,$$

$$M_0' = \Phi_{\gamma_a}^{-1}(\alpha)(V_a - \Phi_{V_{pc}}^{-1}(1-\alpha)) + M_{payload} + M_{pe};$$

$$M_{ch} = 0.3M_{airframe}\Phi_{K_{eq}}^{-1}(1-\alpha);$$

$$M_{airframe} = (1 + \Phi_{K_m}^{-1}(1-\alpha))q_s A_{wet}.$$

**Model B:**

$$\begin{cases} \min\ inf_{\alpha_{M_0}}[M_0],\ \max r, \\ P(\sup_{\alpha_{M_{fuel}}}[M_{fuel}] \ge r) \ge P_{M_{fuel}},\ P_{M_{fuel}} \in [0,1], \\ M(\gamma_a \le 500) \ge \alpha_{\gamma_a}^{\le}, \\ M(\gamma_a \ge 400) \ge \alpha_{\gamma_a}^{\ge}, \\ P(M(M_0 - 0.8M_{fuel} \le M_{landing}) \ge \alpha_{M_{landing}}) \ge P_{M_{landing}},\ P_{M_{landing}} \in [0,1], \end{cases}$$

where $P_{M_{fuel}}$ is level of probability that $\sup_{\alpha_{M_{fuel}}}[M_{fuel}] \ge r$, $\alpha_{M_{fuel}}$ is level of expert belief degree for quantile criterion.

As the objective functions are used:

$$inf_{\alpha_{M_0}}[M_0] = \Phi_{\gamma_a}^{-1}(\alpha_{M_0})(V_a - \Phi_{V_{pc}}^{-1}(1-\alpha_{M_0})) + M_{payload} + M_{pe},$$

$$r,$$

subject to

$$P(M_0^{'} - M_{airframe} - M_{ch} - M_{pp} - \Phi_{M_{ec}}^{-1}(\alpha_{M_{fuel}}) -$$

$$- M_{rae} - M_{payload} - M_{pe} \geq r) \geq P_{M_{fuel}};$$

$$M_0^{'} = \Phi_{\gamma_a}^{-1}(1-\alpha_{M_{fuel}})(V_a - \Phi_{V_{pc}}^{-1}(\alpha_{M_{fuel}})) + M_{payload} + M_{pe};$$

$$M_{ch} = 0.3 M_{airframe} \Phi_{K_{eq}}^{-1}(\alpha_{M_{fuel}});$$

$$M_{airframe} = (1+\Phi_{K_m}^{-1}(\alpha_{M_{fuel}}))q_s A_{wet}.$$

Note that the first criterion does not depend on random variables and is deterministic.

We account for constraints for both models as follows. The constraint $M(\gamma_a \leq 500) \geq \alpha_{\gamma_a}^{\leq}$ is equivalent to $\Phi_{\gamma_a}^{-1}(\alpha_{\gamma_a}^{\leq}) \leq 500$. The constraint $M(\gamma_a \geq 400) \geq \alpha_{\gamma_a}^{\geq}$ is equivalent to $\Phi_{\gamma_a}^{-1}(1-\alpha_{\gamma_a}^{\geq}) \geq 400$.

To check constraint on landing weight of aircraft, we calculate probability that belief degree in constraint performing will be greater than the specified level:

$$P(M(M_0 - 0.8 M_{fuel} \leq M_{landing}) \geq \alpha_{M_{landing}}) \geq P_{M_{landing}}.$$

This inequality is equivalent to inequality (Theorem 2d, Appendix):

$$P(M_0 - 0.8 M_{fuel} - M_{landing} \leq 0) \geq P_{M_{landing}},$$

$$M_0 = \Phi_{\gamma_a}^{-1}(\alpha_{M_{landing}})(V_a - \Phi_{V_{pc}}^{-1}(1-\alpha_{M_{landing}})) + M_{payload} + M_{pe},$$

$$M_{fuel} = M_0 - M_{airframe} - M_{ch} - M_{pp} - \Phi_{M_{ec}}^{-1}(\alpha_{M_{landing}}) - M_{rae} - M_{payload} - M_{pe},$$

$$M_{ch} = 0.3 M_{airframe} \Phi_{K_{eq}}^{-1}(\alpha_{M_{landing}}),$$

$$M_{airframe} = (1+\Phi_{K_m}^{-1}(\alpha_{M_{landing}}))q_s A_{wet}.$$

To calculate weight parameters of aircraft, we use multicriteria genetic algorithm and statistical modeling.

