Adv Syst Sci Appl 2019; **03**; 52-64
Published online at http://ijassa.ipu.ru/index.php/ijassa/article/view/713

Optimal Values in an Uncertain Optimal Control Model with Application to Capital Asset Management

Tolulope Latunde[1*]

[1)] *Department of Mathematics, Federal University, Oye-Ekiti, Nigeria*

E-mail: tolulope.latunde@fuoye.edu.ng

Received March 21, 2019; Revised September 6, 2019; Published October 1, 2019

Abstract: A new model of capital asset management was developed under the assumptions of hyperbolic absolute risk aversion, and employing the basic skills of mathematical modelling. The solution to the model was sought by formulating a continuous-time utility portfolio model satisfying some uncertainty criteria where the investment is continuous, the investor does not possess enough power to determine price and the investor can borrow money for a given period at a particular interest rate. The model was solved using analytical method and numerical method and optimal values of some input factors are derived.

Keywords: Capital asset management, optimal control, uncertainty theory, optimal values.

1. INTRODUCTION

Capital asset models have been subject to an enormous number of empirical studies since Lintner and Sharpe in the mid-1960. Among the most notable early tests of the models are those in [1, 4]. Other contributors are included in [3, 5, 13, 14]. The earliest researchers found the relationships between the risk-free rate and market risk premium. Since Merton developed and solve a portfolio selection model under uncertainty for the case of infinite lifetimes and finite lifetimes: the continuous-time case for risk and risk-free assets, capital asset management has adapted the portfolio theories, see [12].

This results to some researchers formulating asset management models using different mathematical approaches such that optimal values of controls and input factors are obtained. A stochastic optimal control approach was utilised to model debt crisis so as to evaluate debt crisis in international finance, thus, the optimal debt to preclude crisis was obtained [15]. The work [19] was based on the concepts of uncertainty theory where uncertain optimal control was applied to solve a portfolio selection model and obtained a fundamental result called equation of optimality for uncertain optimal control, thus obtain the optimal value of the control. Formulation of two instances of uncertain multidimensional optimal control models with n jumps based on uncertainty theory from the perspectives of agents (

consumers) and principal (government) was carried out in the work [2] - the optimal control problems were applied to Research and Development fiscal policy and optimal control decisions were obtained.

It is understood from existing works that the research carried out in [12,19, 2] examined the selection of risk-free asset and risk asset together in the formulation of the models. Meanwhile, it was examined in [15], the selection of risky assets only with stochastic optimal control approach without considering depreciation and taxation as input factors. The choice of Uncertainty theory over the conventional probability theory exists when the sample size is

[*] Corresponding author: tolulope.latunde@fuoye.edu.ng

small to estimate a probability distribution and degree belief are ascertained from experts to work in place of frequency since human beings always overweight unlikely events Consequently, a new model of asset management based on uncertainty theory was formulated in [6] where the selection of capital assets, classified as risky assets is examined. Thus, depreciation and taxation are considered as input factors in the formulation of the models.

However, this work is an extension of the work [6, 7] such that we seek to provide solutions to a problem of capital assets using a real life situation. The proposed model deals with a case where a nation invests her wealth in capital assets, for a particular period of time. It is assumed that there is difficulty in deciding the fraction of the nation's net worth to be incurred on the investments of capital assets, thus leading to the problem of how to optimize the expected present value of the utility of assets.

2. PRELIMINARY

Uncertainty theory is a branch of mathematics for modelling belief degrees established by Liu, [8] and refined in [11]. The choice of Uncertainty theory over the conventional probability theory exists when the sample size is small to estimate a probability distribution and degree belief are ascertained from experts to work in place of frequency since human beings always overweight unlikely events. For the sake of this work, the following concepts are utilized.
Let Γ be a nonempty set and L a σ- algebra over Γ such that (Γ, L) be a measurable space. Each element $\Lambda \in L$ is called an event.

Definition 2.1 [8]:
A set function M defined on the σ-algebra over L is called an uncertain measure if it satisfies the following axioms:

Axiom 1. *(Normality Axiom)*: $M\{\Lambda\} = 1$ *for the universal set Γ.*

Axiom 2. *(Duality Axiom)*: $M\{\Lambda\} + M\{\Lambda^c\} = 1$ *for any event* Λ.

