Adv Syst Sci Appl 2018; **4**:39–51
http://ijassa.ipu.ru/index.php/ijassa/article/view/572

Estimation of Stress-Strength Reliability for Quasi Lindley Distribution

M.M. Mohie El-Din[1], A. Sadek [1*], Shaimaa H. Elmeghawry[2]

[1] *Dep. of Mathematics, Faculty of Science, Al-Azhar University, Egypt*
[2] *Faculty of Engineering, Benha University, Egypt*

Received April 14, 2018; Revised October 17, 2018; Published December 31, 2018

Abstract: This paper discussed the problem of estimating of the stress-strength reliability $R = Pr(Y < X)$. It is assumed that the strength of a system X, and the environmental stress applied on it Y, follow the Quasi Lindley Distribution(QLD). Stress-strength reliability is studied using the maximum likelihood, and Bayes estimations. Asymptotic confidence interval for reliability is obtained. Bayesian estimations were proposed using two different methods: Importance Sampling technique, and MCMC technique via Metropolis-Hastings algorithm, under symmetric loss function (squared error) and asymmetric loss functions (linex, general entropy). The behaviors of the maximum likelihood and Bayes estimators of stress-strength reliability have been studied through the Monte Carlo simulation study. Finally analysis of a real data set has also been presented.

Keywords: Quasi Lindley distribution; Stress-strength reliability; Maximum likelihood estimation; Asymptotic confidence interval; Bayesian estimation; Importance sampling technique; MCMC technique via Metropolis-Hastings algorithm.

1. INTRODUCTION

The stress-strength models have been widely used for reliability design of systems. In these models the reliability is defined as the probability that the strength is larger than the stress $R = Pr(Y < X)$. As the strength X is larger than the stress Y, the system work efficiently, otherwise the the system fails. Estimation of stress-strength reliability was studied by several authors, for example, stress-strength model and its generalizations has been discussed in [10]. The estimation of R when X and Y are normally distributed was introduced by Church and Harris [7]. Krishnamoorthy et al. [11] introduced an inference on reliability in two-parameter exponential stress-strength model. Al-Mutairi et al. [1, 2] presented the stress-strength reliability for Lindley and weighted Lindley distributions respectively. Stress-strength reliability estimation for generalized Lindley distribution has been introduced by Singh et al. [18]. Recentely Khan and Jan [9] studied the estimation of stress-strength reliability model using finite mixture of two parameter Lindley distributions.

This paper is focused upon studying the problem of the estimation of the stress-strength reliability for the Quasi Lindley Distribution (QLD) introduced by Shanker et al. [17] of which the Lindley distribution is a particular case. We estimated the parameter of the stress-strength reliability R using the maximum likelihood, and Bayesian estimation methods. We

*Corresponding author: a_sadek@azhar.edu.eg

construct the asymptotic confidence interval of R based on the asymptotic distribution of the MLE of R. In Bayesian estimation we introduced two sampling methods (importance sampling and MCMC).
The QLD has the following probability density function (PDF):

$$f(x) = \frac{\theta}{\alpha+1}(\alpha+\theta x)\, e^{-\theta x}\ , \tag{1.1}$$

and the following cumulative distribution function (CDF):

$$F(x) = 1 - \left[\frac{(1+\alpha+\theta x)}{\alpha+1}\, e^{-\theta x}\right]\ , \tag{1.2}$$

where;

$$x > 0, \theta > 0, \alpha > -1.$$

This paper is organized as follows. In Section 2, stress-strength reliability issue is studied to obtain the reliability function of the parameters of QLD distribution. Maximum likelihood estimation for stress-strength reliability is discussed in Section 3. Asympototic confidence interval of reliability is proposed in section 4. In Section 5, a general procedure for deriving the Bayesian estimator of reliability is introduced, wherein we applied the importance sampling and MCMC techniques to compute the approximation of this estimator. Section 6 presented simulation study to investigate and compare the performance of each method of estimation. Section 7 presented analysis of a real data set for illustrative purposes. Finally, conclusions appear in Section 7.

2. STRESS-STRENGTH RELIABILITY

Assume $X \sim QLD(\theta_1, \alpha_1)$ and $Y \sim QLD(\theta_2, \alpha_2)$ are independent random variables with PDF f(x) and g(y), respectively. Then the stress strength reliability can be obtained as:

$$\begin{aligned}
R &= Pr(Y < X). \\
&= \int_0^\infty \int_0^x f(x)g(y)\ dydx. \\
&= \int_0^\infty f(x)G(x)\ dx. \\
&= \int_0^\infty \frac{\theta_1(\alpha_1+\theta_1 x)}{\alpha_1+1}\, e^{-\theta_1 x}\left[1 - \left[\frac{(1+\alpha_2+\theta_2 x)}{\alpha_2+1}\, e^{-\theta_2 x}\right]\right]\ dx. \\
&= \int_0^\infty \frac{\theta_1(\alpha_1+\theta_1 x)}{\alpha_1+1}\, e^{-\theta_1 x} - \int_0^\infty \frac{\theta_1(\alpha_1+\theta_1 x)(1+\alpha_2+\theta_2 x)\, e^{-(\theta_1+\theta_2)x}}{(\alpha_1+1)(\alpha_2+1)}\ dx. \\
&= 1 - \frac{\theta_1\left(2\theta_1\theta_2 + (\theta_1+\theta_2)\left(\alpha_2\theta_1+\alpha_1\theta_2+\theta_1\right) + \alpha_1\left(\alpha_2+1\right)\ (\theta_1+\theta_2)^2\right)}{\left(\alpha_1+1\right)\left(\alpha_2+1\right)\left(\theta_1+\theta_2\right)^3}.
\end{aligned} \tag{2.3}$$

From Eq.(2.3), we noticed that R is a function of parameters $\underline{\Theta} = (\theta_1, \alpha_1, \theta_2, \alpha_2)$. Therefore, for maximum likelihood estimate (MLE) of R we need to obtain the MLEs of these parameters.

