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Chaotic Analysis and Improved Finite-Time Adaptive Stabilization of a Novel 4-D Hyperchaotic System

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Abstract: Chaotic systems are evidently very sensitive to slight perturbations in their algebraic structures and initial conditions, which can result in unpredictability of their future states. This characteristics has rendered them very useful in modelling and design of engineering and non-engineering systems. Using the Burke-Shaw chaotic system as a reference template, a special case of a novel 4-D hyperchaotic system is proposed. The system consists of 10 terms and 9 bounded parameters. In this paper, after the realization of a mathematical model of the novel system, we designed an autonomous electronic circuit equivalent of the model and subsequently proposed an improved adaptive finite-time stabilizing controller which incorporates some augmented strength coefficients in the derived controller structures. These augmented coefficients greatly constrained transient overshoots and resulted in a faster convergence time for the controlled trajectories of the novel system. This novel system is suitable for application in the modelling and design of information security systems such as image encryption and multimedia security systems, due to its good bifurcation property.

Keywords: chaotic analysis, adaptive finite-time stabilization, hyperchaos, Lyapunov stability.

1. INTRODUCTION

Chaos is a unique phenomenon that often occur in a class of dynamic systems that are sensitive to perturbation in their initial conditions or mathematical models, consequently resulting in unpredictability of their future states [1]. Chaos has been found to exists in natural and man-made systems, and its dynamics have been used in the modelling and studies of practical and hypothetical systems in communications engineering [2], medicines [3], robotics [4], thermodynamics [5], waste water treatment plant [6], electric power system [7], amongst others. However, for chaos to be of practical use, it must be controllable. Consequently, during the past decades since the seminal works of Ott, Grebogi and Yorke [8], extensive research on chaos control has resulted in the proposition of different methods in the literature and their applications by researchers in controlling and stabilizing chaotic dynamics such as active control [9], sliding mode control [10], fuzzy control [11] and adaptive hybrid control [12], amongst others. Classical stability concepts such as the Lyapunov stability, BIBO stability and asymptotic stability concepts [13], [14] do not take into consideration the time intervals of stabilization of systems. Rather, the main objective is the stabilization which can continue into infinite time. In recent years however, the concept of finite time stability and stabilization has gained wide attention due to its usefulness in time-critical system designs including secure communication systems, image and data encryption systems. A dynamic system is said to be finite-time stable (or synchronized), if given a bound on the initial conditions, its states does not exceed a certain threshold during a specified interval. Finite-time stability is essential to ensure that the trajectories of a targeted system does not overshoot a specified bound in order to avoid undesirable consequences

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during applications. There is a distinction between finite time stability and Lyapunov stability. A system which is finite-time stable may be Lyapunov asymptotically stable, whereas a system that is Lyapunov asymptotically stable may not be finite-time stable, if during transient it uncontrolled dynamics exceeds a prescribed threshold of time [15]. Many works on finite time control have appeared in recent year [16]–[18]. In some of the reference works listed under Section Four, however, the settling times are comparably large, even when the controllers featured the use of sign function. In the proposed adaptive controller in this paper, we proposes a control structure that offers a faster rate of convergence of the controlled trajectories of a novel special case of a hyperchaotic system [19], which was derived from the Burke – Shaw atmospheric model [20]. Certain characteristics distinguished the novel system from several others in the literature. These include its unique bifurcation properties where the plots of control parameters depict bifurcation diagrams with fewer periodic windows, thus making the system a good candidate for application in chaos-based cryptosystem modelling and design. The elegance of the algebraic structure makes it possible to further reduce the algebraic structure to consists of fewer parameters, yet would still exhibit hyperchaotic features with some degrees of novelty in its qualitative features. The novel system is also highly sensitive to slightest change in its initial conditions and system parameters. Thus, several attractors may be evolved through key sensitivity.

1.1 Concepts and Preliminaries

Throughout this paper, the following existing definitions and Lemmas are used in the analysis and synthesis of the improved finite-time controller design for stabilization. Consider a class of nonlinear systems given by

$$\dot{x} = f(x,t),\ x(t_0) = x(0) = x_0 \tag{1.1}$$

Where $\dot{x} = f(x,t),\ x(t_0) \in \mathbb{R}^n$ is the system state and $\dot{x} = f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function. Assume that the origin O is an equilibrium point, $0(0,0,0,0)$ of (1). If there exist a constant $T > 0$ where T may be influenced by the initial condition x_0 such that $\lim_{t \to T} \|x(t)\| \equiv 0$, $\lim_{t \to T} \|x(t)\| \equiv 0$, if $t > T$, then the system (1) is finite-time stable.

Definition 1.1 [15], [21]

The origin of (1.1) is a finite-time stable equilibrium, if the origin is Lyapunov stable and there exist a function $T : \mathbb{R}^n \to \mathbb{R}^+$ called the settling time function, such that for every $x_0 \in \mathbb{R}^n$, the solution $x(t, x_0)$ of (1.1) is defined on $[0, T(x_0), x(t, x_0) \in \mathbb{R}^n\ \forall t \in [0, T(x_0)]]$ and $\lim_{t \to T(x_0)} x(t, x_0) = 0$.

Lemma 1.1 [22], [23]

If there exist a continuous positive definite function $V(t) : \mathbb{R}^n \to \mathbb{R}^n$, such that $V(t)$ is radially bounded (i.e. $V(x(t)) \to \infty$ as $x(t) \to \infty$), and satisfies the following differential inequality:

$$V(t) \le -\varphi(V(t))^q,\ \forall t \ge 0,\ T \ge t_0,\ V(t_0) \ge 0, \tag{1.2}$$

Where $\varphi > 0$ and $0 < q > 1$ are two positive numbers. It follows that for any t_0, $V(t)$ satisfies the inequality:

$$V^{1-q}(t) \le V^{1-q}(t_0) - \varphi(1-q)(t - t_0),\ t_0 \le t \le T \tag{1.3}$$

and $V(t) \equiv 0,\ \forall t \ge T$ *. We conclude that the origin of (1.1) is globally stable in finite time* T *and the settling time* T *is given by the relationship*:

$$T = t_0 + \frac{1}{\varphi(1-q)} V^{1-q}(t_0) \tag{1.4}$$

Assumption 1 [24], [25]

Assume $a_1, a_2, ..., a_n \in \mathbb{R}^n$ *and* $0 < \gamma < 1$ *are all real numbers. Then the following inequality holds:*

$$(|a_1| + |a_2| + ... + |a_n|)^r \le |a_1|^r + |a_2|^r + ... + |a_n|^r \tag{1.5}$$