Axiom 3. *(Subadditivity Axiom)*: For every countable sequence of events, $\Lambda_1, \Lambda_2, \cdots$, we have

$$M\{\cup_{i=1}^{\infty} \Lambda_i\} \leq \sum_{i=1}^{\infty} M\{\Lambda_i\} \tag{2.1}$$

Axiom 4. *(Product Axiom)*: Let (Γ_k, L_k, M_k) be uncertainty spaces for $k = 1,2, \cdots$The product uncertain measure M is an uncertain measure satisfying

$$M\{\prod_{k=1}^{\infty} \Lambda_k\} = \min_{1 \leq k \leq \infty} M_k\{\Lambda_k\} \tag{2.2}$$

where Λ_k are arbitrarily chosen events from L_k for $k = 1,2, \cdots$, respectively.

Definition 2.2 [10]:
An uncertain process C_σ is said to be a canonical Liu process if

(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,

(ii) C_σ has stationary and independent increments,

(iii) every increment $C_{s+\sigma} - C_s$ is a normal uncertain variable with expected value 0 and variance σ^2. The uncertainty distribution of C_σ is

$$\Phi_\sigma(x) = \left[1 + \exp\left(\frac{-\pi x}{\sqrt{3}\sigma}\right)\right]^{-1}, \quad x \in \Re \tag{2.3}$$

and the inverse distribution is

$$\Phi_\sigma^{-1}(y) = \frac{\sigma\sqrt{3}}{\pi} \ln \frac{y}{1-y}, \quad y \in \Re \tag{2.4}$$

Definition 2.3 [8]:
Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} M\{\xi \geq x\} dx - \int_{-\infty}^{0} M\{\xi \leq x\} dx \tag{2.5}$$

provided that at least one of the two integrals is finite

Definition 2.4 [9]:
An uncertain process X_t is said to have independent increments if

$$X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \cdots, X_{t_k} - X_{t_{k-1}}$$

are independent uncertain variables where $t_1, t_2, \cdots, t_k$ *are any times with* $t_0 < t_1 < \cdots < t_k$

That is, an independent increment process means that its increments are independent uncertain variables whenever the time intervals do not overlap. It is noted that the increments are also independent of the initial state.

Definition 2.5 [9]:

Suppose C_t *is a canonical Liu process, and* f *and* g *are two functions. Then*

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t \tag{2.6}$$

is called an uncertain differential equation. A solution is a Liu process X_t *that satisfies (2.3) and (2.4) identically in* t.

Definition 2.6 [9]:

Let X_t *be an uncertain process. Then for each* $\gamma \in \Gamma$, *the function* $X_t(\gamma)$*is called a sample path of* X_t.

Definition 2.7 [11]:

An uncertain process X_t *is said to be sample-continuous if almost all sample paths are continuous functions with respect to time* t.

Definition 2.8 Uncertainty Distribution of Solution [16]:

Let α *be a number with* $0 < \alpha < 1$. *An uncertain differential equation*

$$dX(t) = f(t, X(t))dt + g(t, X(t))dC(t)$$

is *said to have an* α*-path* $X(t)^\alpha$ *if it solves the corresponding ordinary differential equation*

$$dX(t)^\alpha = f(t, X(t)^\alpha)dt + \left|g\big(t, X(t)\big)\right|\Phi^{-1}(\alpha)dt \tag{2.7}$$

where $\Phi^{-1}(\alpha)$ *is the inverse uncertainty distribution of standard normal uncertain variable, that is,*

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}, \quad \alpha \in \Re$$

Theorem 2.1 Extreme Value of Solution [18]:

Let $X(t)$ *and* $X(t)^\alpha$ *be the solution and* α*-path of the uncertain differential equation (2.6). Then, for any time* $t > 0$ *and strictly increasing function* $J(x)$, *the supremum*

$$\sup_{t_0 \le t \le t_n} J(X(t))$$

has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \sup_{t_0 \le t \le t_n} J(X(t)^\alpha) \tag{2.8}$$

and the infimum

$$\inf_{t_0 \le t \le t_n} J(X(t))$$

has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \inf_{t_0 \le t \le t_n} J(X(t)^\alpha) \tag{2.9}$$

Theorem 2.2 [16]:

Let $X(t)$ *and* $X(t)^\alpha$ *be the solution and* α*-path of the uncertain differential equation (2.6). Then*

$$M\{X(t) \le X(t)^\alpha, \forall t\} = \alpha, \quad M\{X(t) > X(t)^\alpha, \forall t\} = 1 - \alpha. \tag{2.10}$$

3. MODEL FORMULATION

A model of capital asset management is presented herein such that it is assumed that an investor invests his wealth in capital asset, $A(t)$, of a large business for time, t, from t_0 to t_f.