3. MAXIMUM LIKELIHOOD ESTIMATION FOR RELIABILITY

Suppose that $x_1, x_2, \cdots, x_n$ is random sample from $QLD(\theta_1, \alpha_1)$, and $y_1, y_2, \cdots, y_m$ is random sample from $QLD(\theta_2, \alpha_2)$, then the log likelihood function can be written as:

$$\begin{aligned} logL(x,y;\underline{\Theta}) &= n\log(\theta_1) + m\log\ (\theta_2) - n\log(\alpha_1+1) - m\log(\alpha_2+1) \\ &\quad -\theta_1\sum_{i=1}^{n} x_i - \theta_2\sum_{j=1}^{m} y_j + \sum_{i=1}^{n}\log(\alpha_1+\theta_1 x_i) + \sum_{j=1}^{m}\log(\alpha_2+\theta_2 y_j). \end{aligned} \tag{3.4}$$

The MLE of $\underline{\Theta} = (\theta_1, \alpha_1, \theta_2, \alpha_2)$ can be obtained as a solution of the following equations:

$$\frac{\partial L}{\partial \theta_1} = \frac{n}{\theta_1} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\frac{x_i}{\alpha_1+\theta_1 x_i} = 0,$$

$$\frac{\partial L}{\partial \theta_2} = \frac{m}{\theta_2} - \sum_{j=1}^{m} y_j + \sum_{j=1}^{m}\frac{y_j}{\alpha_2+\theta_2 y_j} = 0,$$

$$\frac{\partial L}{\partial \alpha_1} = -\frac{n}{\alpha_1+1} + \sum_{i=1}^{n}\frac{1}{\alpha_1+\theta_1 x_i} = 0,$$

and

$$\frac{\partial L}{\partial \alpha_2} = -\frac{m}{\alpha_2+1} + \sum_{j=1}^{m}\frac{1}{\alpha_2+\theta_2 y_j} = 0.$$

Solving these equations numerically using an iterative process as Newton Raphson to get $\hat{\theta}_1, \hat{\alpha_1}, \hat{\theta}_2, \hat{\alpha_2}$, then the MLE of R can be obtained as following:

$$\hat{R} = 1 - \frac{\hat{\theta}_1\left(2\hat{\theta}_1\hat{\theta}_2 + \left(\hat{\theta}_1+\hat{\theta}_2\right)\left(\hat{\alpha_2}\hat{\theta}_1 + \hat{\alpha_1}\hat{\theta}_2 + \hat{\theta}_1\right) + \hat{\alpha_1}\left(\hat{\alpha_2}+1\right)\ \left(\hat{\theta}_1+\hat{\theta}_2\right)^2\right)}{\left(\hat{\alpha_1}+1\right)\left(\hat{\alpha_2}+1\right)\left(\hat{\theta}_1+\hat{\theta}_2\right)^3}. \tag{3.5}$$

4. ASYMPTOTIC CONFIDENCE INTERVAL OF R

The asymptotic variance-covariance matrix of all parameters can be approximated by the inverse of observed information matrix, and then derive the asymptotic distribution of $\hat{R}$. Based on the asymptotic distribution of $\hat{R}$, we obtain the asymptotic confidence interval of R. The Fisher information matrix of $\Theta = (\theta_1, \alpha_1, \theta_2, \alpha_2)$ is given as:

$$\begin{aligned} I(\Theta) &= -\begin{pmatrix} E(\frac{\partial^2 L}{\partial {\theta_1}^2}) & E(\frac{\partial^2 L}{\partial\theta_1\partial\alpha_1}) & E(\frac{\partial^2 L}{\partial\theta_1\partial\theta_2}) & E(\frac{\partial^2 L}{\partial\theta_1\partial\alpha_2}) \\ E(\frac{\partial^2 L}{\partial\alpha_1\partial\theta_1}) & E(\frac{\partial^2 L}{\partial {\alpha_1}^2}) & E(\frac{\partial^2 L}{\partial\alpha_1\partial\theta_2}) & E(\frac{\partial^2 L}{\partial\alpha_1\partial\alpha_2}) \\ E(\frac{\partial^2 L}{\partial\theta_2\partial\theta_1}) & E(\frac{\partial^2 L}{\partial\theta_2\partial\alpha_1}) & E(\frac{\partial^2 L}{\partial {\theta_2}^2}) & E(\frac{\partial^2 L}{\partial\theta_2\partial\alpha_2}) \\ E(\frac{\partial^2 L}{\partial\alpha_2\partial\theta_1}) & E(\frac{\partial^2 L}{\partial\alpha_2\partial\alpha_1}) & E(\frac{\partial^2 L}{\partial\alpha_2\partial\theta_2}) & E(\frac{\partial^2 L}{\partial {\alpha_2}^2}) \end{pmatrix}. \\ &= \begin{pmatrix} I_{11} & I_{12} & I_{13} & I_{14} \\ I_{21} & I_{22} & I_{23} & I_{24} \\ I_{31} & I_{32} & I_{33} & I_{34} \\ I_{41} & I_{42} & I_{43} & I_{44} \end{pmatrix}. \end{aligned}$$