Lemma 1.2 [25]

For the dynamic system (1.1), if there exists a continuous differentiable function $V:[0,\infty)\times D \to \mathbb{R}$, *class* K_∞ *function* $a(.)$ *and* $b(.)$ *, a function* $k:[0,\infty) \to \mathbb{R}_+$ *, such that* $k(t) > 0$ *for almost all* $t \in [0,\infty)$, *a real number* $v \in (0,1)$ *and an open neighbourhood* $M \subseteq D$ *of the origin, such that:*

$$\begin{aligned} &V(t,0) = 0, t \in [0,\infty) \\ &a(\|x\|) \le V(t,x) \le b(\|x\|), t \in [0,\infty), x \in M, \\ &\dot{V}(t,x) \le -k(t)(V(t,x))^{\sigma}, t \in [0,\infty), x \in M, \end{aligned} \tag{1.6}$$

holds, then the equilibrium point $x(t) = 0$ *of the system (1.1) is uniformly finite-time stable with settling time function* $T(t_0, x_0)$ *satisfying* $T(t_0,x_0) \le K^{-1}[\frac{V(t_0,x_0)^{1-\kappa}}{1-\kappa}], (K \ne \kappa)$, *where* $K(t) = \int_{t_0}^{t} k(s)ds$ *. If* $M = D = \mathbb{R}^n$ *, then the equilibrium* $x(t) = 0$ *of the system (1.1) is globally uniformly finite-time stable.*

2. MAIN RESULTS

It is well known in practical system applications that system parameters are not always known in advance due to the uncertainties that inevitable arises during operations. As a result, practical controllers are designed with uncertainties in focus. In this section, the aim is to design a finite-time adaptive stabilizing control laws that will stabilizing the unstable dynamics of the novel system and update the unknown parameters in a uniform finite time.

2.1 Mathematical model of the novel system

Consider a general structure of a novel 4D hyperchaotic system which is inspired by the Burke-Shaw atmospheric model template [19],

$$\begin{cases} \dot{x}_1 = -\alpha_1(x_1 + x_2) + \alpha_2 x_3 \\ \dot{x}_2 = -\alpha_3 x_1 x_3 + \alpha_4 x_2 + \alpha_5 x_4 \\ \dot{x}_3 = \alpha_6 x_1 x_2 + \alpha_7 \\ \dot{x}_4 = -\alpha_8 x_1 - \alpha_9 x_2 - \alpha_{10} x_3 - \alpha_{11} x_4 \end{cases} \tag{2.1}$$

Where $\alpha_1 - \alpha_{11}, \forall \alpha_i, \alpha_i > 0$ are bounded parameter vectors and $x \in [x_1, x_2, ..., x_4]$ are the state variables. System (2.1) may be considered as a generic structure of a family of the novel hyperchaotic systems which can produce special cases that still exhibit hyperchaos, yet possessing qualitative and quantitative properties which are conditioned by nullifying one or more system parameters, in addition to appropriate selection of system parameter values. In this paper, we nullified two terms $\alpha_2 = \alpha_{11} = 0$, resulting in a new 10 terms and 9 parameter system given by

$$\begin{cases} \dot{x}_1 = -\alpha_1(x_1 + x_2) \\ \dot{x}_2 = -\alpha_3 x_1 x_3 + \alpha_4 x_2 + \alpha_5 x_4 \\ \dot{x}_3 = \alpha_6 x_1 x_2 + \alpha_7 \\ \dot{x}_4 = -\alpha_8 x_1 - \alpha_9 x_2 - \alpha_{10} x_3 \end{cases} \tag{2.2}$$

When $\alpha_1 = 7, \alpha_3 = 3.1,$ $\alpha_4 = 3.5, \alpha_5 = 0.95,$ $\alpha_6 = 1, \alpha_7 = 13,$ $\alpha_8 = 0.01, \alpha_9 = 3, \alpha_{10} = 0.01$, system (2.2) exhibit hyperchaotic behaviour depicted in Fig. 1 (a) – (f) after 200s.