Suppose he starts with a known initial net worth $X_0(t)$. At time t, what ratio of his net worth, ψ, must he select to utilize on capital asset in the presence of liability such that the expected net present value of the utility of asset, $J(\psi)$, is optimized ?

Table 3.1. Definition of Parameters of the Objective function to the model

Parameter	Description
U	Utility function
$A(t)$	Capital asset at time t
η	subjective discount rate, e.g., $\frac{A}{\eta+1} = \text{Presentvalue}$
λ	degree of relative risk, where $(1-\lambda)$ is the risk aversion
ψ	Capital asset ratio (control) $\psi \in \mathfrak{R}$
$X(t)$	Net worth at time (state variable)t

Table 3.2: Definition of Parameters of the constraint to the model

Parameter	Description
$\sigma_r(t)$	Diffusion volatility of liability (with variance σ_r^2 per unit time)
ψ	Capital asset ratio (control) $\psi \in \mathfrak{R}$
$\sigma_b(t)$	Diffusion volatility of asset (with variance σ_b^2 per unit time)
$\kappa(t)$	Capital gain on asset due to inflation at time t
$\sigma_p(t)$	Diffusion volatility on asset price (with variance σ_p^2 per unit time)
$\beta(t)$	Mean rate of return on asset
ω	Mean interest rate of liability
$C(t)$	uncertain process at time t
$\mu(t)$	Consumption level at time t
$j(t)$	Tax ratio at time t
$g(t)$	Depreciation ratio at time t
$h(t)$	Asset supplies ratio at time t

Theorem 3.1 [12]:

If $\lambda > 0$ and U is such that the integral $E_C\left[\int_{t_0}^{t_f} U(A,t)dt\right]$ is absolutely convergent, then the maximization or minimization of $E_C\left[\int_{t_0}^{t_f} U(A,t)dt\right]$ is equivalent to the maximization or minimization of $E_0\int_{t_0}^{t_f} e^{-\eta t}U(A,t)dt$ where E_C is the conditional expectation operator over all random variables excluding λ.

By Theorem 3.1, an investor who faces an exponentially-distributed uncertain investment of capital asset invests as if there is no terminal period, but with a subjective rate of time preference equal to the investments terminals.

$$J = \text{opt}E_C\left[\int_{t_0}^{t_f} e^{-\eta t}U(A(t))dt\right] \tag{3.1}$$

where E_C denotes conditional expectation, $\eta \in (0,1)$ is the arbitrary discount rate. In selecting the discount rate, the effective length of time is inversely proportional to the discount rate in the sense that a high discount rate implies a short time interval, [15]. Utility function $U(A)$ measures satisfaction of an investor as a function of usage or efficiency of capital assets with respect to risk aversion. However, risk aversion is the behaviour of the investors when exposed to attempt to reduce the uncertainty in their investments. There are various measures of risk aversion under expected utility theory.

Thus, a special case of Hyperbolic absolute risk aversion (HARA) is considered as the model's utility function which helps in focusing more on ratios and its assumption also lowers the dimension of dynamic system for the model to be effortlessly solved analytically unlike some other utility functions. That is

$$U(A) = \begin{cases} \frac{1}{\lambda} A^{\lambda}, & 0 < \lambda < 1 \\ \ln A, & \lambda = 0 \end{cases} \tag{3.2}$$

where $1 - \lambda > 0$ is the investor's relative risk aversion. The larger the λ, the more reluctant to own a risky asset, [12].