### *3.2. Methods and results*

We apply a genetic algorithm of multicriteria optimization with values $\alpha_{\gamma_a}^{\leq}$, $\alpha_{\gamma_a}^{\geq}$, $\alpha_{M_{sb}}$, $P_{M_{sb}}$ given by the designer. We calculate the probabilistic characteristic of uncertain-random values based on the Monte Carlo statistical modeling method. As an example, we present the calculation algorithm for model A presented in Section 3.1. For each combination of variable design parameters, we calculate the values of objective functions using following formulas:

$$E[M_0] = \int_0^1 (\Phi_{\gamma_a}^{-1}(\alpha)(V_a - \Phi_{V_{pc}}^{-1}(1-\alpha)) + M_{payload} + M_{pe}) d\alpha,$$

$$E[M_{fuel}] = \frac{1}{N}\sum_{i=1}^{N}\int_0^1 (M_0^{'} - M_{airframe} - M_{ch} - M_{pp} - \Phi_{M_{ec}}^{-1}(1-\alpha) -$$

$$- M_{rae} - M_{payload} - M_{pe}) d\alpha,$$

$$M_0^{'} = \Phi_{\gamma_a}^{-1}(\alpha)(V_a - \Phi_{V_{pc}}^{-1}(1-\alpha)) + M_{payload} + M_{pe},$$

$$M_{ch} = 0.3 M_{airframe} \Phi_{K_{eq}}^{-1}(1-\alpha),$$

$$M_{airframe} = (1+\Phi_{K_m}^{-1}(1-\alpha))q_s A_{wet},$$

$$q_s = \{k_{qs0}\}_i + (\{k_{qs1}\}_i + \lg V_a)\lg V_a,$$

$$M_{pp} = \{\gamma_{eng}\}_i \{T\}_i (\{k_{eng0}\}_i + \{k_{eng1}\}_i \frac{V_a}{A_{wet}}).$$

where $\Phi^{-1}_{\gamma_a}$, $\Phi^{-1}_{V_{pc}}$, $\Phi^{-1}_{M_{ec}}$, $\Phi^{-1}_{K_{eq}}$, $\Phi^{-1}_{K_m}$ are inverse uncertainty distributions for the parameters $\gamma_a$, $V_{pc}$, $M_{ec}$, $K_{eq}$, $K_m$; $N$ is number of value combinations for parameter $k_{qs0}$, $k_{qs1}$, $k_{eng0}$, $k_{eng1}$, $\gamma_{eng}$, $T$ while we form the values of the parameters in accordance with the specified probability distributions; $\{\bullet\}_i$ is the $i$-th element of value set for parameter $\{\bullet\}$ formed by statistical modeling.

The constraint $M(\gamma_a \le 500) \ge \alpha^{\le}_{\gamma_a}$ is equivalent to $\Phi^{-1}_{\gamma_a}(\alpha^{\le}_{\gamma_a}) \le 500$. The constraint $M(\gamma_a \ge 400) \ge \alpha^{\ge}_{\gamma_a}$ is equivalent to $\Phi^{-1}_{\gamma_a}(1-\alpha^{\ge}_{\gamma_a}) \ge 400$. To check the restrictions on the landing weight, an approximate probability value is calculated:

$$P(M(M_0 - 0.8M_{fuel} \le M_{landing}) \ge \alpha_{M_{landing}}) = \frac{1}{N}\sum_{i=1}^{N} I[M_0 - 0.8M_{fuel}],$$

where

$$I[M_0 - 0.8M_{fuel}] = \begin{cases} 1, M_0 - 0.8M_{fuel} \le M_{landing} \\ 0, \text{иначе} \end{cases},$$

$$M_0 = \Phi^{-1}_{\gamma_a}(\alpha_{M_{landing}})(V_a - \Phi^{-1}_{V_{pc}}(1-\alpha_{M_{landing}})) + M_{payload} + M_{pe},$$

$$M_{fuel} = M_0 - M_{airframe} - M_{ch} - M_{pp} - \Phi^{-1}_{M_{ec}}(\alpha_{M_{landing}}) - M_{rae} - M_{payload} - M_{pe},$$

$$M_{ch} = 0.3M_{airframe}\Phi^{-1}_{K_{eq}}(\alpha_{M_{landing}}),$$

$$M_{airframe} = (1 + \Phi^{-1}_{K_m}(\alpha_{M_{landing}}))q_s A_{wet},$$

The constraint is satisfied subject to

$$P(M(M_0 - 0.5M_{fuel} \le M_{landing}) \ge \alpha_{M_{landing}}) \ge P_{M_{landing}}.$$