Where:

$$I_{13} = I_{31} = 0;\ I_{14} = I_{41} = 0,$$

$$I_{23} = I_{32} = 0;\ I_{24} = I_{42} = 0,$$

$$I_{11} = -\frac{n}{\theta_1}^2 - \sum_{i=1}^{n} \frac{{x_i}^2}{(\alpha_1 + \theta_1 x_i)^2},$$

$$I_{22} = \sum_{i=1}^{n} \frac{1}{(\alpha_1 + \theta_1 x_i)^2} - \frac{n}{(\alpha_1 + 1)^2},$$

$$I_{33} = -\frac{m}{\theta_2}^2 - \sum_{j=1}^{m} \frac{{y_j}^2}{(\alpha_2 + \theta_2 y_j)^2},$$

$$I_{44} = \sum_{j=1}^{m} \frac{1}{(\alpha_2 + \theta_2 y_j)^2} - \frac{m}{(\alpha_2 + 1)^2},$$

$$I_{12} = I_{21} = \sum_{i=1}^{n} \frac{x_i}{(\alpha_1 + \theta_1 x_i)^2},$$

and

$$I_{34} = I_{43} = -\sum_{j=1}^{m} \frac{y_j}{(\alpha_2 + \theta_2 y_j)^2}.$$

Using the Central limit theorem, we obtain the following theorem :

Theorem 1: As $n \to \infty$, $m \to \infty$; then

$$(\sqrt{n}(\hat{\theta_1} - \theta_1), \sqrt{n}(\hat{\alpha_1} - \alpha_1), \sqrt{m}(\hat{\theta_2} - \theta_2), \sqrt{m}(\hat{\alpha_2} - \alpha_2)) \xrightarrow{d} N(0, I^{-1}(\Theta)).$$

Where $\xrightarrow{d}$ means converge in distribution, and $I^{-1}(\Theta)$ is the inverse of the matrix $I(\Theta)$.

In order to establish the asymptotic normality of R, we first define:

$$d(\Theta) = (\frac{\partial R}{\partial \theta_1}, \frac{\partial R}{\partial \alpha_1}, \frac{\partial R}{\partial \theta_2}, \frac{\partial R}{\partial \alpha_2})^T = (d_1, d_2, d_3, d_4)^T,$$

where T is transpose operation, and

$$\begin{aligned} d_1 \quad = \quad & -\frac{\theta_1(2\alpha_1(\alpha_2+1)(\theta_1+\theta_2) + \alpha_1\theta_2 + (\alpha_2+1)(\theta_1+\theta_2) + \alpha_2\theta_1 + \theta_1 + 2\theta_2)}{(\alpha_1+1)(\alpha_2+1)(\theta_1+\theta_2)^3} \\ & + \frac{3\theta_1\left(\alpha_1(\alpha_2+1)(\theta_1+\theta_2)^2 + (\theta_1+\theta_2)(\alpha_1\theta_2 + \alpha_2\theta_1 + \theta_1) + 2\theta_1\theta_2\right)}{(\alpha_1+1)(\alpha_2+1)(\theta_1+\theta_2)^4} \\ & - \frac{\alpha_1(\alpha_2+1)(\theta_1+\theta_2)^2 + (\theta 1+\theta_2)(\alpha_1\theta_2 + \alpha_2\theta_1 + \theta_1) + 2\theta_1\theta_2}{(\alpha_1+1)(\alpha_2+1)(\theta_1+\theta_2)^3}, \end{aligned}$$

$$
\begin{aligned}
d_2 &= \frac{\theta_1\left(\alpha_1(\alpha_2+1)(\theta_1+\theta_2)^2+(\theta_1+\theta_2)(\alpha_1\theta_2+\alpha_2\theta_1+\theta_1)+2\theta_1\theta_2\right)}{(\alpha_1+1)^2(\alpha_2+1)(\theta_1+\theta_2)^3} \\
&\quad -\frac{\theta_1\left((\alpha_2+1)(\theta_1+\theta_2)^2+\theta_2(\theta_1+\theta_2)\right)}{(\alpha_1+1)(\alpha_2+1)(\theta_1+\theta_2)^3},
\end{aligned}
$$

$$
\begin{aligned}
d_3 &= \frac{3\theta_1\left(\alpha_1(\alpha_2+1)(\theta_1+\theta_2)^2+(\theta_1+\theta_2)(\alpha_1\theta_2+\alpha_2\theta_1+\theta_1)+2\theta_1\theta_2\right)}{(\alpha_1+1)(\alpha_2+1)(\theta_1+\theta_2)^4} \\
&\quad -\frac{\theta_1(2\alpha_1(\alpha_2+1)(\theta_1+\theta_2)+\alpha_1(\theta_1+\theta_2)+\alpha_1\theta_2+\alpha_2\theta_1+3\theta_1)}{(\alpha_1+1)(\alpha_2+1)(\theta_1+\theta_2)^3},
\end{aligned}
$$

and

$$
\begin{aligned}
d_4 &= \frac{\theta_1\left(\alpha_1(\alpha_2+1)(\theta_1+\theta_2)^2+(\theta_1+\theta_2)(\alpha_1\theta_2+\alpha_2\theta_1+\theta_1)+2\theta_1\theta_2\right)}{(\alpha_1+1)(\alpha_2+1)^2(\theta_1+\theta_2)^3} \\
&\quad -\frac{\theta_1\left(\alpha_1(\theta_1+\theta_2)^2+\theta_1(\theta_1+\theta_2)\right)}{(\alpha_1+1)(\alpha_2+1)(\theta_1+\theta_2)^3}.
\end{aligned}
$$

Hence; using Theorem 1, the asymptotic distribution of $\hat{R}$, the MLE of R is defined as

$$\sqrt{n+m}(\hat{R}-R) \xrightarrow{d} N(0,B),$$

where

$$
\begin{aligned}
B &= Var(\hat{R}) \\
&= d^T(\Theta)I^{-1}(\Theta)d(\Theta).
\end{aligned}
\tag{4.6}
$$

Therefore, using Eq.(4.6), an asymptotic $100(1-\alpha)\%$ confidence interval for R can be obtained as:

$$\hat{R} \pm Z_{\frac{\alpha}{2}}\sqrt{Var(\hat{R})},$$

where $Z_{\frac{\alpha}{2}}$ is the upper $\frac{\alpha}{2}$ precentile of the standard normal distribution.