The efficiency or performance of the capital asset, ψ, is expressed as the ratio of capital asset, $A(t)$, and net worth, $X(t)$, such that $X(t) \neq 0$, [15]. That is,

$$\psi = \frac{A(t)}{X(t)} \tag{3.3}$$

$$A(t) = \psi X(t)$$

From the proceeding, the model of risky capital asset is an optimal control of the form.

$$J(\psi) = \max_{\psi} E_C \left[\int_{t_0}^{t_n} \frac{1}{\lambda} e^{-\eta t} (\psi X(t))^{\lambda} dt \right] \tag{3.4}$$

subject to

$$dX(t) = [(\kappa + \beta)\psi - (\omega(\psi - 1) + \mu + h - j - g)]X(t)dt$$

$$+ \left[\psi \sigma_p + \psi \sigma_b - \sigma_r(\psi - 1)\right] X(t) dC(t) \ \ [6] \tag{3.5}$$

4. APPLICATION OF THE CAPITAL ASSET MODEL

Here, the model is analysed using real life data in order to provide some optimal solutions satisfying the optimality criteria.

Utilizing the model in international finance, the revenue is taken to be the Gross Domestic Product of a nation (GDP) or value added, the risky capital asset as the capital investment or the Gross Fixed Capital Formation, Consumption as Household and Government Consumption, Depreciation is taken as the Consumption of Fixed capital. The debt is also taken as a study case of liability to be considered.

Table 4.1: Definition of parameters according to the model application

Parameter	Description
J	Expected present value of utility of GFCF

η	subjective discount rate, e.g., $\frac{A}{\eta+1}=\text{Presentvalue}$
λ	degree of relative risk, where $(1-\lambda)$ is the risk aversion
ψ	GFCF ratio (control) $\psi\in\Re$
$\tau(t)$	Debt ratio (control) at time t, $\tau=\psi-1$
$X(t)$	Net worth at time t (GFCF minus Debt)
$\kappa(t)$	Capital gain on GFCF with net worth at time t
$\beta(t)$	Mean rate of return on GFCF with net worth
$\omega(t)$	Mean rate of debt with net worth
$\sigma_p(t)$	Diffusion volatility on asset price (with variance σ_p^2 per unit time)
$\sigma_b(t)$	Diffusion volatility of GFCF (with variance σ_b^2 per unit time)
$\sigma_r(t)$	Diffusion volatility of Debt (with variance σ_r^2 per unit time)
$\sigma(t)$	Diffusion volatility of the whole process $(\sigma_p+\sigma_b-\sigma_\omega)$, [15]
$\mu(t)$	Consumption level with net worth at time t
$s(t)$	Net foreign supplies - net worth ratio at time t
$j(t)$	Tax-net worth ratio at time t
$g(t)$	Depreciation-net worth ratio at time t
$C(t)$	Liu canonical process at time t

In order to examine the debt crisis in Nigeria and propose a warning signal, the data that are available after the Paris Debt forgiveness in 2006 are used. Thus, Tables 4.2 and 4.3 below represent the base parameter set for the case study.

Table 4.2: Nigeria Net worth Profile

Year	GDP	Debt	GFCF	Net worth	Consumption	Indirect Tax	Depreciation	Supplies
2007	166.451	22.330	15.396	-6.934	149. 152	2.553	3.738	5.089
2008	208.065	21.399	17.318	-4.081	161.035	3.436	3.853	30.988
2009	169.481	25.817	20.487	-5.330	147.601	3.180	2.952	0.445
2010	369.062	40.100	61.099	21.860	293.507	5.623	16.079	28.662
2011	411.744	47.898	63.960	16.062	323.540	4.516	18.815	38.719
2012	460.953	48.496	65.283	16.787	348.597	5.686	24.260	86.210
2013	514.966	64.510	72.964	8.454	453.699	7.929	23.857	26.280
2014	568.499	67.726	85.737	18.011	464.696	6.857	25.272	32.499
2015	481.066	65.429	71.329	5.900	417.560	5.362	23.097	0.000
2016	405.083	57.392	73.261	15.869	206.414	3.188	10.332	-3.341

Source:
Columns 1 and 3. The world bank (http://data.worldbank.org/indicator)
Column 2. Debt Management Office of Nigeria (https://www.dmo.gov.ng/).
Columns 5, 6, 7, 8 and 9. National Bureau of Statistics (http://nigerianstat.gov.ng/)
Table 4.3 is derived from Table 4.2.

