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Threshold analysis of a stochastic epidemic model with delay and temporary immunity

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Abstract: A stochastic susceptible-infected-recovered model is formulated and investigated when the temporary immunity is fixed for the population in this paper. The existence and uniqueness of the global positive solution has been checked with probability one for any initial value. And the sufficient conditions for the extinction and the persistence of the stochastic epidemic model with temporary immunity are derived by constructing Lyapunov functions and the generalized Ito's formula, where the threshold of the persistence does not depend on the temporary immunity, while the densities of the infected and recovered are obviously dependent on the temporary immunity when given a perturbation. Illustrative examples and simulations show that the perturbations make the properties of the stochastic epidemic model different from the deterministic one.

Keywords: stochastic epidemic model; delay; persistence; extinction; threshold

1. INTRODUCTION

The susceptible-infected-removed epidemic model is one of the most important models in epidemiological patterns and disease control. Epidemic models with delay always make the description of the epidemiological patterns more realistic, more interesting and more complicated. Some authors investigated the effects of disease latency or immunity, in which the models are described by systems of ordinary differential equations with delay, for instance [1–10], and systems without delay [11–15]. Among these recent results we would like to mention the work by Wen and Yang [10], in which they considered a type of delayed susceptible-infected-recovered epidemic model with temporary immunity:

$$\begin{cases} \dot{S}(t) &= \Lambda - \mu_1 S(t) - \beta S(t)I(t) + \gamma I(t-\tau)e^{-\mu_3\tau}, \\ \dot{I}(t) &= \beta S(t)I(t) - (\mu_2+\gamma)I(t), \\ \dot{R}(t) &= \gamma I(t) - \gamma I(t-\tau)e^{-\mu_3\tau} - \mu_3 R(t), \end{cases} \tag{1.1}$$

where $S(t)$ is the number of the susceptible individuals to the disease, $I(t)$ represents the number of the individuals who are infected and $R(t)$ denotes the recovered individuals who have been removed from the possibility of infection, Λ denotes a constant input of new members into the population, μ_1, μ_2, μ_3 represent the death rates of the susceptible, the infected, the recovered, and they assume that $\mu_1 \leq \min\{\mu_2, \mu_3\}$ from the biological point of view, β denotes the transmission coefficient between compartments S and I, γ denotes the recovery rate of the infected individuals, *i.e.*, the rate at which the individuals move from the infected compartment to the recovered compartment, $\tau > 0$ is the length of immunity

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period. Wen and Yang [10] assume that $\Lambda, \mu_i, \beta, \gamma$ are all positive constants in model (1.1), and they obtain the expression of the basic reproduction number in terms of parameters, that is, $R_0 = \frac{\beta\Lambda}{\mu_1(\mu_2+\gamma)}$, which measures how fast the diseases spread in the deterministic model provided that the epidemics take place.

Some common phenomena in the real world, such as the weather fluctuations, temperature changes and perturbations caused by human beings, can not been ignored when it comes to the dynamics of the infectious diseases. From both biological and mathematical perspectives, the stochastic models have more reasonable patterns and have been investigated by different approaches presented in the recent literatures [16–20]. Motivated by model (1.1) and the approaches mentioned in [16–20], we try to investigate several properties of model (1.2) when transmission coefficient β in (1.1) is perturbed by the white noise $\beta + \sigma\xi(t)$:

$$\begin{cases} dS(t) &= \big(\Lambda - \mu_1 S(t) - \beta S(t)I(t) + \gamma I(t-\tau)e^{-\mu_3\tau}\big)dt - \sigma S(t)I(t)dB(t), \\ dI(t) &= \big(\beta S(t)I(t) - (\mu_2+\gamma)I(t)\big)dt + \sigma S(t)I(t)dB(t), \\ dR(t) &= \big(\gamma I(t) - \gamma I(t-\tau)e^{-\mu_3\tau} - \mu_3 R(t)\big)dt, \end{cases} \tag{1.2}$$

where $\xi(t) = \frac{dB(t)}{dt}$, and $B(t)$ is the independent Brownian motion, and σ is the intensity of the white noise when model (1.2) works on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and increasing function while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets).