5. BAYESIAN ESTIMATION

In this section, we provide the Bayes estimate of R where $\theta_1, \theta_2, \alpha_1, \alpha_2$ are unknown parameters and all of these parameters having independent gamma prior distributions as following:

$$\pi(\theta_1) \sim Gamma(a_1, b_1),$$

$$\pi(\theta_2) \sim Gamma(a_2, b_2),$$

$$\pi(\alpha_1) \sim Gamma(a_3, b_3),$$

and

$$\pi(\alpha_2) \sim Gamma(a_4, b_4).$$

The joint posterior PDF is defined as

$$g(\theta_1,\theta_2,\alpha_1,\alpha_2/data) = \frac{L(x,y/\theta_1,\theta_2,\alpha_1,\alpha_2)\pi(\theta_1)\pi(\theta_2)\pi(\alpha_1)\pi(\alpha_2)}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty L(x,y/\theta_1,\theta_2,\alpha_1,\alpha_2)\pi(\theta_1)\pi(\theta_2)\pi(\alpha_1)\pi(\alpha_2)d\theta_1 d\theta_2 d\alpha_1 d\alpha_2}.$$

Then

$$\begin{aligned} g(\theta_1,\theta_2,\alpha_1,\alpha_2/data) \;\propto\; & \frac{{\theta_1}^n}{(\alpha_1+1)^n}\frac{{\theta_2}^m}{(\alpha_2+1)^m}\prod_{i=1}^n(\alpha_1+\theta_1 x_i)\prod_{j=1}^m(\alpha_2+\theta_2 y_j)\; e^{-\theta_1\sum_{i=1}^n x_i} \\ & \times\, e^{-\theta_2\sum_{j=1}^m y_j}\;{\theta_1}^{a_1-1}e^{-b_1\theta_1}{\theta_2}^{a_2-1}e^{-b_2\theta_2}{\alpha_1}^{a_3-1}e^{-b_3\alpha_1}{\alpha_2}^{a_4-1}e^{-b_4\alpha_2}. \end{aligned} \tag{5.7}$$

5.1. Bayes estimators under Symmetric and Asymmetric loss function:

The Bayes estimate of reliability formula depending on the choice of the loss function. Two different loss functions are used, symmetric and asymmetric loss function. If the amount of loss assigned by a loss function to a positive error is equal to the negative error of the same magnitude, then the loss function is called a symmetric loss function. In most of the studies on estimation and prediction problems, authors prefer to use the squared error loss function which is symmetric in nature. However, the use of the squared error loss function is not appropriate particularly in these cases, where the losses are not symmetric. Thus in order to make the statistical inferences more practical and applicable, we often needs to choose an asymmetric loss function. A number of asymmetric loss functions proposed for use, first is the linex loss function which suggested by Varian [20], and studied by several others including Basu and Ebrahimi [3], Zellner [21]. Second is the general entropy loss function which introduced by Calabria and Pulcini [5]. These asymmetric loss functions are also studied by Braess and Dette [4], Pandey and Rao [14], Parsian and Kirmani [15], Sanku Dey [16] and Soliman [19], who have used these loss function in different estimation and prediction problem.

Then the equation of the Bayes estimate of the reliability is depend on the loss function, here is the general equation for each type:

-The Bayes estimate under the squared error loss function, which is the posterior mean of R, is given by:

$$\hat{R}_{Se} = \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty R\; g(\theta_1,\theta_2,\alpha_1,\alpha_2/data)d\theta_1 d\theta_2 d\alpha_1 d\alpha_2.$$

-The Bayes estimate under the linex loss function is given by:

$$\hat{R}_{Lx} = \frac{-1}{c}\, ln\Big[\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-cR}\; g(\theta_1,\theta_2,\alpha_1,\alpha_2/data)d\theta_1 d\theta_2 d\alpha_1 d\alpha_2\Big],$$

where c is constant, $c > 0$, see [21].

-The Bayes estimate under the general entropy loss function is given by:

$$\hat{R}_{Ge} = \Big[\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty R^{-q}\; g(\theta_1,\theta_2,\alpha_1,\alpha_2/data)d\theta_1 d\theta_2 d\alpha_1 d\alpha_2\Big]^{-1/q},$$

where q is constant, $q > 0$, see [5].

It is impossible to compute these integrals analytically. Two approaches can be used to approximate these integrals, namely, Importance Sampling technique and MCMC technique.

5.2. Importance Sampling Technique

Importance Sampling Technique has suggested by Chen and Shao [6]. In statistics, importance sampling is the name for the general technique of determining the properties of a distribution by drawing samples from another distribution. The focus of importance sampling here is to determine as easily and accurately as possible the properties of the posterior from a representative sample from the second distribution.