Data on uncertain and random parameters obtained from experts and statistics collection are approximated by uncertainty and probability distribution functions. Normal distributions are most often used in the design of technical objects for approximation. We will conduct research for these types of distribution of uncertainty and probability.

To generate the values of each random parameter, we used the standard normal distribution.

$$\psi_{(\bullet)}(x) = \frac{1}{2}\left(1 + erf\left(\frac{(x-\mu_{(\bullet)})^2}{2\sigma^2_{\mu_{(\bullet)}}}\right)\right),\ erf(x) = \frac{1}{\sqrt{\pi}}\int_0^x e^{-t^2}dt.$$

For each epistemic parameter, we set the normal uncertainty distribution function:

$$\Phi_{(\bullet)}(x) = \left(1 + exp\left(\frac{\pi(e_{(\bullet)} - x)}{\sqrt{3}\sigma_{e_{(\bullet)}}}\right)\right)^{-1},\ \Phi^{-1}(\alpha) = e_{(\bullet)} + \frac{\sigma_{e_{(\bullet)}}\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}.$$

We used the nominal (mean) values of random and epistemic parameters as the $\mu_{(\bullet)}$ and $e_{(\bullet)}$ respectively. We set the standard deviations $\sigma_{\mu(\bullet)}$ and $\sigma_{e(\bullet)}$ as a percentage of $\mu_{(\bullet)}$ and $e_{(\bullet)}$.

We investigate the influence of model parameters that reflect the characteristics of uncertainty and preferences of design maker.

We present in Fig. 1 and 2 the results of applying the optimization models A and B for different variants of the initial data: standard deviations of parameters $\sigma_{\mu(\bullet)}$, $\sigma_{e(\bullet)}$, and values related to reliability $\alpha_{M_{landing}}$, $P_{M_{landing}}$, $\alpha_{M_0}$, $\alpha_{M\ fuel}$, $P_{M\ fuel}$, $\alpha^{\leq}_{\gamma_a}$, $\alpha^{\geq}_{\gamma_a}$. The upper line in Fig. 1-3 is the result of applying the optimization model with deterministic values of uncertain and random parameters equal to their average values. The Pareto-front, which is the result of the application of model A, practically coincides with the upper line, but narrows with increasing reliability. The values $\alpha_{M_{landing}}$, $P_{M_{landing}}$, $\alpha_{M_0}$, $\alpha_{M\ fuel}$, $P_{M\ fuel}$, $\alpha^{\leq}_{\gamma_a}$, $\alpha^{\geq}_{\gamma_a}$ are equal and vary from 0.6 to 0.9 with a step of 0.1 (bottom four lines in Fig. 1 and 2).

Set for all random parameters $\sigma_{\mu(\bullet)} = 0{,}03\mu_{(\bullet)}$, for all epistemic parameters $\sigma_{e(\bullet)} = 0{,}03e_{(\bullet)}$. Then we obtained the Pareto fronts shown in Fig.1.