What we concern about in model (1.2) in this paper are listed below, of course, for any positive initial value herewith:

- Existence and uniqueness of a global nonnegative solution with probability one;
- The sufficient conditions that guarantees the extinction of the diseases;
- Figuring out the threshold presented in terms of model parameters;
- Demonstrating illustrative examples and their realizations to support the validity of the main results.

2. EXISTENCE AND UNIQUENESS OF THE POSITIVE SOLUTION

For model (1.2), the sum of three equations gives

$$\frac{d(S(t)+I(t)+R(t))}{dt} \leq \Lambda - \mu_1(S(t)+I(t)+R(t)), \tag{2.3}$$

then one can obtain that

$$S(t)+I(t)+R(t) \leq \begin{cases} \dfrac{\Lambda}{\mu_1}, & S(0)+I(0)+R(0) \leq \dfrac{\Lambda}{\mu_1}, \\ S(0)+I(0)+R(0), & S(0)+I(0)+R(0) > \dfrac{\Lambda}{\mu_1}. \end{cases} \tag{2.4}$$

We denote

$$N = \max\left\{\frac{\Lambda}{\mu_1}, S(0)+I(0)+R(0)\right\}, \tag{2.5}$$

therefore, $S(t) \leq N, I(t) \leq N, R(t) \leq N$. We notice that the first two equations in model (1.2) do not depend on the third equation, we omit them without loss of generality. Thus, we

only discuss the simplified model of having two equations:

$$\begin{cases} dS(t) &= \big(\Lambda - \mu_1 S(t) - \beta S(t)I(t) + \gamma I(t-\tau)e^{-\mu_3\tau}\big)dt - \sigma S(t)I(t)dB(t), \\ dI(t) &= \big(\beta S(t)I(t) - (\mu_2+\gamma)I(t)\big)dt + \sigma S(t)I(t)dB(t). \end{cases} \tag{2.6}$$

In order to investigate other properties, we firstly study the fundamental property of the solution of model (2.6). Next we will present the existence and uniqueness of the solution of model (1.2).

Theorem 2.1:
For any initial value $S(0) > 0$ and $I(\zeta) \geq 0$ for all $\zeta \in [-\tau, 0)$ with $I(0) > 0$, model (2.6) admits a unique positive solution $(S(t), I(t))$ on $t > 0$ and the solution will remain in $\mathbb{R}^2_+$ with probability one, that is to say, $(S(t), I(t)) \in \mathbb{R}^2_+$ for all $t > 0$ almost surely.

Proof
According to the approach mentioned in [13–15, 21]. The proof will go as follows. Since the coefficients of model (2.6) satisfy the local Lipschitz conditions, then for any initial value $S(0) > 0$ and $I(\zeta) \geq 0$ for all $\zeta \in [-\tau, 0)$ with $I(0) > 0$, there is a unique local solution $(S(t), I(t))$ on the half-closed interval $[-\tau, \tau_e)$, where τ_e represents the explosion time of the solution. To verify the solution is global, what we prepare to check that explosion time $\tau_e = \infty$ holds almost surely. To this end, let $k_0 \geq 1$ be sufficiently large such that each component of the initial value $(S(0), I(\zeta))$ always lies within the interval $[\frac{1}{k_0}, k_0]$. For each integer $k \geq k_0$, let us define the following stopping time

$$\tau_k = \inf\left\{t \in [0, \tau_e) \middle| S(t) \notin \left(\frac{1}{k}, k\right) \text{ or } I(t) \notin \left(\frac{1}{k}, k\right)\right\}. \tag{2.7}$$

Throughout this paper, we set $\inf \emptyset = \infty$ (as usual $\emptyset$ represents the empty set). Obviously, τ_k is an increasing function as $k \to \infty$. Let τ_∞ be the limit of τ_k as $k \to \infty$. It is obvious that $\tau_\infty \leq \tau_e$ is valid almost surely. In order to complete the proof, we only need to show $\tau_\infty = \infty$ a.s.. We start the proof by contradiction, if this assertion is false, then there exists a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$\mathbb{P}\{\tau_\infty \leq T\} > \varepsilon. \tag{2.8}$$