Using Importance Sampling Technique, Eq.(5.7) can be written as

$$g(\theta_1,\theta_2,\alpha_1,\alpha_2/data) \propto g_1(\theta_1/data)g_2(\theta_2/data)g_3(\alpha_1/data)g_4(\alpha_2/data)h(\theta_1,\theta_2,\alpha_1,\alpha_2/data),$$

where:

$$g_1(\theta_1/data) \propto Gamma(n+a_1, b_1+\sum_{i=1}^{n} x_i),$$

$$g_2(\theta_2/data) \propto Gamma(m+a_2, b_2+\sum_{j=1}^{m} y_j),$$

$$g_3(\alpha_1/data) \propto Gamma(a_3,b_3),$$

$$g_4(\alpha_2/data) \propto Gamma(a_4,b_4),$$

and

$$h(\theta_1,\theta_2,\alpha_1,\alpha_2/data) = \frac{\prod_{i=1}^{n}(\alpha_1+\theta_1 x_i) \quad \prod_{j=1}^{m}(\alpha_2+\theta_2 y_j)}{(\alpha_1+1)^n \quad (\alpha_2+1)^m}.$$

As shown, all the above functions from $g_1(\theta_1/data)$ to $g_4(\alpha_2/data)$ follow gamma distributions with different parameters, so it is quite simple to generate QLD parameters from them. Assuming that $a_1,\cdots,a_4$ and $b_1,\cdots,b_4$ are known, and assuming initial values for $\theta_1,\theta_2,\alpha_1,\alpha_2$. we can use the following Importance Sampling Algorithm:

- Step1: Generate θ_{11} from $g_1(\theta_1/data)$.
- Step2: Generate θ_{21} from $g_2(\theta_2/data)$.
- Step3: Generate α_{11} from $g_3(\alpha_1/data)$.
- Step4: Generate α_{21} from $g_4(\alpha_2/data)$.
- Step5: Repeat steps from 1 to 4, N times to obtain the vector $(\theta_{11},\theta_{21},\alpha_{11},\alpha_{21}),\cdots,(\theta_{1N},\theta_{2N},\alpha_{1N},\alpha_{2N})$.

Then
-An approximate Bayes estimate of R under squared error loss function can be obtained as

$$\tilde{R}_{imp_{Se}} = \frac{\sum_{i=1}^{N} R_i \; h(\theta_{1i},\theta_{2i},\alpha_{1i},\alpha_{2i}/data)}{\sum_{i=1}^{N} h(\theta_{1i},\theta_{2i},\alpha_{1i},\alpha_{2i}/data)},$$

- An approximate Bayes estimate of R under linex loss function can be obtained as

$$\tilde{R}_{imp_{Lx}} = \frac{-1}{c}\log\left[\frac{\sum_{i=1}^{N} e^{-cR_i} \; h(\theta_{1i},\theta_{2i},\alpha_{1i},\alpha_{2i}/data)}{\sum_{i=1}^{N} h(\theta_{1i},\theta_{2i},\alpha_{1i},\alpha_{2i}/data)}\right],$$

-An approximate Bayes estimate of R under general entropy loss function can be obtained as:

$$\tilde{R}_{imp_{Ge}} = \left[\frac{\sum_{i=1}^{N} R_i^{-q} \; h(\theta_{1i},\theta_{2i},\alpha_{1i},\alpha_{2i}/data)}{\sum_{i=1}^{N} h(\theta_{1i},\theta_{2i},\alpha_{1i},\alpha_{2i}/data)}\right]^{-1/q},$$

where $R_i = R(\theta_{1i},\theta_{2i},\alpha_{1i},\alpha_{2i})$. as defined in Eq.(2.3), for $i=1,\cdots,N$.

5.3. MCMC Technique

The most general MCMC algorithm is the Metropolis-Hastings (MH) algorithm, which was originally introduced by Metropolis et al. [13], and Hastings [8]. The Metropolis-Hastings (MH) algorithm simulates samples from a probability distribution by making use of the full joint density function and (independent) proposal distributions for each of the variables of interest.
The joint posterior density function of $\theta_1, \theta_2, \alpha_1$, and α_2 is given in Eq.(5.7). It is easily seen that the posterior density functions of $\theta_1, \theta_2, \alpha_1$, and α_2 are, respectively:

$$\pi_1(\theta_1/data) \propto Gamma\left(n + a_1,\ b_1 + \sum_{i=1}^{n} x_i\right), \tag{5.8}$$

$$\pi_2(\theta_2/data) \propto Gamma\left(m + a_2,\ b_2 + \sum_{j=1}^{m} y_j\right), \tag{5.9}$$

$$\pi_3(\alpha_1/\theta_1, data) \propto \frac{{\alpha_1}^{a_3-1} e^{-b_3\alpha_1} \prod_{i=1}^{n}(\alpha_1 + \theta_1 x_i)}{(\alpha_1+1)^n}, \tag{5.10}$$

and

$$\pi_4(\alpha_2/\theta_2, data) \propto \frac{{\alpha_2}^{a_4-1} e^{-b_4\alpha_2} \prod_{j=1}^{m}(\alpha_2 + \theta_2 y_j)}{(\alpha_2+1)^m}. \tag{5.11}$$

Therefore, easily samples of θ_1 and θ_2 can be generated by using Gamma distribution as shown in Eqs.(5.8), and (5.9) respectively. However, the posterior distribution of α_1 , α_2 cannot be generated from a well known distributions. The Metropolis-Hastings algorithm, can be used to solve this problem, as shown in the following algorithm.