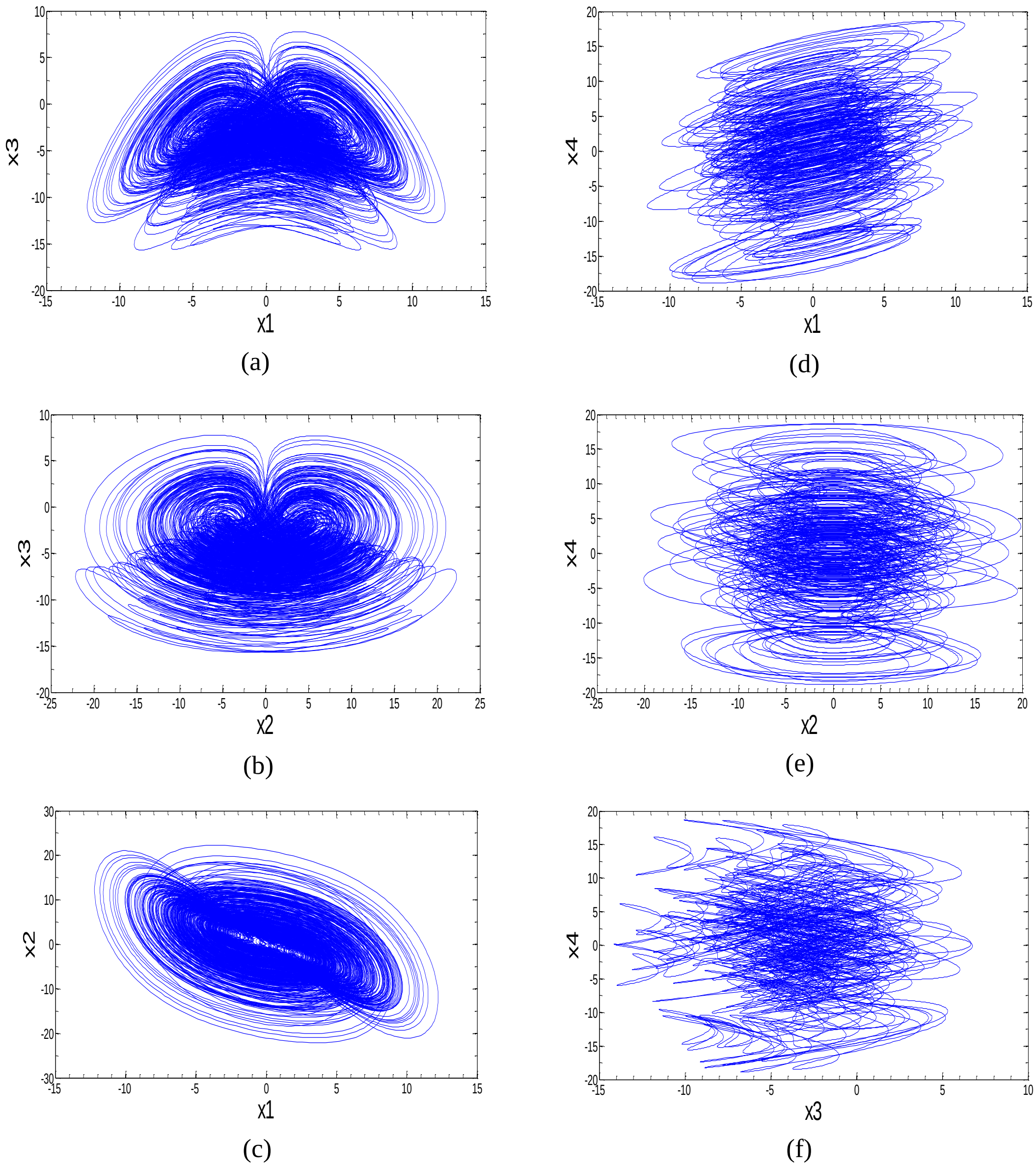

Fig. 2.1. 2-D phase portraits of the novel hyperchaotic system

2.2 Qualitative and quantitative properties of the hyperchaotic system

2.2.1 Dissipativity

We can examine whether the system is dissipative or otherwise by applying the Liouville divergence theorem [26], [27]. Let the vector notation of the 4-D hyperchaotic system (2.2) be denoted by

$$\frac{d\dot{x}_i}{dt} = f(x_i) = \begin{pmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ f(x_4) \end{pmatrix} = \begin{cases} f(x_1) = -a_1(x_1 + x_2) \\ f(x_2) = -a_3 x_1 x_3 + a_4 x_2 + a_5 x_4 \\ f(x_3) = a_6 x_1 x_2 + \alpha_7 \\ f(x_4) = -a_8 x_1 - a_9 x_2 - a_{10} x_3 \end{cases} \tag{2.3}$$

Suppose Ψ is a region in the phase space $\mathbb{R}^4$ with a smooth boundary and $\Psi(t) = \Phi_t$ where Φ_t is the flow of the system (2.3). If V is a hypervolume of the phase space at time $t = 0$, then by Liouville's divergence theorem [28],

$$\frac{dV(t)}{dt} = \int_{\Psi_t} \nabla . f dx_1 dx_2 dx_3 dx_4 \tag{2.4}$$

Where $\nabla . f = \sum_{i=1}^{4} \frac{df(x_i)}{dx_i}$ is the divergence of the vector field f of the system (2.3). Using the theorem, the rate of volume contraction is given by the Lie derivative [33]

$$\frac{1}{V}\frac{dV}{dt} = \sum_i \frac{\partial \dot{\Phi}_i}{\partial \Phi_i} \quad i = 1, 2, 3... \tag{2.5}$$

Where $\Phi_1 = x_1, \Phi_2 = x_2, \Phi_3 = x_3, \Phi_4 = x_4$ are the state variables of the system (2.3). The divergence of the vector field f on $\mathbb{R}^4$ can be obtained by using the relationship

$$\nabla . f = \frac{df(x_1)}{dx_1} + \frac{df(x_2)}{dx_2} + \frac{df(x_3)}{dx_3} + \frac{df(x_4)}{dx_4} \tag{2.6}$$

By using (2.3), (2.4) and (2.6), the divergence is

$$\nabla f = -\alpha_1 + \alpha_4 = -(\alpha_1 - \alpha_4) \tag{2.7}$$

For $\alpha_1 = 7, \alpha_4 = 3.5$, $\nabla f = -3.5$. Since the divergence is negative, it implies that the hyperchaotic system is dissipative.

2.2.2 Equilibria and local stability

Hyperchaotic systems are essentially nonlinear models, and are therefore linearized in order to study their local stability for different parameters. The stability is determined by the sign of the real part of the eigenvalues of the Jacobian matrix. The Jacobian matrix is the matrix of the partial derivatives of the right-hand side with respect to state variables where all derivatives are evaluated at the equilibrium point $x = x_e$ and is expressed by

$$J(x)=\begin{pmatrix} \frac{\partial f(x_1)}{\partial x_1} & \frac{\partial f(x_1)}{\partial x_2} & \frac{\partial f(x_1)}{\partial x_3} & \frac{\partial f(x_1)}{\partial x_4} \\ \frac{\partial f(x_2)}{\partial x_1} & \frac{\partial f(x_2)}{\partial x_2} & \frac{\partial f(x_2)}{\partial x_3} & \frac{\partial f(x_2)}{\partial x_4} \\ \frac{\partial f(x_3)}{\partial x_1} & \frac{\partial f(x_3)}{\partial x_2} & \frac{\partial f(x_3)}{\partial x_3} & \frac{\partial f(x_3)}{\partial x_4} \\ \frac{\partial f(x_4)}{\partial x_1} & \frac{\partial f(x_4)}{\partial x_2} & \frac{\partial f(x_4)}{\partial x_3} & \frac{\partial f(x_4)}{\partial x_4} \end{pmatrix} \tag{2.8}$$

When $x=0$, the equilibrium is at the origin, i.e. $E(x_1=0, x_2=0, x_3=0, x_4=0)$. The equilibrium point of (2.3) at any point $x \in R^4$ (other than the origin), that is, $E(x_1^* \neq 0, x_2^* \neq 0, x_3^* \neq 0, x_4^* \neq 0)$ is calculated by using the matrix (2.8).

$$J(x)=\begin{pmatrix} -\alpha_1 & -\alpha_1 & 0 & 0 \\ -\alpha_3 x_3^* & \alpha_4 & -\alpha_3 x_1^* & \alpha_5 \\ \alpha_6 x_2^* & \alpha_6 x_1^* & 0 & 0 \\ -\alpha_8 & -\alpha_9 & -\alpha_{10} & 0 \end{pmatrix} \tag{2.9}$$

Analyzing (2.9), the equilibrium points are located at

$$\begin{aligned} &E_+(B, -B, AB/\alpha_{10}, CB-DA) \\ &E_-(-B, B, -AB/\alpha_{10}, -CB-DA) \end{aligned} \tag{2.10}$$