Table 4.3: Parameters for the Nigeria Net worth Profile

Year	κ	β	ω	σ_p	σ_b	σ_r	σ	μ	h	j	g

2007	-7.25	-51.72	-6.10	-3.78	-21.47	-2.46	-22.79	-21.51	-0.73	-0.37	-0.54
2008	-12.32	-87.88	-10.36	-6.42	-36.48	-4.17	-38.73	-39.46	-7.59	-0.77	-0.94
2009	-9.43	-67.29	-7.93	-4.92	-27.93	-3.20	-29.65	-27.69	0.08	-0.60	-0.55
2010	2.30	16.41	1.93	1.20	6.81	0.78	7.23	13.43	1.31	0.26	0.74
2011	3.13	22.33	2.63	1.63	9.27	1.06	9.84	20.14	2.14	0.28	1.17
2012	2.30	21.37	2.52	1.56	8.87	1.02	9.41	20.77	5.14	0.34	1.45
2013	5.95	42.42	5.00	3.10	17.61	2.02	18.69	53.67	3.11	0.94	2.82
2014	2.79	19.91	1.43	1.46	8.27	0.95	8.78	25.80	1.92	0.37	1.40
2015	8.52	60.79	4.37	4.44	25.23	2.89	26.78	70.77	0.00	0.91	3.91
2016	3.17	22.60	1.63	1.65	9.38	1.07	9.96	13.01	-0.21	0.20	0.65

Measurements

The measurements considered in obtaining data in Tables 4.1 – 4.3 are described below.
All the values of parameters are measured in Billion US Dollars except the following parameters: $C(t)$ - measures the uncertainty process which exists in the interval $0 < C(t) < 1$; λ - is used to measure risk which exists in the interval $0 < \lambda < 1$; and η - measures discount rate which exists in the interval $0 < \eta < 1$. The debt is calculated as the total debt of the nation by summing the external debt stock (federal government and state) and Domestic debt (federal government and state) together. The net worth is also calculated by deducting the debt from the GFCF. The CBN Official Exchange rate of USD at 31st December of each year is used while current market prices from the national account are used in the computations.

4.1 Solution to the model

Here, the analytical and numerical solutions are derived.

For the analytic solution, the required problem under consideration is

$$J(\psi) = \min_{\psi} E_C \left[\int_{t_0}^{t_n} \frac{1}{\lambda} e^{-\eta t} (\psi X(t))^{\lambda} dt \right]$$

subject to

$$dX(t) = [(\kappa + \beta)\psi - (\omega(\psi - 1) + \mu + s - j - g)]X(t)dt$$

$$+[\psi\sigma_p + \psi\sigma_b - \sigma_r(\psi - 1)]X(t)dC(t)$$

with α-path equation

$$dX(t)^{\alpha} = [(\kappa + \beta)\psi - (\omega(\psi - 1) + \mu + h - j - g)]X(t)^{\alpha}dt + |[\psi\sigma_p + \psi\sigma_b - \sigma_r(\psi - 1)]X(t)^{\alpha}|\Phi^{-1}(\alpha)dt.$$

The analytical solution to the constraint is

$$X(t) = X_0 \exp\big([(\kappa+\beta)\psi - (\omega(\psi-1)+\mu+s-j-g)]t + [\psi\sigma_p + \psi\sigma_b - \sigma_r(\psi - 1)]C(t)\big)$$

and its inverse uncertainty distribution is

$$\Psi(t)^{-1}(\alpha) = X_0 \exp([(\kappa+\beta)\psi - (\omega(\psi-1)+\mu+s-j-g)]t + \frac{[\psi\sigma_p + \psi\sigma_b - \sigma_r(\psi-1)]t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\Big)$$

Hence, by Theorem 2.1,

$$\Psi(t)^{-1}(\alpha) = E(X(t)^{\alpha})$$

Numerical solutions are presented via trapezoidal rule for the objective functional and, Euler method and fourth order Runge-Kutta method for solving uncertain differential equations due to its ability to yield more precise outcomes than other methods for the constraints, [17].

Trapezoidal method:

$$\int_a^b f(x)dx = \frac{h}{2}(f_0 + 2f_1 + 2f_2 + \cdots + 2f_{n-1} + f_n)$$

$$= h(\frac{f_0 + f_n}{2} + \sum_{i=1}^{n-1} f_i),$$

where $f(x_k) \equiv f_k$

The Runge-Kutta method for solving uncertain differential equations was designed in [17] with respect to the following definition and theorems.