Thereby, there is an integer $k_1 \geq k_0$ such that

$$\mathbb{P}\{\tau_k \leq T\} \geq \varepsilon \quad \text{for all } k \geq k_1. \tag{2.9}$$

Let us define a $\mathcal{C}^2$-function $V : \mathbb{R}^2_+ \to \mathbb{R}_+$ as follows:

$$V(S(t), I(t)) = S(t) - a - a\ln\frac{S(t)}{a} + I(t) - 1 - \ln I(t) + \gamma e^{-\mu_3\tau}\int_{t-\tau}^{t} I(r)dr, \tag{2.10}$$

where a is a positive constant to be determined later. The nonnegativity of $\mu - 1 - \log\mu$ is obvious for all $\mu > 0$. Let $k \geq k_0$ and $T > 0$ be arbitrary positive constants. The application of Itô's formula gives

$$\begin{aligned} dV(S(t), I(t)) &= \left(1 - \frac{a}{S(t)}\right)dS(t) + \frac{a}{2S^2(t)}(dS(t))^2 + \left(1 - \frac{1}{I(t)}\right)dI(t) \\ &\quad + \frac{1}{2I^2(t)}(dI(t))^2 + \gamma I(t)e^{-\mu_3\tau}dt - \gamma I(t-\tau)e^{-\mu_3\tau}dt \\ &:= \mathcal{L}V(S(t), I(t))dt + \sigma(aI(t) - S(t))dB(t), \end{aligned} \tag{2.11}$$

where $\mathcal{L}V(S(t), I(t))$ is defined from $\mathbb{R}^2_+$ to $\mathbb{R}$ and computed as

$$\begin{aligned}
&\mathcal{L}V(S(t), I(t))\\
=\ & \left(1-\frac{a}{S(t)}\right)\big(\Lambda-\mu_1 S(t)-\beta S(t)I(t)+\gamma I(t-\tau)e^{-\mu_3\tau}\big)\\
&+\frac{a\sigma^2}{2}I^2(t)+\left(1-\frac{1}{I(t)}\right)\big(\beta S(t)I(t)-(\mu_2+\gamma)I(t)\big)\\
&+\frac{\sigma^2}{2}S^2(t)+\gamma I(t)e^{-\mu_3\tau}-\gamma I(t-\tau)e^{-\mu_3\tau}\\
=\ & \Lambda-\mu_1 S(t)-(\mu_2+\gamma)I(t)+\gamma I(t-\tau)e^{-\mu_3\tau}\\
&-\frac{a\Lambda}{S(t)}+\mu_1 a+a\beta I(t)-a\gamma e^{-\mu_3\tau}\frac{I(t-\tau)}{S(t)}-\beta S(t)+\mu_2+\gamma\\
&+\gamma I(t)e^{-\mu_3\tau}-\gamma I(t-\tau)e^{-\mu_3\tau}+\frac{\sigma^2}{2}(S^2(t)+aI^2(t))\\
\leq\ & \Lambda+\mu_1 a+\mu_2+\gamma+(a\beta+\gamma e^{-\mu_3\tau}-(\mu_2+\gamma))I(t)+\frac{\sigma^2}{2}(1+a)N^2.
\end{aligned} \tag{2.12}$$

Choosing a positive constant

$$a=\frac{\mu_2+\gamma(1-e^{-\mu_3\tau})}{\beta} \tag{2.13}$$

such that $a\beta+\gamma e^{-\mu_3\tau}=\mu_2+\gamma$, then we have

$$\mathcal{L}V(S(t), I(t))\leq \Lambda+\mu_1 a+\mu_2+\gamma+\frac{\sigma^2}{2}(1+a)N^2:=K, \tag{2.14}$$

where $K>0$ is a constant. The remainder of the proof follows the similar approach given in [22]. □

Remark 2.1:
Theorem 2.1 *implies that the stochastic model* (1.2) *admits a unique solution* $(S(t), I(t), R(t))$ *for any initial value* $(S(0), I(0), R(0))\in\mathbb{R}^3_+, I(\zeta)\geq 0$ *for all* $\zeta\in[-\tau, 0)$. *Each component of the solution* $(S(t), I(t), R(t))$ *is positive for all* $t\geq 0$ *almost surely. According to the statement of Theorem* 2.1*, if the sum of each component with the initial value is almost surely bounded, say* $S(0)+I(0)+R(0)\leq\frac{\Lambda}{\mu_1}$*, so the region*

$$\Gamma^*=\left\{(S(t), I(t), R(t))\in\mathbb{R}^3_+ : S(t)+I(t)+R(t)\leq\frac{\Lambda}{\mu_1},\ t\geq 0\right\} \tag{2.15}$$

is the positively invariant set of model (1.2).