Where

$$\begin{aligned} A &= \alpha_8 - \alpha_9 \\ B &= \sqrt{\alpha_7/\alpha_6} \\ C &= \alpha_4/\alpha_5 \\ D &= \alpha_3\, \alpha_7/\alpha_5\alpha_6\alpha_{10} \end{aligned} \tag{2.11}$$

To test for the type of stability associated with each equilibrium point, eq. (2.10) were computed as follows:

$$E_+ = \begin{bmatrix} 3.61 \\ -3.61 \\ 1078 \\ -12670.28 \end{bmatrix}, \quad E_- = \begin{bmatrix} -3.61 \\ 3.61 \\ -1078 \\ -12696.88 \end{bmatrix} \tag{2.12}$$

Using E_+ and E_- respectively in the matrix (2.9) gives an indication of the nature of stability of these equilibrium points i.e.

$$J(E_+)=\begin{bmatrix} -7 & -7 & 0 & 0 \\ -3341.8 & 3.5 & -11.191 & 0.95 \\ -3.61 & 3.61 & 0 & 0 \\ -0.01 & -3 & -0.01 & 0 \end{bmatrix} \tag{2.13}$$

The eigenvalues of (2.13) are $\lambda_1 = 0.000, \lambda_2 = -154.911, \lambda_3 = 150.886, \lambda_4 = 0.0251$, which implies that it is a saddle and unstable. Next, we have

$$J(E_-)=\begin{bmatrix} -7 & -7 & 0 & 0 \\ 3341.8 & 3.5 & 11.191 & 0.95 \\ 3.61 & -3.61 & 0 & 0 \\ -0.01 & -3 & -0.01 & 0 \end{bmatrix} \tag{2.14}$$

The eigenvalues of (2.14) are $\lambda_1 = 0.000, \lambda_2 = -155.032, \lambda_3 = 151.000, \lambda_4 = 0.025$ which implies it is a saddle and unstable.

2.2.3 Lyapunov exponents and Kaplan-Yorke dimension

The Lyapunov exponent measures qualitatively the rate of exponential divergence or convergence of nearby trajectories of the system in state space, while the Kaplan-Yorke dimension (Lyapunov dimension) gives a quantitative measure of this divergence. The Lyapunov exponents was calculated based on the Wolf algorithm [29], in conjunction with the procedure which implements the Gram-Schmidt ortho-normalization in the MATLAB environment. The numerically computed values are $LE_1 = 2.216,\ LE_2 = 1.280,\ LE_3 = 0.000,$ and $LE_4 = -6.997$. The system has two positive Lyapunov exponents, a null and a negative Lyapunov exponent (+, +, 0, -), which confirmed its hyperchaoticity. Moreover, $LE_1 > LE_2 > LE_3 > LE_4$ and $LE_1 + LE_2 + LE_3 + LE_4 < 0$, hence the novel system is dissipative. The Kaplan-Yorke dimension [30] is given by

$$D_{KY} = D + \frac{1}{|LE_{D+1}|} \sum_{j=1}^{D} LE_j, \quad j = 1, 2, 3 \tag{2.15}$$

Where D is the topological dimension of the attractor, and must satisfy $\sum_{j=1}^{D} LE_j \geq 0$. For regular chaos with three dimensions, the topological dimension is 2, while for hyperchaos, it is 3. By using the numerically generated values of Lyapunov exponents, the Kaplan-Yorke dimension is calculated as

$$D_{KY} = 3 + \frac{LE_1 + LE_2 + LE_3}{|LE_4|} = 3.4996$$

2.2.4. Bifurcation diagrams

Bifurcation diagram shows how the dynamics of a system changes with variation of control parameters. In this paper, we explored the parameter space to discover which parameters influence the dynamic characteristics of the system. Two parameters α_1 and α_3 influences the route to chaos. Fig. 2.2 depicts the bifurcation diagrams plotted by varying the two control parameters for the range $6.3 \leq \alpha_1 \leq 6.5$ and $3.5 \leq \alpha_1 \leq 4.15$.

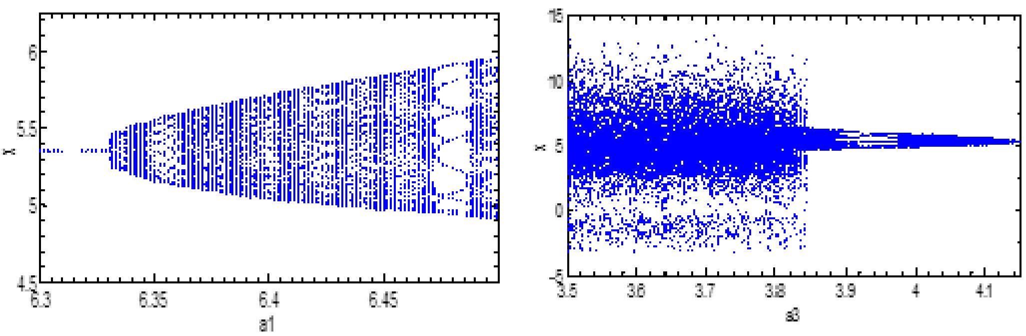

Fig. 2.2. Bifurcation diagrams of control parameters α_1, α_3

3. FINITE-TIME ADAPTIVE STABILIZATION

The objective of finite-time adaptive stabilization is to design an adaptive control law and parameter update laws such that the state and parameter update trajectories of the controlled system (2.16) converged in uniform finite time for any initial condition. Let the controlled form of system (2.3) be given as follows:

$$\begin{cases} \dot{x}_1 = -\hat{\alpha}_1(x_1 + x_2) + u_1 \\ \dot{x}_2 = -\hat{\alpha}_3 x_1 x_3 + \hat{\alpha}_4 x_2 + \hat{\alpha}_5 x_4 + u_2 \\ \dot{x}_3 = \hat{\alpha}_6 x_1 x_2 + \hat{\alpha}_7 + u_3 \\ \dot{x}_4 = -\hat{\alpha}_8 x_1 - \hat{\alpha}_9 x_2 - \hat{\alpha}_{10} x_3 + u_4 \end{cases} \quad (3.1)$$

Where $\hat{\alpha}_1, \hat{\alpha}_3 - \hat{\alpha}_{10}$ are the unknown parameters to be estimated, $u_i = D_{x_i} - A_{x_i}, (i = 1, 2..4)$ is the finite-time adaptive control functions associated with each coupled equation and comprises of D_{x_i} , the model equation-based derivative of the control functions and $A_{x_i} = -x_i^{\kappa} - \xi_{u_i} \operatorname{sgn}(x_i)|x_i|^{\kappa}, (i = 1, 2..4)$, the augmented control functions and ξ_{u_i} are the augmented controller strength coefficients.