- Step1: Start with initial value of α_1,α_2 such that ${\alpha_1}^{(0)} = \hat{\alpha_1}$, and ${\alpha_2}^{(0)} = \hat{\alpha_2}$.
- Step2: Set $i = 1$.
- Step3: Generate ${\theta_1}^{(i)}$ from $\pi_1(\theta_1/data)$.
- Step4: Generate ${\theta_2}^{(i)}$ from $\pi_2(\theta_2/data)$.
- Step5: Generate ${\alpha_1}^{(i)}$ from $\pi_3(\alpha_1/\theta_1, data)$ using the Metropolis-Hastings algorithm with the proposal distribution q_1 as following:
  - Generate ${\alpha_1}^{(*)}$ from the proposal distribution $q_1 = N({\alpha_1}^{(i-1)}, Var({\alpha_1}^{(i-1)}))$.
  - Calculate the acceptance probability
    $r_1({\alpha_1}^{(i-1)}, {\alpha_1}^{(*)}) = Min[1, \frac{\pi_3({\alpha_1}^{(*)}/{\theta_1}^{(i)}, data)}{\pi_3({\alpha_1}^{(i-1)}{\theta_1}^{(i)}, data)}]$.
  - Generate U from Uniform(0, 1).
  - If $U \le r_1({\alpha_1}^{(i-1)}, {\alpha_1}^{(*)})$, accept the proposal distribution and set ${\alpha_1}^{(i)} = {\alpha_1}^{(*)}$, otherwise set ${\alpha_1}^{(i)} = {\alpha_1}^{(i-1)}$.
- Step6: Generate ${\alpha_2}^{(i)}$ from $\pi_4(\alpha_2/\theta_2, data)$ using the Metropolis-Hastings algorithm with the proposal distribution q_2 as following:

- Generate ${\alpha_2}^{(*)}$ from the proposal distribution $q_2 = N({\alpha_2}^{(i-1)}, Var({\alpha_2}^{(i-1)}))$.
- Calculate the acceptance probability
  $r_2({\alpha_2}^{(i-1)}, {\alpha_2}^{(*)}) = Min[1, \frac{\pi_4({\alpha_2}^{(*)}/{\theta_2}^{(i)}, data)}{\pi_4({\alpha_2}^{(i-1)}{\theta_2}^{(i)}, data)}]$.
- Generate U from Uniform(0, 1).
- If $U \leq r_2({\alpha_2}^{(i-1)}, {\alpha_2}^{(*)})$, accept the proposal distribution and set ${\alpha_2}^{(i)} = {\alpha_2}^{(*)}$, otherwise set ${\alpha_2}^{(i)} = {\alpha_2}^{(i-1)}$.

- Step7: Compute $R^{(i)}$ at $({\theta_1}^{(i)}, {\theta_2}^{(i)}, {\alpha_1}^{(i)}, {\alpha_2}^{(i)})$ using Eq.(2.3).
- Step8: Set $i = i + 1$.
- Step9: Repeat steps from $(3 - 8)$ N times.

Then;
-An approximate Bayes estimate of R under squared error loss function is given as:

$$\tilde{R}_{MH_{Se}} = \frac{1}{N - M} \sum_{i=M+1}^{N} R^{(i)}.$$

-An approximate Bayes estimate of R under linex loss function is given as:

$$\tilde{R}_{MH_{Lx}} = \frac{-1}{c} \log\left[\frac{1}{N - M} \sum_{i=M+1}^{N} e^{-cR^{(i)}}\right].$$

-An approximate Bayes estimate of R under general entropy loss function is given as:

$$\tilde{R}_{MH_{Ge}} = \left[\frac{1}{N - M} \sum_{i=M+1}^{N} \left(R^{(i)}\right)^{-q}\right]^{-1/q}.$$

where M is the burn-in units, N is the MCMC samples.

6. SIMULATION STUDY

In this section, we mainly present some simulation experiments to see the performance of the mentioned methods for different sample sizes, $(n, m) = (10, 10), (20, 20), (30, 30), (50, 50), (70, 70), (100, 100)$. we simulated 1000 complete samples from quasi lindely distribution with the parameter values; $\theta_1 = 0.2,\ \theta_2 = 1.5,\ \alpha_1 = 2,\ \alpha_2 = 0.8$ with true reliability value is 0.87399. We also compute the 95% confidence intervals of R based on the observed Fisher information matrix. We compared the performances of the MLE and the Bayes estimates in terms of mean squared errors (MSE's). Also two different techniques of Bayesian estimation (Importance, MCMC) are compared for different loss error functions. Bayesian estimation for different loss error functions was proposed with many values of c, q such that; $c_1 = -3(Lx1)$, $c_2 = 5(Lx2)$, $q_1 = -3$ (Ge1), $q_2 = 5$ (Ge2).
Bayesian estimation studied under the informative gamma priors. For choosing suitable hyper-parameters, the experimenters can incorporate their prior guess in terms of location and precision for the parameter of interest. The gamma distribution for the priors has $mean = a/b$, and $variance = a/b^2$. We assume a small value of prior variance (0.01), and take the mean equal to the true value of the parameter of interest. For each parameter prior we solve the two equations of the mean and the variance, we obtain the following values of

hyper-parameters :
$a_1 = 4,\ a_2 = 225,\ a_3 = 400,\ a_4 = 64,$ and $b_1 = 20,\ b_2 = 150,\ b_3 = 200,\ b_4 = 80.$ We also computed the Bayes estimates based on 11000 samples and discard the first 1000 values as burn-in.
The maximum likelihood estimator and asymptotic confidence intervals of R for different (n,m) are obtained in Table 6.1. Bayes estimates of R using different techniques under different loss error functions are obtained in Table 6.2.
Therefore, from this study of the simulation results we observed that:

- The performance of the Bayes estimators is better than maximum likelihood for all different sample sizes.
- Mean squared error(MSE's) for all estimation methods decreased as sample size increased.
- As sample size increased, the asymptotic confidence intervals for R are improving, and their lengths are decreasing. That means the estimated reliability becomes in the most accurate interval.
- When the sample size increased, both Bayesian and maximum likelihood results become close to each other.
- For Bayes estimators, importance sampling technique gives less MSE's values, so it is better than MCMC technique for the same priors values, and same number of generated samples.
- General entropy, and linex loss error functions gave less MSE's at specified values of c, q. As shown Lx2, Ge2 acheived the best results for MCMC, but for importance sampling technique Lx1, Ge1 are the best methods .