3. THE SUFFICIENT CONDITIONS OF THE EXTINCTION OF THE DISEASES

Let

$$\widetilde{R}_0=\frac{\beta\Lambda}{\mu_1(\mu_2+\gamma)}-\frac{\sigma^2\Lambda^2}{2\mu_1^2(\mu_2+\gamma)}, \tag{3.16}$$

which can be presented in terms of the basic reproduction number of the deterministic model (1.1):

$$\widetilde{R}_0=R_0-\frac{\sigma^2\Lambda^2}{2\mu_1^2(\mu_2+\gamma)}. \tag{3.17}$$

For the stochastic model (1.2), we will investigate the sufficient conditions of the extinction of the disease. In other words, the infected individuals will eventually recover from the attack of the disease, and the capacity of the total population will keep kind of constant in the long run.

Theorem 3.1:
Let $(S(t), I(t), R(t))$ *be the solution of model* (1.2) *with initial value* $(S(0), I(0), R(0)) \in \Gamma^*$. *If the intensity of the white noise satisfies*

$$\sigma^2 > \frac{\beta^2}{2(\mu_2+\gamma)} \tag{3.18}$$

or

$$\tilde{R}_0 < 1,\ \sigma^2 \leq \frac{\beta\mu_1}{\Lambda}, \tag{3.19}$$

then the density of the infected declines exponentially

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} \leq -(\mu_2+\gamma) + \frac{\beta^2}{2\sigma^2} < 0 \ \ a.s. \tag{3.20}$$

or

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} \leq (\mu_2+\gamma)(\tilde{R}_0 - 1) < 0 \ \ a.s. \tag{3.21}$$

respectively; and

$$\limsup_{t\to\infty} S(t) = \frac{\Lambda}{\mu_1} \ \ a.s.. \tag{3.22}$$

Proof

Adding up those three equations of model (1.2) gives

$$d\bigg(S(t)+I(t)+\gamma e^{-\mu_3\tau}\int_{t-\tau}^{t} I(r)dr\bigg) = [\Lambda - \mu_1 S(t) - (\mu_2+\gamma(1-e^{-\mu_3\tau}))I(t)]dt. \tag{3.23}$$

We integrate the above from 0 to t and divided by t, which implies

$$\begin{aligned} &\Lambda - \mu_1\langle S(t)\rangle - (\mu_2+\gamma(1-e^{-\mu_3\tau}))\langle I(t)\rangle \\ =\ & \frac{1}{t}\bigg(S(t)+I(t)+\gamma e^{-\mu_3\tau}\int_{t-\tau}^{t} I(r)dr - S(0) - I(0) - \gamma e^{-\mu_3\tau}\int_{-\tau}^{0} I(r)dr\bigg), \end{aligned} \tag{3.24}$$

and then derives that

$$\langle S(t)\rangle = \frac{\Lambda}{\mu_1} - \frac{\mu_2+\gamma(1-e^{-\mu_3\tau})}{\mu_1}\langle I(t)\rangle + \varphi(t), \tag{3.25}$$

where

$$\varphi(t) = \frac{1}{\mu_1 t}\bigg(S(t)+I(t)+\gamma e^{-\mu_3\tau}\int_{t-\tau}^{t} I(r)dr - S(0) - I(0) - \gamma e^{-\mu_3\tau}\int_{-\tau}^{0} I(r)dr\bigg). \tag{3.26}$$