Lemma 3.1

The unknown parameters of system (3.1) can be estimated if there exists parametric error functions $\bar{\alpha}_i, (i = 1, 3, ...10)$*, where :*

$$\begin{aligned} &\bar{\alpha}_1 = \alpha_1 - \hat{\alpha}_1,\ \bar{\alpha}_3 = \alpha_3 - \hat{\alpha}_3,\ \bar{\alpha}_4 = \alpha_4 - \hat{\alpha}_4, \\ &\bar{\alpha}_5 = \alpha_5 - \hat{\alpha}_5,\ \bar{\alpha}_6 = \alpha_6 - \hat{\alpha}_6,\ \bar{\alpha}_7 = \alpha_7 - \hat{\alpha}_7, \\ &\bar{\alpha}_8 = \alpha_8 - \hat{\alpha}_8,\ \bar{\alpha}_9 = \alpha_9 - \hat{\alpha}_9, \bar{\alpha}_{10} = \alpha_{10} - \hat{\alpha}_{10} \end{aligned} \quad (3.2)$$

$\hat{\alpha}_1, \hat{\alpha}_3 - \hat{\alpha}_{10}$ *are the unknown parameters to be estimated. Taking the derivatives of (3.2) yields the following time-varying functions*:

$$\begin{aligned} &\dot{\bar{\alpha}}_1 = -\dot{\hat{\alpha}}_1,\ \dot{\bar{\alpha}}_3 = -\dot{\hat{\alpha}}_3, \dot{\bar{\alpha}}_4 = -\dot{\hat{\alpha}}_4,\ \ \dot{\bar{\alpha}}_5 = -\dot{\hat{\alpha}}_5,\ \dot{\bar{\alpha}}_6 = -\dot{\hat{\alpha}}_6 \\ &\dot{\bar{\alpha}}_7 = -\dot{\hat{\alpha}}_7,\ \dot{\bar{\alpha}}_8 = -\dot{\hat{\alpha}}_8,\ \dot{\bar{\alpha}}_9 = -\dot{\hat{\alpha}}_9, \dot{\bar{\alpha}}_{10} = -\dot{\hat{\alpha}}_{10} \end{aligned} \quad (3.3)$$

Remark 3.1

Several works have featured the use of sign function in finite-time controller design due to its good tracking properties [18], [31] *etc. In these references, the designed controllers exhibit good noise rejection and good convergence time than controllers that do not feature the signum variables. However, their convergence time are still appreciably large. In this present work therefore, we introduced coefficient terms called augmented controller strength coefficients and augmented parameter update strength coefficients respectively, in conjunction with signum variables to produces controller and parameter update structures that drastically cut down on the uniform convergence time of the dynamics of the controlled system.*

3.1 The proposed controller and parameter update structures

The controlled hyperchaotic system (3.1) can be finite-time adaptive stabilized and the unknown parameters can be accurately estimated in uniform finite time, if the following adaptive stabilizing control laws and parameter update laws are applied:

$$\begin{cases} u_1 = \hat{\alpha}_1(x_1+x_2) - x_1^{\kappa} - \xi_{u_1}\,\mathrm{sgn}(x_1)|x_1|^{\kappa} \\ u_2 = \hat{\alpha}_3 x_1 x_3 - \hat{\alpha}_4 x_2 - \hat{\alpha}_5 x_4 - x_2^{\kappa} - \xi_{u_2}\,\mathrm{sgn}(x_2)|x_2|^{\kappa} \\ u_3 = -\hat{\alpha}_6 x_1 x_2 - \hat{x}_7 - x_3^{\kappa} - \xi_{u_3}\,\mathrm{sgn}(x_3)|x_3|^{\kappa} \\ u_4 = \hat{\alpha}_8 x_1 + \hat{\alpha}_9 x_2 + \hat{\alpha}_{10} x_3 - x_4^{\kappa} - \xi_{u_4}\,\mathrm{sgn}(x_4)|x_4|^{\kappa} \\ \dot{\hat{\alpha}}_1 = x_1(x_1+x_2) + \xi_{\alpha_1}\bar{\alpha}_1 \\ \dot{\hat{\alpha}}_3 = x_1 x_2 x_3 + \xi_{\alpha_3}\bar{\alpha}_3 \\ \dot{\hat{\alpha}}_4 = -x_2^2 + \xi_{\alpha_4}\bar{\alpha}_4 \\ \dot{\hat{\alpha}}_5 = -x_2 x_4 + \xi_{\alpha_5}\bar{\alpha}_5 \\ \dot{\hat{\alpha}}_6 = -x_1 x_2 x_3 + \xi_{\alpha_6}\bar{\alpha}_6 \\ \dot{\hat{\alpha}}_7 = -x_3 + \xi_{\alpha_7}\bar{\alpha}_7 \\ \dot{\hat{\alpha}}_8 = x_1 x_4 + \xi_{\alpha_8}\bar{\alpha}_8 \\ \dot{\hat{\alpha}}_9 = x_2 x_4 + \xi_{\alpha_9}\bar{\alpha}_9 \\ \dot{\hat{\alpha}}_{10} = x_3 x_4 + \xi_{\alpha_{10}}\bar{\alpha}_{10} \end{cases} \tag{3.4}$$

Where ξ_{α_1} are the augmented parameter update strength coefficients (ξ_{α_i} has the same definition with ξ_{u_i}) and $0<\kappa<1$ is a non-negative index.