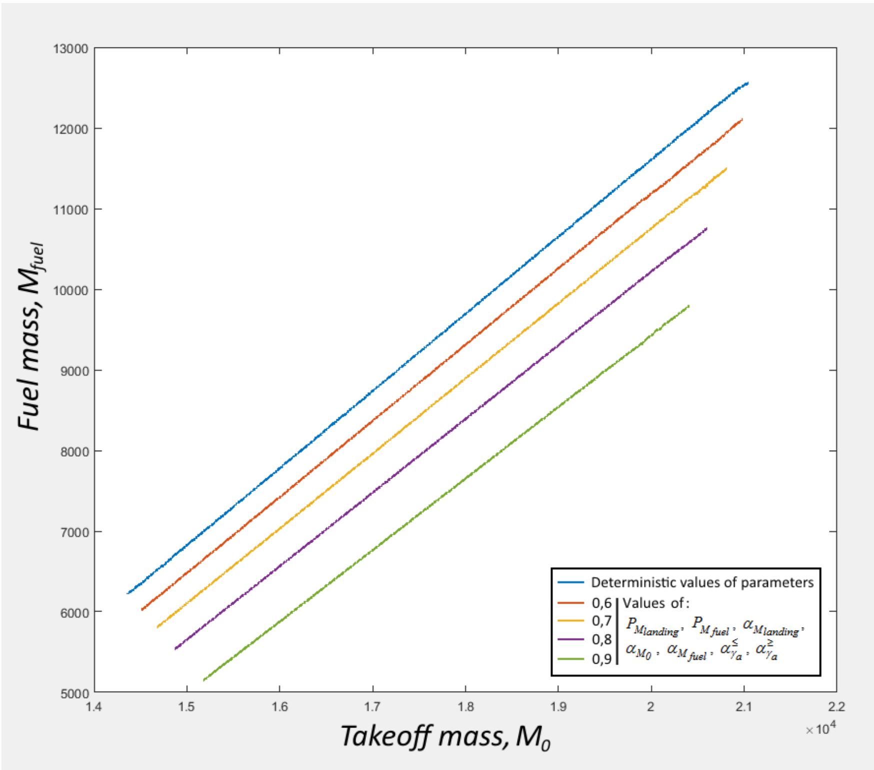


**Fig. 1.** Pareto fronts for model B at $\sigma_{\mu_{(\bullet)}} = 0{,}03\mu_{(\bullet)}$ and $\sigma_{e_{(\bullet)}} = 0{,}03e_{(\bullet)}$ (the parameters are random and epistemic)

Set for all random parameters $\sigma_{\mu_{(\bullet)}} = 0{,}05\mu_{(\bullet)}$, for all epistemic parameters $\sigma_{e(\bullet)} = 0{,}05e_{(\bullet)}$. The parameters $P_{M_{landing}}$, $P_{M\ fuel}$, $\alpha_{M_{landing}}$, $\alpha_{M_0}$, $\alpha_{M\ fuel}$, $\alpha^{\leq}_{\gamma_a}$, $\alpha^{\geq}_{\gamma_a}$ are equal and vary from 0.6 to 0.9 with a step of 0.1. Then we obtained the Pareto fronts shown in Fig.2.

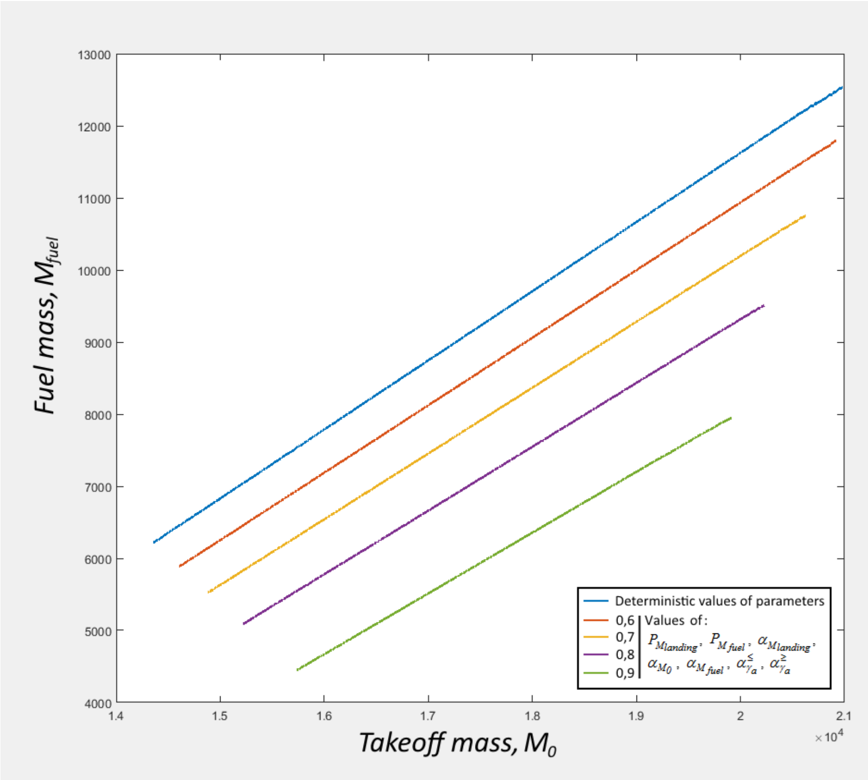


**Fig. 2.** Pareto fronts for model B at $\sigma_{\mu_{(\bullet)}} = 0{,}05\mu_{(\bullet)}$ and $\sigma_{e_{(\bullet)}} = 0{,}05e_{(\bullet)}$ (the parameters are random and epistemic)