Runge-Kutta method is an effective method for solving ordinary differential equations. The generally used Runge-Kutta formula is a fourth-order formula. It should be noted that there is a wide range of fourth-order schemes and here, just one common structure is exhibited. For an ordinary differential equation with initial value X_0

$$dX(t) = F(t, X(t))dt.$$

The scheme uses the following formula

$$X(t_{n+1}) = X(t_n) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = hF(t_n, X_n),$$

$$k_2 = hF(t_n + \frac{h}{2}, X_n + \frac{1}{2}k_1),$$

$$k_3 = hF(t_n + \frac{h}{2}, X_n + \frac{1}{2}k_2),$$

$$k_4 = hF(t_n + h, X_n + k_3)$$

and h is the step size which is assumed to be constant for all steps.

However, based on Theorem 2.4, a Runge-Kutta method for uncertain differential equations was designed as

$$X_{i+1}^{\alpha} = X_i^{\alpha} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = h(f(t_i, X_i^{\alpha}) + |g(t_i, X_i^{\alpha)}|\Phi^{-1}(\alpha)),$$

$$k_2 = h(f(t_i + \frac{h}{2}, X_i^{\alpha} + \frac{1}{2}k_1) + |g(t_i + \frac{h}{2}, X_i^{\alpha} + \frac{1}{2}k_1)|\Phi^{-1}(\alpha)),$$

$$k_3 = h(f(t_i + \frac{h}{2}, X_i^{\alpha} + \frac{1}{2}k_2) + |g(t_i + \frac{h}{2}, X_i^{\alpha} + \frac{1}{2}k_2)|\Phi^{-1}(\alpha)),$$

$$k_4 = h(f(t_i + h, X_i^{\alpha} + k_3) + |g(t_i + h, X_i^{\alpha} + k_3)|\Phi^{-1}(\alpha)).$$

For the proposed optimal control model of net risky capital asset with an uncertain differential equation

$$dX(t) = [(\kappa + \beta)\psi - (\omega(\psi - 1) + \mu + h - j - g)]X(t)dt + [\psi\sigma_p + \psi\sigma_b - \sigma_r(\psi - 1)]X(t)dC(t)$$

with initial value X_0 and its α-path equation.

i.e.,

$$dX(t)^{\alpha} = [(\kappa + \beta)\psi - (\omega(\psi - 1) + \mu + h - j - g)]X(t)^{\alpha}dt$$

$$+|[\psi\sigma_p + \psi\sigma_b - \sigma_r(\psi - 1)]X(t)^{\alpha}|\Phi^{-1}(\alpha)dt,$$

This is solved using the algorithm below.

4.2. *Algorithm 4.1: Runge-Kutta method for solving the model*

Step 1. Given time interval $t, [a, b]$, iteration number N, step length $h = \frac{b-a}{N}$. Set $t_i = a + ih$, $i = 0,1,\cdots,N$ and $\alpha > 0$.

Step 2. Compute the corresponding differential equation

$$dX(t)^{\alpha} = [(\kappa + \beta)\psi - (\omega(\psi - 1) + \mu + s - j - g)]X(t)^{\alpha}dt$$

$$+|[\psi\sigma_p + \psi\sigma_b - \sigma_r(\psi - 1)]X(t)^{\alpha}|\frac{\sigma\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}dt,$$

$X_0^{\alpha} = X_0$, with the Runge-Kutta method for solving uncertain differential equations.

Step 3. Set $i = i + 1$, repeat Step 2 and step 3 for N times, then $X(t)^{\alpha}$ is derived. Go back to step 1 until $t_i = b$,

Table 4.4: Results of Analytical solution to the Model with $h = 0.05$, $a \le t \le b$, $a = 0$, $b = 1$, $\eta = 0.9$, $\lambda = 0.1$ and $X_0 = 18.011$

ψ	X	J
-9	-1175.785	16.657
-7	-28.783	11.209
-5	-0.277	6.812
-3	-0.118	5.941
-1	-1.612	6.916
1	19.050	8.854
3	95.377	11.609
5	91.492	12.166
7	77242.383	24.684
9	4.159×10^6	37.710

Table 4.5: Results of Numerical solution to the Model ($h = 0.05$, $a = 0$, $b = 1$, $\eta = 0.9$, $\lambda = 0.1$ and $X_0 = 18.011$)

ψ	X	J
-9	-1638.001	17.218
-7	-31.576	11.313

-5	-0.322	6.916
-3	-0.150	6.088
-1	-3.037	7.368
1	19.007	8.852
3	93.596	11.587
5	84.931	12.076
7	74991.214	24.612
9	3.524×10^{6}	37.090

Furthermore, using the available data, the numerical and analytical solution to the asset-liability management problem was presented. Using the net worth of the year 2007 to 2016 were to study the behaviour of the optimal control in each year and provide a control policy to the problem.