The strong law of large numbers and the fact $S(0)+I(0) \leq \frac{\Lambda}{\mu_1}$ show that

$$\lim_{t\to\infty} \varphi(t) = 0 \ \text{ a.s..} \tag{3.27}$$

For the second equation of model (1.2), the generalized Itô's formula leads to

$$d\ln I(t) = \left(\beta S(t) - (\mu_2+\gamma) - \frac{\sigma^2 S^2(t)}{2}\right)dt + \sigma S(t)dB(t). \tag{3.28}$$

Integrating (3.28) from 0 to t and divided by t gives

$$\frac{\ln I(t)}{t} = \beta\langle S(t)\rangle - (\mu_2+\gamma) - \frac{\sigma^2}{2}\langle S^2(t)\rangle + \frac{\sigma}{t}\int_0^t S(r)dB(r) + \frac{\ln I(0)}{t}. \tag{3.29}$$

We denote

$$M(t) := \sigma\int_0^t S(r)dB(r), \tag{3.30}$$

where $M(t)$ is a local continuous martingale with $M(0)=0$. Moreover,

$$\limsup_{t\to\infty}\frac{\langle M(t),M(t)\rangle_t}{t} \le \frac{\sigma^2\Lambda^2}{\mu_1^2} < \infty \ \text{ a.s..} \tag{3.31}$$

The strong law of large numbers for martingales presented in [23] thus implies that

$$\limsup_{t\to\infty}\frac{M(t)}{t} = 0 \ \text{ a.s..} \tag{3.32}$$

If $\sigma^2 > \frac{\beta^2}{2(\mu_2+\gamma)}$, rewriting (3.29) gives

$$\begin{aligned}\frac{\ln I(t)}{t} &= -\frac{\sigma^2}{2t}\int_0^t\left(S(r)-\frac{\beta}{\sigma^2}\right)^2 dr - (\mu_2+\gamma) + \frac{\beta^2}{2\sigma^2} + \frac{\sigma}{t}\int_0^t S(r)dB(r) + \frac{\ln I(0)}{t}\\ &\le -(\mu_2+\gamma) + \frac{\beta^2}{2\sigma^2} + \frac{\sigma}{t}\int_0^t S(r)dB(r) + \frac{\ln I(0)}{t}.\end{aligned} \tag{3.33}$$

Taking superior limit on both sides of (3.33), then we get that

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t} \le -(\mu_2+\gamma) + \frac{\beta^2}{2\sigma^2} < 0 \ \text{ a.s..} \tag{3.34}$$

If $\tilde{R}_0 < 1$ and $\sigma^2 \leq \frac{\beta\mu_1}{\Lambda}$ hold, the Schwarz inequality and substituting (3.25) into (3.29) yield that

$$\begin{aligned}
\frac{\ln I(t)}{t} &\leq \beta\langle S(t)\rangle - (\mu_2+\gamma) - \frac{\sigma^2}{2}\langle S(t)\rangle^2 + \frac{\sigma}{t}\int_0^t S(r)dB(r) + \frac{\ln I(0)}{t} \\
&= \frac{\beta\Lambda}{\mu_1} - (\mu_2+\gamma) - \frac{\beta(\mu_2+\gamma(1-e^{-\mu_3\tau}))}{\mu_1}\langle I(t)\rangle + \beta\varphi(t) \\
&\quad + \frac{\sigma}{t}\int_0^t S(r)dB(r) - \frac{\sigma^2}{2}\left[\frac{\Lambda}{\mu_1} - \frac{\mu_2+\gamma(1-e^{-\mu_3\tau})}{\mu_1}\langle I(t)\rangle + \varphi(t)\right]^2 \\
&= \frac{\beta\Lambda}{\mu_1} - (\mu_2+\gamma) - \frac{\beta(\mu_2+\gamma(1-e^{-\mu_3\tau}))}{\mu_1}\langle I(t)\rangle + \beta\varphi(t) \\
&\quad + \frac{\sigma}{t}\int_0^t S(r)dB(r) - \frac{\sigma^2}{2}\left[\frac{\Lambda^2}{\mu_1^2} + \left(\frac{\mu_2+\gamma(1-e^{-\mu_3\tau})}{\mu_1}\langle I(t)\rangle\right)^2\right. \\
&\quad + \varphi^2(t) - \frac{2\Lambda(\mu_2+\gamma(1-e^{-\mu_3\tau}))}{\mu_1^2}\langle I(t)\rangle \\
&\quad \left. + \frac{2\Lambda - 2(\mu_2+\gamma(1-e^{-\mu_3\tau}))\langle I(t)\rangle}{\mu_1}\varphi(t)\right]^2,
\end{aligned} \tag{3.35}$$