The figures show that the Pareto-fronts obtained as a result of performing optimization calculations using the model B with increasing $P_{M_{landing}}$, $P_{M\,fuel}$, $\alpha_{M_{landing}}$, $\alpha_{M_0}$, $\alpha_{M\,fuel}$, $\alpha^{\le}_{\gamma_a}$, $\alpha^{\ge}_{\gamma_a}$ are shifted to the region of the worst values of the objective functions. While ensuring high reliability of the solution ($P$ and $\alpha$ close to 1) at fixed take-off weight the difference in fuel mass can reach more than 4000 kg.

For comparison, consider the case where the parameters $K_m$, $K_{eq}$, $V_{pc}$, $M_{ec}$, $\gamma_a$, $k_{qs0}$, $k_{qs1}$, $k_{eng0}$, $k_{eng1}$, $\gamma_{eng}$, $T$ are random. We use the following optimization model:

$$\begin{cases} \min\ r_1,\ \max\ r_2, \\ P(M_0 \le r_1) \ge P_{M_0},\ P_{M_0} \in [0,1], \\ P(M_{fuel} \ge r_2) \ge P_{M\,fuel},\ P_{M\,fuel} \in [0,1], \\ P(\gamma_a \le 500) \ge P^{\le}_{\gamma_a},\ P^{\le}_{\gamma_a} \in [0,1], \\ P(\gamma_a \ge 400) \ge P^{\ge}_{\gamma_a},\ P^{\ge}_{\gamma_a} \in [0,1], \\ P(M_0 - 0.8M_{fuel} \le M_{landing}) \ge P_{M_{landing}},\ P_{M_{landing}} \in [0,1]. \end{cases}$$

Let for all random parameters $\sigma_{\mu_{(\bullet)}} = 0{,}03\mu_{(\bullet)}$. The parameters $P_{M_0}$, $P_{M_{landing}}$, $P_{M\,fuel}$, $P^{\le}_{\gamma_a}$, $P^{\ge}_{\gamma_a}$ are equal and vary from 0.6 to 0.9 with a step of 0.1 (bottom four lines). Then we obtained the Pareto fronts shown in Fig.3. The upper line in Figures 3 is the result of applying the optimization model with deterministic values of uncertain and random parameters equal to their average values.