Proof. Firstly, by using (3.2) and (3.4) in (3.1), the controlled system (3.1) becomes

$$\begin{cases} \dot{x}_1 = \bar{\alpha}_1(x_1+x_2) - x_1^{\kappa} - \xi_{u_1}\,\mathrm{sgn}(x_1)|x_1|^{\kappa} \\ \dot{x}_2 = \bar{\alpha}_3 x_1 x_3 - \bar{\alpha}_4 x_2 - \bar{\alpha}_5 x_4 - x_2^{\kappa} - \xi_{u_2}\,\mathrm{sgn}(x_2)|x_2|^{\kappa} \\ \dot{x}_3 = -\bar{\alpha}_6 x_1 x_2 - \bar{x}_7 - x_3^{\kappa} - \xi_{u_3}\,\mathrm{sgn}(x_3)|x_3|^{\kappa} \\ \dot{x}_4 = \bar{\alpha}_8 x_1 + \bar{\alpha}_9 x_2 + \bar{\alpha}_{10} x_3 - x_4^{\kappa} - \xi_{u_4}\,\mathrm{sgn}(x_4)|x_4|^{\kappa} \end{cases} \tag{3.5}$$

Secondly, based on Lemma 1.2, a Lyapunov function candidate is given by

$$\begin{aligned} V(t) &= V_x + V_p = \sum_{i=1}^{4}\frac{1}{2}x_i^2 + \sum_{i=1}^{10}\frac{1}{2}\bar{\alpha}_i^2 \\ &= \frac{1}{2}(x_1^2+x_2^2+x_3^2+x_4^2) + \frac{1}{2}(\bar{\alpha}_1^2+\bar{\alpha}_3^2+\bar{\alpha}_4^2+\bar{\alpha}_5^2+\bar{\alpha}_6^2+\bar{\alpha}_7^2+\bar{\alpha}_8^2+\bar{\alpha}_9^2+\bar{\alpha}_{10}^2) \end{aligned} \tag{3.6}$$

The partial derivative of (3.6) produces

$$\begin{aligned} \dot{V}(t) &= x_1\dot{x}_1 + x_2\dot{x}_2 + x_3\dot{x}_3 + x_4\dot{x}_4 + \bar{\alpha}_1\dot{\bar{\alpha}}_1 + \bar{\alpha}_3\dot{\bar{\alpha}}_3 + \bar{\alpha}_4\dot{\bar{\alpha}}_4 + \bar{\alpha}_5\dot{\bar{\alpha}}_5 + \ldots \\ &\ldots + \bar{\alpha}_6\dot{\bar{\alpha}}_6 + \bar{\alpha}_7\dot{\bar{\alpha}}_7 + \bar{\alpha}_8\dot{\bar{\alpha}}_8 + \bar{\alpha}_9\dot{\bar{\alpha}}_9 + \bar{\alpha}_{10}\dot{\bar{\alpha}}_{10} \end{aligned} \tag{3.7}$$

By making use of (3.3), (3.4) and (3.5) in (3.7), the derivative becomes

$$\begin{aligned} \dot{V}(t) &= x_1(\bar{\alpha}_1(x_1+x_2) - x_1^{\kappa} - \xi_{u_1}^{\delta}\,\mathrm{sgn}(x_1)|x_1|^{\kappa}) + x_2(\bar{\alpha}_3 x_1 x_3 - \bar{\alpha}_4 x_2 - \bar{\alpha}_5 x_4 - \ldots \\ &\ldots - x_2^{\kappa} - \xi_{u_2}^{\delta}\,\mathrm{sgn}(x_2)|x_2|^{\kappa}) + x_3(-\bar{\alpha}_6 x_1 x_2 - \bar{x}_7 - x_3^{\kappa} - \xi_{u_3}^{\delta}\,\mathrm{sgn}(x_3)|x_3|^{\kappa}) + \ldots \\ &\ldots + x_4(\bar{\alpha}_8 x_1 + \bar{\alpha}_9 x_2 + \bar{\alpha}_{10} x_3 - x_4^{\kappa} - \xi_{u_4}^{\delta}\,\mathrm{sgn}(x_4)|x_4|^{\kappa}) - \bar{\alpha}_1\dot{\hat{\alpha}}_1 - \bar{\alpha}_3\dot{\hat{\alpha}}_3 - \ldots \\ &\ldots - \bar{\alpha}_4\dot{\hat{\alpha}}_4 - \bar{\alpha}_5\dot{\hat{\alpha}}_5 - \bar{\alpha}_6\dot{\hat{\alpha}}_6 - \bar{\alpha}_7\dot{\hat{\alpha}}_7 - \bar{\alpha}_8\dot{\hat{\alpha}}_8 - \bar{\alpha}_9\dot{\hat{\alpha}}_9 - \bar{\alpha}_{10}\dot{\hat{\alpha}}_{10} \end{aligned} \tag{3.8}$$

Using the following convention

$$\begin{aligned}&\operatorname{sgn}(x_i)=\frac{|x_i|}{x_i};\ \operatorname{sgn}(x_i)|x_i|^{\kappa}\Leftrightarrow\frac{x_i|x_i|}{x_i}=|x_i|\\&x_i\operatorname{sgn}(x_i)|x_i|^{\kappa}\Leftrightarrow|x_i|^{1+\kappa}\end{aligned}\tag{3.9}$$

Eq. (3.8) is then transformed to

$$\begin{aligned}\dot V(t)=&-x_1^{1+\kappa}-x_2^{1+\kappa}-x_3^{1+\kappa}-x_4^{1+\kappa}-\xi_{u_1}|x_1|^{\kappa}-\xi_{u_2}|x_2|^{\kappa}-\xi_{u_3}|x_3|^{\kappa}-\xi_{u_4}|x_4|^{\kappa}+\ldots\\&\ldots+\bar\alpha_1x_1(x_1+x_2)+\bar\alpha_3x_1x_2x_3-\bar\alpha_4x_2^2-\bar\alpha_5x_2x_4-\bar\alpha_6x_1x_2x_3-\bar x_7x_3+\bar\alpha_8x_1x_4+\bar\alpha_9x_2x_4+\ldots\\&\ldots+\bar\alpha_{10}x_3x_4-\bar\alpha_1\dot{\hat\alpha}_1-\bar\alpha_3\dot{\hat\alpha}_3-\bar\alpha_4\dot{\hat\alpha}_4-\bar\alpha_5\dot{\hat\alpha}_5-\bar\alpha_6\dot{\hat\alpha}_6-\bar\alpha_7\dot{\hat\alpha}_7-\bar\alpha_8\dot{\hat\alpha}_8-\bar\alpha_9\dot{\hat\alpha}_9-\bar\alpha_{10}\dot{\hat\alpha}_{10}\end{aligned}\tag{3.10}$$

And by using (3.9) in (3.10), the derivative reduces to

$$\dot V(t)=\begin{pmatrix}-x_1^{1+\kappa}-x_2^{1+\kappa}-x_3^{1+\kappa}-x_4^{1+\kappa}-\xi_{u_1}|x_1|^{\kappa}-\xi_{u_2}|x_2|^{\kappa}-\xi_{u_3}|x_3|^{\kappa}-\xi_{u_4}|x_4|^{\kappa}-\xi_{\alpha_1}\bar\alpha_1-\ldots\\\ldots-\xi_{\alpha_3}\bar\alpha_3-\xi_{\alpha_4}\bar\alpha_4-\xi_{\alpha_5}\bar\alpha_5-\xi_{\alpha_6}\bar\alpha_6-\xi_{\alpha_7}\bar\alpha_7-\xi_{\alpha_8}\bar\alpha_8-\xi_{\alpha_9}\bar\alpha_9-\xi_{\alpha_{10}}\bar\alpha_{10}\end{pmatrix}\le 0\tag{3.11}$$