which gives that

$$\begin{aligned}
\frac{\ln I(t)}{t} &\leq \frac{\beta\Lambda}{\mu_1} - \frac{\sigma^2\Lambda^2}{2\mu_1^2} - \frac{\mu_2+\gamma(1-e^{-\mu_3\tau})}{\mu_1}\left(\beta - \frac{\sigma^2\Lambda}{\mu_1}\right)\langle I(t)\rangle \\
&\quad -(\mu_2+\gamma) + \psi(t),
\end{aligned} \tag{3.36}$$

where

$$\begin{aligned}
\psi(t) &= \beta\varphi(t) + \frac{\sigma}{t}\int_0^t S(r)dB(r) \\
&\quad -\frac{\sigma^2}{2}\left[\varphi^2(t) + \frac{2\Lambda - 2(\mu_2+\gamma(1-e^{-\mu_3\tau}))\langle I(t)\rangle}{\mu_1}\varphi(t)\right].
\end{aligned} \tag{3.37}$$

It is easy to check that

$$\lim_{t\to\infty}\psi(t) = 0 \ \text{ a.s..} \tag{3.38}$$

We take superior limit on both sides of (3.35) and derive that

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t} \leq (\mu_2+\gamma)(\tilde{R}_0 - 1) < 0 \ \text{ a.s..} \tag{3.39}$$

Further, the density of the infected individuals declines to zero, and the density of the susceptible individuals keeps constant in a long time run, mathematically speaking, these two expressions

$$\lim_{t\to\infty} I(t) = 0, \quad \lim_{t\to\infty} S(t) = \frac{\Lambda}{\mu_1} \tag{3.40}$$

hold almost surely when condition (3.18) or condition (3.19) is satisfied. □

Remark 3.1:
Wen and Yang [10] had shown that the disease-free equilibrium $E_0(\frac{\Lambda}{\mu_1}, 0, 0)$ *of the deterministic model* (1.1) *was globally asymptotically stable if the basic reproduction number* $R_0 < 1$ (*i.e., the disease disappeared under the condition* $R_0 < 1$). *While for the stochastic*

model (1.2)*, we obtain that the disease disappears when conditions* $\tilde{R}_0 < 1$ *and* $\sigma^2 \leq \frac{\beta\mu_1}{\Lambda}$ *are being satisfied in this chapter. This indicates that the condition of extinction to the stochastic model* (1.2) *is weaker than that of the corresponding deterministic model* (1.1).

4. THE THRESHOLD FOR THE PERSISTENCE OF THE DISEASES

Theorem 4.1:
Let $(S(t), I(t), R(t))$ *be any solution of model* (1.2)*,* $(S(0), I(0), R(0)) \in \Gamma^*$ *be any initial value with* $I(\zeta) \geq 0$ *for all* $\zeta \in [-\tau, 0)$*. We assume that* $\tilde{R}_0 > 1$*, and also assume that the intensity of the white noise satisfies*

$$\sigma^2 \leq \frac{\beta\mu_1}{\Lambda}, \tag{4.41}$$

then the densities of the infected and the recovered individuals have the following properties:

$$\tilde{I}_* \leq \liminf_{t\to\infty}\langle I(t)\rangle \leq \limsup_{t\to\infty}\langle I(t)\rangle \leq \tilde{I}^* \text{ a.s.} \tag{4.42}$$

and

$$\tilde{R}_* \leq \liminf_{t\to\infty}\langle R(t)\rangle \leq \limsup_{t\to\infty}\langle R(t)\rangle \leq \tilde{R}^* \text{ a.s.}, \tag{4.43}$$

where

$$\tilde{I}_* = \frac{\mu_1(\mu_2+\gamma)}{\beta(\mu_2+\gamma(1-e^{-\mu_3\tau}))}(\tilde{R}_0-1), \quad \tilde{R}_* = \frac{\gamma(1-e^{-\mu_3\tau})}{\mu_3}\tilde{I}_*, \tag{4.44}$$

and

$$\tilde{I}^* = \frac{\mu_1^2(\mu_2+\gamma)}{(\beta\mu_1-\sigma^2\Lambda)(\mu_2+\gamma(1-e^{-\mu_3\tau}))}(\tilde{R}_0-1), \quad \tilde{R}^* = \frac{\gamma(1-e^{-\mu_3\tau})}{\mu_3}\tilde{I}^*. \tag{4.45}$$