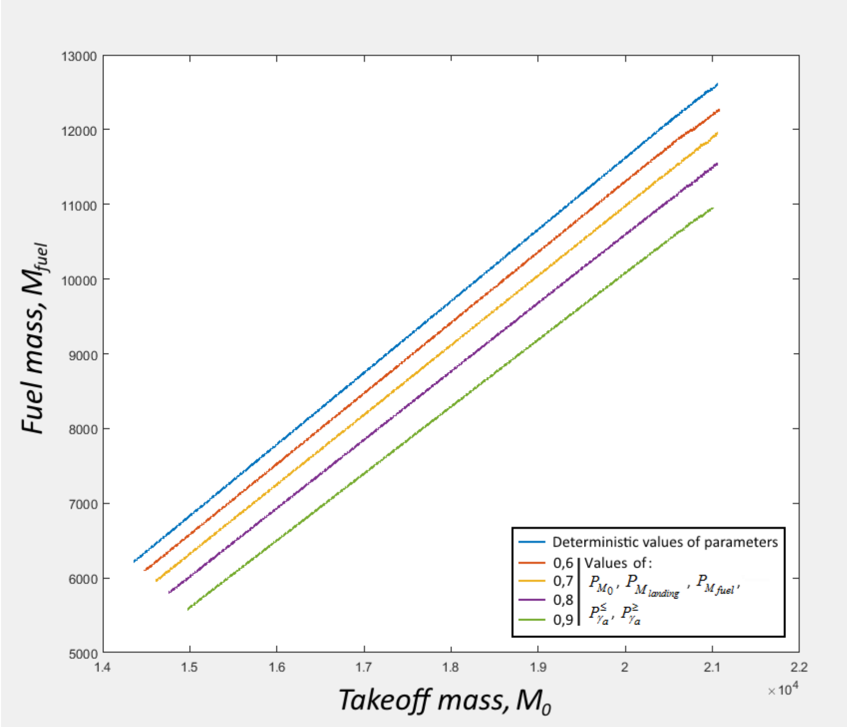


**Fig. 3.** Pareto fronts at $\sigma_{\mu_{(\bullet)}} = 0{,}03\mu_{(\bullet)}$ (all parameters are random)

Pareto fronts also move to the area of the worst values of the objective functions, but less than in Fig. 1 and 2.

The explorations show that the greater the degree of uncertainty and the greater the level of reliability, the more conservative obtained solutions. The obtained results do not contradict the experience and intuition of the designer, which confirms the adequacy of the models.

## 5. CONCLUSION

The models of the optimization problem with constraints under conditions of parametric mixed uncertainty are proposed. Uncertain values introduced in the theory of uncertainty models parameters with epistemic uncertainty. Under certain conditions (for rather wide class of functions), optimization models with mixed uncertainty are reduced to deterministic models or to stochastic programming models with probabilistic criteria. We formalize and solve the task of preliminary aerodynamic design in conditions of parametric mixed uncertainty using the proposed models.

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## APPENDIX [1]

**Definition 1:**

Let *Γ* be a nonempty set, let *Λ* be a σ-algebra over *Γ*, and let *M* be an uncertain measure. Then the triplet (*Γ*, *L*, *M*) is called an uncertainty space.

A number $M\{\Lambda\}$ will be assigned to each event *Λ* to indicate the belief degree with which we believe *Λ* will happen. The measure of uncertainty M is dual, subadditive, the measure of the product of events is equal to the minimum of the measures of these events.

**Definition 2:**

An uncertain variable is a function *ξ* from *Γ* to the set of real numbers such that $\{\gamma \mid \xi(\gamma) \in B\}$ is an event for any Borel set *B* of real numbers.

**Definition 3:**

The uncertainty distribution of an uncertain variable *ξ* is defined by $\Phi(x) = M\{\xi \le x\}$ for any real number *x*.

**Definition 4:**

An uncertainty distribution *Φ*(x) is said to be regular if it is a continuous and strictly increasing function with respect to *x* at which $0 < \Phi(x) < 1$, and

$$\lim_{x \to -\infty} \Phi(x) = 0, \lim_{x \to +\infty} \Phi(x) = 1.$$

**Definition 5:**

Let *ξ* be an uncertain variable with regular uncertainty distribution *Φ*(*x*). Then the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ.

**Definition 6:**

The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if

$$M\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\} = \bigwedge_{k=1}^{n} M(\xi_i \in B_i)$$

for any Borel sets $B_1$, $B_2$, …, $B_n$ of real numbers.

**Theorem 1:**

Let $\xi_1$, $\xi_2$, …, $\xi_n$ be uncertain variables, and let $f$ be a real-valued measurable function. Then $f(\xi_1, \xi_2, ..., \xi_n)$ is an uncertain variable.

**Definition 7:**

Let $\xi$ be an uncertain variable. Then the expected value of $\xi$ is defined by $E[\xi] = \int_0^{+\infty} M\{\xi \geq r\}dr - \int_{-\infty}^{0} M\{\xi \leq r\}dr.$

**Definition 8:**

Let $\xi$ be an uncertain variable with finite expected value $E[\xi]$. Then the variance of $\xi$ is $V[\xi] = E[(\xi - E[\xi])]^2$.

**Theorem 2:**

Let $f(\xi_1, \xi_2, ..., \xi_n)$ be continuous, strictly increasing with respect to $\xi_1, \xi_2, ..., \xi_m$ continuous, strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2,...,} \xi_n$. If $\xi_1, \xi_2, ..., \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1$, $\Phi_2$, …, $\Phi_n$, respectively, then:

*a)* $\xi = f(\xi_1, \xi_2, ..., \xi_n)$ is uncertain variable with inverse uncertainty distribution: $\Psi^{-1}(\xi) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), ..., \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), ..., \Phi_n^{-1}(1-\alpha)),$

*b)* $E[\xi] = \int f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), ..., \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), ..., \Phi_n^{-1}(1-\alpha))d\alpha,$