Table 6.1. Average estimate (mean squared error) for MLE, and average confidence length of the simulated 95%confidence intervals of R. all MSE values are multiplied by 10^{-3}

Estimator	MLE	C.I.L	C.I.U	C.I. length
(10,10)	0.866095 (3.2129)	0.734952	0.997237	0.262
(20,20)	0.878225 (1.1262)	0.787222	0.969227	0.182
(30,30)	0.874831 (0.7609)	0.797586	0.952077	0.155
(50,50)	0.875794 (0.459333)	0.816741	0.93706	0.120
(70,70)	0.875149 (0.3349)	0.823703	0.926595	0.103
(100,100)	0.874276 (0.1554)	0.834959	0.919893	0.084

7. REAL DATA ANALYSIS

In this section we present the analysis of real data, introduced by Singh et al. [18]. The data represent the waiting times (in minutes) before customer service of two banks A and B, respectively. The use of Lindley distribution for the waiting times (bank A) data has been originally discussed by Lindley [12]. Since then, many authors have suggested the data under different set-up for Lindley distribution. We are interested in estimating the stress-strength

Table 6.2. Average estimates(mean squared error) of R for different Bayes estimators under different error loss functions. all MSE values are multiplied by 10^{-3}

Est.	Importance Sampling					MCMC Technique				
(m,n)	Se	Lx1	Lx2	Ge1	Ge2	Se	Lx1	Lx2	Ge1	Ge2
(10,10)	0.86713 (0.7516)	0.86862 (0.7058)	0.86457 (0.8421)	0.86829 (0.7134)	0.86346 (0.8934)	0.88787 (0.7611)	0.88908 (0.7748)	0.88577 (0.7459)	0.8888 (0.77)	0.88494 (0.7477)
(20,20)	0.87063 (0.4769)	0.87147 (0.4611)	0.86919 (0.5075)	0.87128 (0.4637)	0.86861 (0.5239)	0.89199 (0.6747)	0.89268 (0.692)	0.89081 (0.6479)	0.89251 (0.6870)	0.89038 (0.6403)
(30,30)	0.87073 (0.3591)	0.87135 (0.3491)	0.86967 (0.3784)	0.87121 (0.3508)	0.86925 (0.3883)	0.89157 (0.5684)	0.89208 (0.5822)	0.89071 (0.5464)	0.89196 (0.5785)	0.8904 (0.5396)
(50,50)	0.87168 (0.2536)	0.87206 (0.2486)	0.87104 (0.2626)	0.87198 (0.2496)	0.87079 (0.2670)	0.88997 (0.4366)	0.89032 (0.4458)	0.88938 (0.4216)	0.89023 (0.4433)	0.88916 (0.4167)
(70,70)	0.873434 (0.2189)	0.873697 (0.217)	0.872995 (0.2226)	0.873636 (0.2173)	0.872827 (0.2244)	0.88874 (0.3755)	0.889014 (0.3823)	0.88828 (0.3645)	0.888946 (0.3805)	0.888114 (0.3609)
(100,100)	0.875249 (0.1384)	0.875418 (0.1387)	0.874965 (0.138)	0.875378 (0.1386)	0.874857 (0.138)	0.885949 (0.2432)	0.886165 (0.2477)	0.885586 (0.2359)	0.886112 (0.2465)	0.885455 (0.2334)

parameter $R = P(Y < X)$ where X and Y denotes the customer service time in Bank A and B (Data set 1, 2) respectively. The data sets are presented below:

Data set 1: X (n=100)
0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2,3.3, 3.5, 3.6,4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5.

Data set 2: Y (m=60)
0.1, 0.2, 0.3, 0.7, 0.9, 1.1, 1.2, 1.8, 1.9, 2.0, 2.2, 2.3, 2.3, 2.3, 2.5, 2.6, 2.7, 2.7, 2.9, 3.1, 3.1, 3.2, 3.4, 3.4, 3.5, 3.9, 4.0, 4.2, 4.5, 4.7, 5.3, 5.6, 5.6, 6.2, 6.3, 6.6, 6.8, 7.3, 7.5, 7.7, 7.7, 8.0, 8.0, 8.5, 8.5, 8.7, 9.5, 10.7, 10.9, 11.0, 12.1, 12.3, 12.8, 12.9, 13.2, 13.7, 14.5, 16.0, 16.5, 28.0.

First,we checked the suitability of Quasi Lindley distribution for the considered real data sets. We, therefore, have provided the Kolmogorov-Smirnov (K-S),Anderson-Darling(A-D) and Cramér-von Mises Statistics to test the goodness-of-fit of above data sets to the Quasi Lindley distribution. The fitting summary has been presented in Table 7.1, which indicates that the QLD fits well to Data Set 1 and Data Set 2.

Table 7.3. P-value (Statistic) of different goodness-of-fit tests for data set 1, 2.