Proof

From (3.28), the second equation of model (1.2) implies that

$$d\ln I(t) \geq \left(\beta S(t) - \mu_2 - \gamma - \frac{\sigma^2\Lambda^2}{2\mu^2}\right)dt + \sigma S(t)dB(t), \tag{4.46}$$

integrating (4.46) on both sides and substituting (3.25) into the integration, which derive that

$$\begin{aligned}\frac{\ln I(t) - \ln I(0)}{t} &\geq \beta\langle S(t)\rangle - \mu_2 - \gamma - \frac{\sigma^2\Lambda^2}{2\mu^2} + \frac{\sigma}{t}\int_0^t S(r)dB(r)\\ &= \frac{\beta\Lambda}{\mu_1} - \left(\mu_2+\gamma+\frac{\sigma^2\Lambda^2}{2\mu^2}\right) - \frac{\beta(\mu_2+\gamma(1-e^{-\mu_3\tau}))}{\mu_1}\langle I(t)\rangle\\ &\quad +\beta\varphi(t) + \frac{\sigma}{t}\int_0^t S(r)dB(r).\end{aligned} \tag{4.47}$$

Inequality (4.47) can be rewritten as

$$\begin{aligned}\langle I(t)\rangle &\geq \frac{\mu_1}{\beta(\mu_2+\gamma(1-e^{-\mu_3\tau}))}\left(\frac{\beta\Lambda}{\mu_1} - \left(\mu_2+\gamma+\frac{\sigma^2\Lambda^2}{2\mu^2}\right)\right.\\ &\quad \left. -\frac{\ln I(t)-\ln I(0)}{t} + \beta\varphi(t) + \frac{\sigma}{t}\int_0^t S(r)dB(r)\right).\end{aligned} \tag{4.48}$$

According to (2.15), we have

$$-\infty < \ln I(t) < \ln \frac{\Lambda}{\mu_1}, \ \lim_{t\to\infty} \frac{\sigma}{t} \int_0^t S(r)dB(r) = 0, \tag{4.49}$$

and

$$\lim_{t\to\infty} \varphi(t) = 0. \tag{4.50}$$

Taking the inferior limit on both sides of (4.48), we have

$$\liminf_{t\to\infty} \langle I(t)\rangle \geq \frac{\mu_1(\mu_2+\gamma)}{\beta(\mu_2+\gamma(1-e^{-\mu_3\tau}))}(\tilde{R}_0 - 1) := \tilde{I}_*. \tag{4.51}$$

On the other hand, integrating the first equation of (4.46) from 0 to t on both sides yields

$$\frac{\ln I(t) - \ln I(0)}{t} = \beta\langle S(t)\rangle - \mu_2 - \gamma - \frac{\sigma^2}{2}\langle S^2(t)\rangle + \frac{\sigma}{t}\int_0^t S(r)dB(r). \tag{4.52}$$

We rewrite (3.35), then we get that

$$\begin{aligned} \langle I(t)\rangle \ \leq \ & \frac{\mu_1^2}{(\mu_2+\gamma(1-e^{-\mu_3\tau}))(\beta\mu_1 - \sigma^2\Lambda)} \\ & \times\left(\frac{\beta\Lambda}{\mu_1} - \left(\mu_2+\gamma+\frac{\sigma^2\Lambda^2}{2\mu_1^2}\right) - \frac{\ln I(t) - \ln(0)}{t} + \psi(t)\right), \end{aligned} \tag{4.53}$$

taking the superior limit on both sides of above equation, then one can obtain that

$$\limsup_{t\to\infty} \langle I(t)\rangle \leq \frac{\mu_1^2}{(\mu_2+\gamma(1-e^{-\mu_3\tau}))(\beta\mu_1 - \sigma^2\Lambda)}(\tilde{R}_0 - 1) := \tilde{I}^*. \tag{4.54}$$