Let ψ_O be the optimal control which implies the rate of capital asset such that the expected present value of the utility of assets, and ψ_A be the actual control which is obtained from the given ratio of capital asset and net worth.

The optimal control ψ_O for each year was derived analytically using the equation of optimality proposed in [19] for uncertain optimal control problem, where

$$\psi_O = \frac{(\mu+j+g+\omega-h)\lambda-\eta}{(1-\lambda)(\kappa+\beta-\omega)}$$

and the actual control is

$$\psi_A = \frac{A(t)}{X(t)}.$$

Table 4.6: Results on performance of Capital Asset

Year	ψ_A	ψ_O	ψ_A - ψ_O
2007	-2.22	-4.09	1.87
2008	-4.24	-5.88	1.64
2009	-3.84	-5.09	1.25
2010	2.80	0.67	2.13
2011	3.98	1.42	2.56
2012	3.89	1.22	2.67
2013	8.63	5.59	3.04
2014	4.76	2.01	2.75
2015	12.09	7.88	4.21
2016	4.62	0.74	3.88

The warning signal will be based on the difference between the actual liability ratio τ_A and the optimal liability ratio τ_O.

Table 4.7: Warning signal

Year	τ_A	τ_O	τ_A - τ_O
2007	-3.22	-5.09	0.87

2008	-5.24	-6.88	0.64
2009	-4.84	-6.09	0.25
2010	1.80	-0.33	1.13
2011	2.98	0.42	1.56
2012	2.89	0.22	2.67
2013	7.63	4.59	2.04
2014	3.76	1.01	1.75
2015	11.09	6.88	3.21
2016	3.62	-0.26	2.88

4.2 Optimal U, λ and η

By hyperbolic absolute risk aversion utility function,

$$U(A) = \begin{cases} \frac{1}{\lambda}A^{\lambda}, & 0 < \lambda < 1 \\ \text{lnA}, & \lambda = 0 \end{cases}$$

$$\frac{dU}{d\lambda} = 0$$

$$-\frac{1}{\lambda^2}A^{\lambda} + \frac{1}{\lambda}A^{\lambda}\ln A = 0$$

$$\frac{1}{\lambda} = \ln A$$

$$\lambda = \frac{1}{\ln A}$$

Hence, the optimal λ is $\lambda^* = \frac{1}{\ln A}$.
Similarly, optimal U is U^* where

$$U^* = \frac{1}{\lambda^*}A^{\lambda^*}$$

$$= (\ln A)A^{\frac{1}{\ln A}}$$

From Table 4.2, Let $A_1 = 15.396$ and $A_2 = 85.737$ where $\lambda = \lambda_1$ and $A = A_1$.
This implies

$$\lambda_1 = \frac{1}{\ln A_1} = 0.366$$

Let $U = U_1$

$$U_1 = (\ln A_1)A_1^{\frac{1}{\ln A_1}} = 7.432$$

and let $\lambda = \lambda_2$, $A = A_2$

$$\lambda_2 = \frac{1}{\ln A_2} = 0.225$$

$$U_2 = (\ln A_2)A_1^{\frac{1}{\ln A_2}} = 12.100$$

Therefore, $\lambda_2 \equiv \lambda^* = 0.225$ and $U_2 \equiv U^* = 12.100$

Also, from equation (3.1)

$$J = \max E_C \left[\int_{t_0}^{t_f} e^{-\eta t} U(A(t)) dt \right]$$

$$= \max U(A) \max \int_0^1 e^{-\eta t} dt$$

$$= U^* \max \frac{1}{\eta}(1 - e^{-\eta}) = \frac{1}{\eta} U^*$$

Given in Table 4.1 is the net present value J as $\frac{A}{\eta+1}$ This implies

$$\frac{A_2}{\eta + 1} = \frac{U^*}{\eta}$$

Thus,

$$\eta^* = \frac{U^*}{A_2 - U^*}$$

$$\Rightarrow \eta^* = 0.164$$

Therefore, using the calculated optimal values, the following results are obtained.