*c)* $V[\xi] = \int_0^1 (f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), ..., \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), ..., \Phi_n^{-1}(1-\alpha)) - E[\xi])^2 d\alpha,$

*d)* for any $\alpha \in [0, 1]$
$M\{f(\xi_1, \xi_2, ..., \xi_n) \leq 0\} \geq \alpha$
is equivalent
$f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), ..., \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), ..., \Phi_n^{-1}(1-\alpha)) \leq 0.$

**Definition 9:**

*a*) The quantiles of the random variable ξ:
$sup_\alpha[\xi] = sup\{r \mid Pr\{\xi \geq r\} \geq \alpha\}$ – $\alpha$-optimistic value,
$inf_\alpha[\xi] = inf\{r \mid Pr\{\xi \leq r\} \geq \alpha\}$ – $\alpha$-pessimistic value,
$\alpha \in [0, 1]$,
*b*) The quantiles of the uncertain variable ξ:
$sup_\alpha[\xi] = sup\{r \mid M\{\xi \geq r\} \geq \alpha\}$ – $\alpha$-optimistic value,
$inf_\alpha[\xi] = inf\{r \mid M\{\xi \leq r\} \geq \alpha\}$ – $\alpha$-pessimistic value,
$\alpha \in [0, 1]$.

**Definition 10:**

Let $(\Omega, A, Pr)$ be a probability space and $(\Gamma, \Lambda, M)$ be an uncertainty space. Then the product $(\Gamma \times \Omega, \Lambda \times A, M \times Pr)$ is called a chance space, where $\Gamma \times \Omega = \{(\gamma, \omega) \mid \gamma \in \Gamma, \omega \in \Omega\}$. Then the chance measure of an event Θ of Λ × A is:

$$Ch\{\Theta\} = \int_0^1 \Pr\{\omega \in \Omega | M\{\gamma \in \Gamma | (\gamma,\omega) \in \Theta\} \geq x\} dx \ .$$

$M\{\gamma \in \Gamma \,|(\gamma, \omega) \in \Theta\}$ is just the uncertain measure of cross section of $\Theta$ at ω. Since $M\{\gamma \in \Gamma \,|(\gamma, \omega) \in \Theta\}$ can be regarded as a function from the probability space ($\Omega$, $A$, $Pr$) to [0, 1], it is a random variable. Thus the chance measure $Ch\{\Theta\}$ is just the expected value (i.e., average value) of this random variable. The chance-measure is monotonous, dual and subadditive.

**Definition 11:**

An uncertain random variable is a function $\xi$ from $\Gamma \times \Omega$ to the set of real numbers such that $\{(\gamma, \omega)|\, \gamma \in \Gamma, \omega \in \Omega \,|\xi(\gamma, \omega) \in B\}$ is an event in $L \times A$ for any Borel set $B$ of real numbers.

**Definition 12:**

Let $\xi$ be an uncertain-random variable. Then its chance distribution is defined by $\Phi(x) = Ch\{\xi \leq x\}$ for any real x.

**Definition 13:**

The expected value of uncertain-random variable $\xi$ is:

$$E[\xi] = \int_0^{+\infty} Ch\{\xi \geq x\} dx - \int_{-\infty}^{0} Ch\{\xi \leq x\} dx$$

provided that at least one of the two integrals is finite.

**Theorem 3:**

Let $\eta_1, \eta_2, \ldots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \ldots, \Psi_m$, and let $\tau_1, \tau_2, \ldots, \tau_n$ be independent uncertain variables with uncertainty distributions $Y_1, Y_2, \ldots, Y_n$, respectively. If $f(\eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_n)$ is a measurable function, then has an expected value $E[\xi]$ is:

$$E[f(\eta_1, \ldots, \eta_m, \tau_1, \ldots, \tau_n)] = \int_{R^m} G(y_1, \ldots, y_m) d\Psi_1(y_1) \ldots d\Psi_m(y_m),$$

where $G(y_1, \ldots, y_m) = E[f(y_1, \ldots, y_m, \tau_1, \ldots, \tau_n)]$ − is the expected value of the uncertain variable $f(y_1, \ldots, y_m, \tau_1, \ldots, \tau_n)$ for any real numbers $y_1, \ldots, y_m$, and is determined by $Y_1, Y_2, \ldots, Y_n$.

 