Eq. (3.11) is negative definite in $\mathbb{R}^{17}$, thus Lemma 1.2 is satisfied, and the equilibrium point $x(t)=0$ is uniformly finite-time stable about the origin because $\dot V(t)\le 0$ and also implies that $\dot V(0)\le 0$ and $\lim_{t\to T}\|x\|=0$. Suppose $\xi_{u_1}=\xi_{u_2}=\xi_{u_3}=\xi_{u_4}=\xi_{u_C}$ (where ξ_{u_C} is known as the controller strength coefficient) and $\xi_{\alpha_1}=\xi_{\alpha_3}=\xi_{\alpha_4}=\ldots=\xi_{\alpha_{10}}=\xi_{\alpha_P}$ (where ξ_{u_P} is known as the parameter update coefficient), then by setting $\xi_{u_C}=\xi_{u_P}=\xi_G$, ($\xi_G$ is known as the global strength coefficient), (3.11) reduces to:

$$\dot V(t)=-\sum_{i=1}^{4}x_i^{1+\kappa}-\sum_{i=1}^{4}\xi_G|x_i|^{1+\kappa}-\sum_{i=1}^{9}\xi_G\bar\alpha_i\le 0\tag{3.12}$$

Using Assumption 1, it can be deducted that

$$\sum_{i=1}^{4}\xi_G|x_i|^{1+\kappa}\ge(\sum_{i=1}^{4}x_i^2)^{\frac{1+\kappa}{2}}=\|x\|^{1+\kappa}\tag{3.13}$$

Substituting (3.3) in (3.12) yields

$$\dot V(t)=-\sum_{i=1}^{4}x_i^{1+\kappa}-\sum_{i=1}^{11}\xi_G\bar\alpha_i-\|x\|^{1+\kappa}\tag{3.14}$$

By virtue of Assumption 1, it is easy to see that

$$\begin{aligned}\dot V(t)&\le-\|x\|^{1+\kappa}=-(\|x\|^2)^{\frac{1+\kappa}{2}}\\&=-2^{\frac{1+\kappa}{2}}(\frac{\|x\|^2}{2})^{\frac{1+\kappa}{2}}=-2^{\frac{1+\kappa}{2}}V^{\frac{1+\kappa}{2}}\end{aligned}\tag{3.15}$$

Setting $\varphi=2^{\frac{1+\kappa}{2}}$, $p=\dfrac{1+\kappa}{2}$, reduces (3.15) to

$$\dot{V}(t) \le -\varphi V^{p} \tag{3.16}$$

Also, by substituting φ and p into (1.4), we have

$$T = t_0 + \frac{V^{\frac{1-\kappa}{2}}(t_0)}{2^{\frac{1+\kappa}{2}}\left(\frac{1-\kappa}{2}\right)} = t_0 + \frac{1}{1-\kappa}\left(2V(t_0)\right)^{\frac{1-\kappa}{2}} \tag{3.17}$$

Where $V(t_0) = \|V_x(0)\|^2 + \|V_p(0)\|^2 = \|x_i(0)\|^2 + \|\bar{\alpha}_i(0)\|^2$

Remark 3.2

Eq (3.17) implies that the finite-time stabilization of the controlled system (3.1) depends on the initial condition φ and rational number p, while the uniform convergence time is either increasing or decreasing as the parameter p is varied as can be observed in the works of [18], [32] . In our case however, due to the complexity introduced by the global strength coefficient ξ_G, it was observed that the bounded value of ξ_G, $\xi_G \in [\xi_{G_{\min}}, \xi_{G_{\max}}]$ has a strong constraining effect on trajectory overshoot and also has domino effects on the uniform convergence time. Accordingly, the following cases were observed when $\kappa_{\min} \le \kappa \le \kappa_{\max}$ and $\xi_{G_{\min}} \le \xi_G \le \xi_{G_{\max}}$.

a. As $\xi_G \to \xi_{G_{\min}}$ and $\kappa \to \kappa_{\min}$, the convergence rate was relatively slower and the uniform convergence time is an increasing function

b. As $\xi_G \to \xi_{G_{\max}}$ and $\kappa \to \kappa_{\max}$, the convergence rate was relatively faster and the uniform convergence time is a decreasing function.

4. NUMERICAL SIMULATION RESULTS

The controlled hyperchaotic system is simulated using MATLAB for the following parameters $\alpha_1 = 7, \alpha_3 = 3.1$ $\alpha_4 = 3.5, \alpha_5 = 0.95, \alpha_6 = 1, \alpha_7 = 13$, $\alpha_8 = 0.01, \alpha_9 = 3, \alpha_{10} = 0.01$, with initial conditions set as $[x_1(0), x_2(0), x_3(0), x_4(0)] = [4, 2, -4, -2]$. The following results were obtained for two different cases.