Table 4.8: Numerical result of Expected present value of utility of asset $J(\psi)^\alpha$ with different α-paths

α	$J(\psi)^\alpha$
10^{-6}	1.152×10^6
.1	25619.235
.24	6056.883
.38	448.529
.52	429.982
.64	3372.794
.8	15826.851
.999999	1.273×10^6

5. CONCLUSION

Based on uncertainty theory, an optimal control model of capital asset was formulated by adapting the expected value operator to quantify the objective rewards in the model. Furthermore, the model was solved by using some optimality criteria to derive the optimal values of some input factors, thus applied to a real life problem. In future work, the objective of the model may be viewed from the perspective of stakeholders or regulators other than that of the investors.

REFERENCES

[1] Black, F., Jensen, M. C. & Scholes, M. (1972). The capital asset pricing model: Some empirical tests. In M. Jensen (Ed.), *Studies in the Theory of Capital Markets* (pp. 79-121). NewYork: Praeger Publishers.

[2] Deng, L. & Zhu, Y. (2012). An uncertain optimal control model with n jumps and application. *ComSIS 9*(4), 1453-1468.

[3] Douglas, G. W. (1968). *Risk in the equity markets: An empirical appraisal of market efficiency*. University Microfilms, Inc., Ann Arbor, Michigan.

[4] Fama, E. F. & MacBeth, J. D. (1973). Risk, return, and equilibrium: Empirical tests. *Journal of Political Economy, 71*, 607-636.

[5] Friend, I. & Blume, M. (1970). Measurement of portfolio performance under uncertainty. *American Economic Review, 60*(4), 607-636.

[6] Latunde T. & Bamigbola O. M. (2016). Uncertain optimal control model for management of net risky capital asset. *IOSR Journal of Mathematics (IOSR-JM), 12*(3), 22-30.

[7] Latunde, T. & Bamigbola, O. M. (2018). Parameter estimation and sensitivity analysis of an optimal control model for capital asset management. *Advances in Fuzzy Systems, 2018*, 1-11. https//doi.org/10.1155/2018/4756520

[8] Liu, B. (2007). *Uncertainty theory* (2nd ed.). Berlin: Springer.

[9] Liu, B. (2008). Fuzzy process, hybrid process and uncertain process, *Journal of Uncertain Systems, 2*(1), 3-16.

[10] Liu, B. (2009). *Theory and practice of uncertain programming* (2nd ed.). Berlin: Springer Verlag.

[11] Liu, B. (2016). *Uncertain theory* (5th ed.). China: Uncertainty Theory Laboratory.

[12] Merton, R. C. (1971). Optimal consumption and portfolio rules in a continuous time model. *Journal of Economic Theory, 42*(3), 373-413.

[13] Miller, M. & Scholes, M. (1972). Rates of return in relation to risk: A re-examination of some recent findings. *Studies in the Theory of Capital Markets*, Michael C. Jensen, ed., Praeger, New York, 47-78.

[14] Stambaugh, R. F. (1982). On the exclusion of assets from tests of two-parameter model: A sensitivity analysis. *Journal of Financial Economics, 10*(3), 237-268.

[15] Stein, J. L. (2003). Stochastic optimal control modelling of debt crises. CESifo (Working Paper 1043), 1-23.

[16] Yao, K. & Chen, X. (2013). A numerical method for solving uncertain differential equations. *Journal of Intelligent Fuzzy Systems, 25*(3), 825-832.

[17] Yang, X. & Shen, Y.(2015). Runge-Kutta method for solving uncertain differential equations, *Journal of Uncertainty Analysis and Applications, 3*(17), 1-12.

[18] Yao, K. (2013). Extreme values and integral of solution of uncertain differential equation. *Journal of Uncertainty Analysis and Applications*, *1*(2), 1-21.

[19] Zhu, Y. (2010). Uncertain optimal control with application to a portfolio selection model. *Cybernetics and Systems: An International Journal, 41*, 535-547.