	K-S	A-D	Cramér-von
data set 1.	0.0654 (0.1290)	0.0217 (3.2033)	0.0501 (0.4610)
data set 2.	0.9287 (0.0677)	0.965 (0.2597)	0.9310 (0.0404)

Based on the MLEs $\hat{\theta_1}, \hat{\alpha_1}, \hat{\theta_2}, \hat{\alpha_2}$ the point estimate of R is 0.59 and the 95% confidence interval of R is (0.25, 0.93). For real data sets, the maximum likelihood and Bayes estimates of the stress-strength parameters and reliability are summarized in Table 7.2.

Table 7.4. The MLEs and Bayes estimates of stress-strength parameters and reliability R from real data sets

Est.	$\hat{\theta_1}$	$\hat{\theta_1}$	$\hat{\alpha_1}$	$\hat{\alpha_2}$	$\hat{R}$
MLE	0.1	0.27	84.09	0.41	0.59
$Bayes_{IMP}$	0.13	0.59	1.44	0.66	0.81
$Bayes_{MH}$	0.1	0.54	22.13	0.69	0.77

8. CONCLUSION

In this paper, maximum likelihood and Bayesian estimation methods for stress-strength reliability R were discussed, when X and Y both follow a quasi lindley distribution with different parameters. We obtained the 95% confidence intervals of R based on the observed Fisher information matrix. We proposed the Bayesian estimation based on independent gamma priors under different error loss functions(squared, linex, and general entropy). we suggested the Importance sampling, and MCMC techniques to generate samples from the posterior distributions and then compute the Bayes estimates. Simulation study has been introduced to investigate the performance and compare among all mentioned methods. Simulation results suggest that the performance of the Bayes estimator is better than maximum likelihood for all different sample sizes. Also, maximum likelihood method provides very satisfactory results as the sample size increased.

REFERENCES

1. Al-Mutairi, D.K., Ghitany, M.E. & Kundu, D. (2013). Inferences on stress-strength reliability from Lindley distributions. *Communications in Statistics-Theory and Methods*, 42, 1443-1463.
2. Al-Mutairi, D.K., Ghitany & M.E.,Kundu, D. (2015). Inferences on stress-strength reliability from weigthed Lindley distributions. *Communications in Statistics-Theory and Methods*, 44, 4096-4113.
3. Basu, A.P & Ebrahimi, N.(1991). Bayesian Approach to life Testing and Reliability Estimation Using Asymmetric Loss Function. *Jour. Stat. Plann. Infer*, 29, 21-31.
4. Braess, D. & Dette, H.(2004). The asymptotic minimax risk for the estimation of constrained binomial and multinomial probabilities. *Sankhyaind.J.statist*.66(4), 707-732.
5. Calabria, R. & Pulcini, G. (1994). An engineering approach to Bayes estimation for the Weibull distribution. *Microelectronic Reliability*, 34 (5),789-802.
6. Chen, M.H. & Shao, Q.M. (1999). Monte Carlo estimation of Bayesian credible and HPD intervals. *Journal of Computational and Graphical Statistics*, 8(1), 69-92.
7. Church, J.D. & Harris, B. (1970). The estimation of reliability from stress-strength relationships. *Technometrics*, 12, 49-54.
8. Hastings, WK. (1970). Monte Carlo sampling methods using Markov chains and their applications. *Biometrika*, 57, 97-109.
9. Khan, Adil H. & Jan, T. R. (2015). Estimation of stress-strength reliability model using finite mixture of two parameter Lindley distributions. *Journal of Statistics Applications and Probability*, 4(1), 147-159.

10. Kotz, S. , Lumelskii,Y. & Penskey,M. (2003). The stress–strength model and its generalizations and applications. World Scientific, Singapore.
11. Krishnamoorthy, K., Mukherjee, S. & Guo, H. (2007). Inference on reliability in two-parameter exponential stress-strength model. *Metrika*, 65, 261-273.
12. Lindley, D.V.(1958) Fiducial distributions and bayes theorem. *Journal of the Royal Statistical Society*, 20, 102-107.
13. Metropolis, N., Rosenbluth, AW., Rosenbluth, MN., Teller, AH. & Teller, E., (1953). Equations of state calculations by fast computing machine. *Journal of Chemical Physics*, 21, 1087-1092.
14. Pandey, H. & Rao, A. K. (2009). Bayesian estimation of the shape parameter of a generalized Pareto distribution under asymmetric loss functions. *Hacettepe Journal of Mathematics and Statistics*, 38(1), 69-83.
15. Parsian, A. & Kirmani, S. N. U. A. (2002). Estimation under LINEX loss function. *Hand book of applied econometrics and statistical inference, Statistics Textbook and Monograph.165. New York: Marcel Dekker Inc*, 53-76.
16. Dey, S. (2010). Bayesian estimation of the shape parameter of the generalized exponential distribution under different loss functions. *Pak. J. Statist. Oper. Res*, 62, 163-174.
17. Shanker, R., Sharma, S. & Shanker, R. (2013). A quasi Lindley distribution. *African Journal of Mathematics and Computer Science Research*, 6(4), 64-71.
18. Singh, S. K., Singh, U., & Sharma, V. K. (2014). Estimation on system reliability in generalized Lindley stress-strength model.*Journal of Statistics Applications and Probability*, 3(1), 61-75.
19. Soliman, A.A. (2000). Comparison of Linex and Quadratic Bayes estimators for the Rayleigh Distribution. *Commun.Statis.Theory Meth*, 29(1), 95-107.
20. Varian, H.R.(1975). A Bayesian Approach to Reliability Real Estate Assessment. *Amsterdam, North Holland*, 195-208.
21. Zellner, A. (1986). Bayesian estimation and prediction using asymmetric loss functions. *Journal of American Statistical Associations*, 81(394), 446-451.