Case 1: $\kappa_{\min} = 0.8, \xi_{G_{\min}} = 1000$

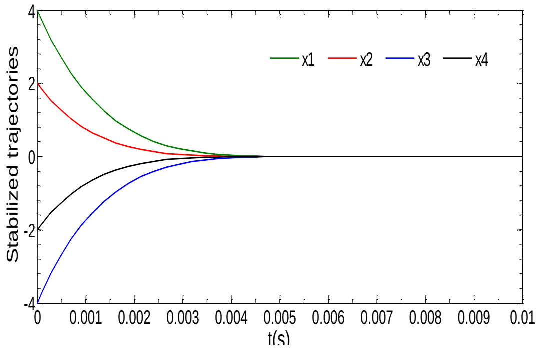

Fig.4.1. Stabilized trajectories of the controlled system

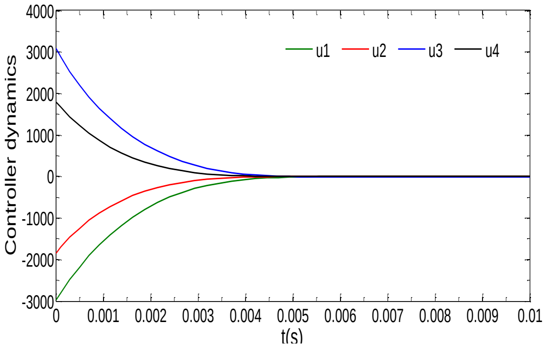

Fig.4.2. Uniformly converged dynamics of the adaptive controller

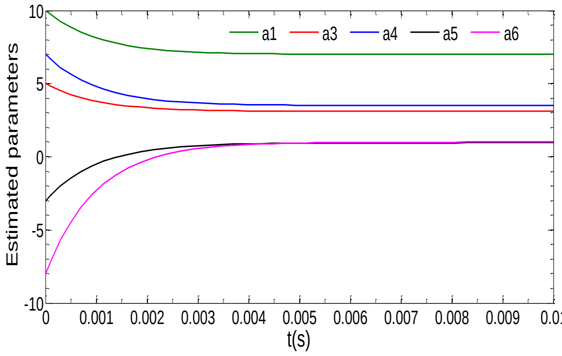

Fig.4.3 (a). Estimated parameters of the controlled system

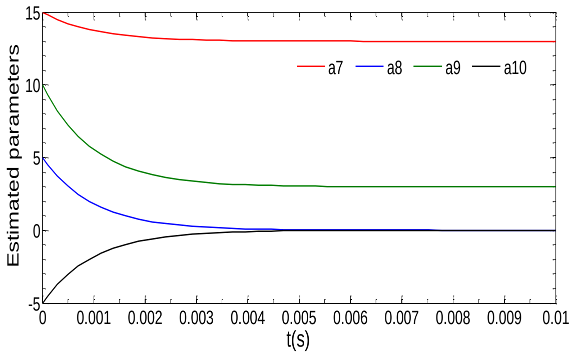

Fig.4.3 (b). Estimated parameters of the controlled system

Case 2: $\kappa_{\max} = 0.99, \xi_{G_{\max}} = 8000$

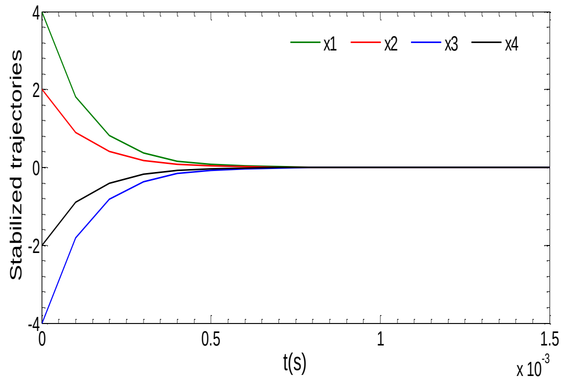

Fig.4.4. Stabilized trajectories of the controlled system

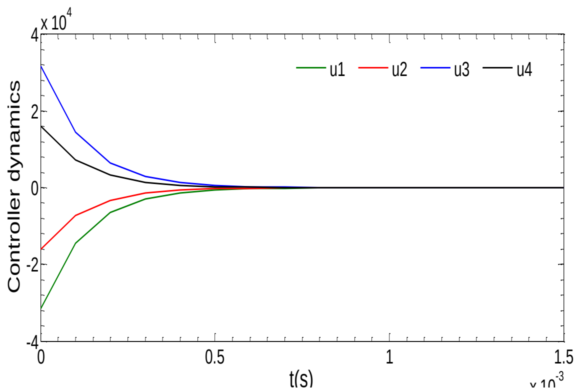

Fig.4.5. Uniformly converged dynamics of the adaptive controller

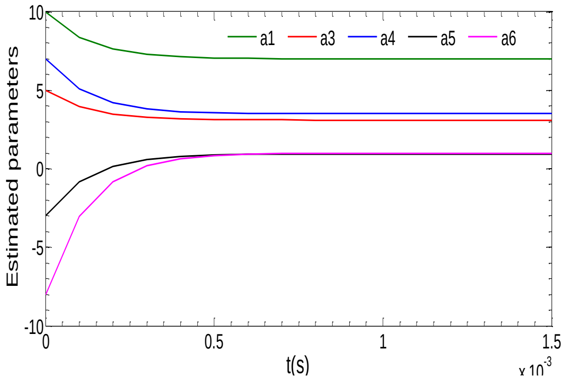

Fig.4.6 (a). Estimated parameters of the controlled system

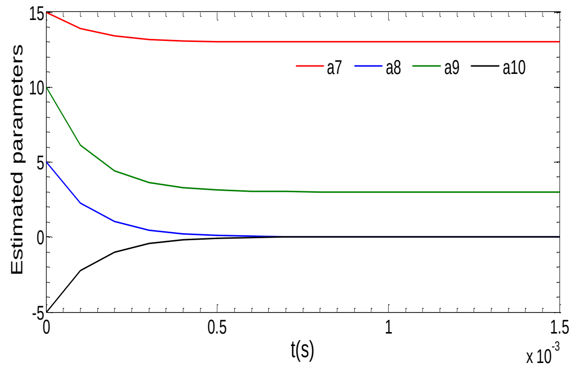

Fig.4.6 (b). Estimated parameters of the controlled system

Remark 4.1

The rate of asymptotic convergence of the system dynamics in Case 1 is slower than those in Case 2 due to the effects of the global strength coefficients. This is in consonance with the Remark 3.2 on the effects of the global strength coefficient. It can be observed that the larger ξ_G, the faster the rate of uniform convergence of the system's trajectories. When compared to controllers that featured in related works listed in Table I, it is obvious that the designed adaptive controller provides a faster convergence time than those in these works.

4.1. Comparison with related works

The rate of uniform convergence of system dynamics using the proposed finite-time adaptive controller was compared with related works in the literature. The different rates reported in the papers are given in Table I.

Table I. Comparison with related works

Related works	Approximated convergence time of error dynamic systems
Proposed work	0.005s
Ref [33]	0.2s
Ref [34]	0.3s
Ref [35]	0.5s
Ref [36]	1.0s
Ref [37]	>1.0s
Ref [38]	>1.0s
Ref [31]	2.0s
Ref [39]	>1.0s
Ref [40]	>2.0s
Ref [41]	8.0s

5. CONCLUSION

In this paper, we proposed an improved finite-time adaptive controller with a fast rate of uniform convergence of system trajectories of hyperchaotic systems. We examined the performance of the controller on a novel special case of a hyperchaotic system which was originally evolved by adding feedback control to the classical Burke-Shaw chaotic system that models some atmospheric phenomena. Numerically obtained results shows that the proposed control structure offered better rate of convergence of the controlled system dynamics when compared to related works in the literature. Overall, these systems can contribute to improve the design characteristics of chaos-based cryptosystems.

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