The last equation of model (1.2) yields

$$\langle R(t)\rangle = \frac{\gamma}{\mu_3}\langle I(t)\rangle - \frac{\gamma e^{-\mu_3\tau}}{\mu_3}\langle I(t-\tau)\rangle - \frac{R(t)-R(0)}{\mu_3 t}, \tag{4.55}$$

then

$$\liminf_{t\to\infty} \langle R(t)\rangle = \frac{\gamma(1-e^{-\mu_3\tau})}{\mu_3}\liminf_{t\to\infty}\langle I(t)\rangle \geq \frac{\gamma(1-e^{-\mu_3\tau})}{\mu_3}\tilde{I}_* = \tilde{R}_*, \tag{4.56}$$

$$\limsup_{t\to\infty} \langle R(t)\rangle = \frac{\gamma(1-e^{-\mu_3\tau})}{\mu_3}\limsup_{t\to\infty}\langle I(t)\rangle \leq \frac{\gamma(1-e^{-\mu_3\tau})}{\mu_3}\tilde{I}^* = \tilde{R}^*. \tag{4.57}$$

□

Remark 4.1:
Both Theorem 2 *and Theorem* 3 *have the common condition* $\sigma^2 \leq \frac{\beta\mu_1}{\Lambda}$ *therewith. While the opposite properties of these two theorems depend on the value of* $\tilde{R}_0$*, that is,* $\tilde{R}_0 < 1$ *indicates the extinction of the disease and* $\tilde{R}_0 > 1$ *means the persistence of the disease. The expression* $\tilde{R}_0$ *plays the role of threshold of model* (1.2).

5. ILLUSTRATIVE EXAMPLES AND THEIR REALIZATIONS

Example 1

Let the parameters of model (1.2) be $\Lambda = 0.2, \mu_1 = \mu_2 = \mu_3 = 0.2, \beta = 0.8, \gamma = 0.5, \sigma = 0.6$, and the initial value be $(S(0), I(0), R(0)) = (0.1, 0.6, 0.3)$.

We can verify that

$$\tilde{R}_0 = R_0 - \frac{\sigma^2\Lambda^2}{2\mu_1^2(\mu_2+\gamma)} = 0.8857 < 1, \ \ \sigma^2 = 0.36 \leq \frac{\beta\mu_1}{\Lambda} = 0.8. \tag{5.58}$$

Condition (3.19) of Theorem 1.2 is satisfied, the solution of model (1.2) has the properties as follows:

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} \leq (\mu_2+\gamma)(\tilde{R}_0 - 1) = -0.08 < 0 \ \text{ a.s.} \tag{5.59}$$

and

$$\limsup_{t\to\infty} S(t) = \frac{\Lambda}{\mu_1} = 1 \ \text{ a.s.} \tag{5.60}$$

Figure 5.1 indicates that the infected vanishes exponentially, and the susceptible individuals reach their maximum when given a long time run.

Obviously, the basic reproduction number of the deterministic model (1.1) can be computed as $R_0 = 1.1429 > 1$ in this case. And the endemic equilibrium E^* of model (1.1) is globally asymptotically stable according to Theorem 5.2 in [10].

We conclude that in the case of medium perturbation, say $\sigma = 0.6$, the density of the susceptible approaches one almost surely, and is much higher than that of the deterministic model. Compared with the deterministic model, the density of the infected declines fast to zero with the exponential rate -0.08 at early time scale 1.5×10^4 days. And the density of the recovered is somehow affected by the infected and ends up at zero at 2.5×10^4 days instead of persistence for the deterministic model.

Example 2

We keep the initial value and other parameters same as shown in Example 1 except for $\sigma = 0.9$. Here

$$\sigma^2 \geq \frac{\beta^2}{2(\mu_2+\gamma)} = 0.4571, \tag{5.61}$$

and condition (3.18) of Theorem 1.2 is being satisfied, then the solution of model (1.2) admits the following property:

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} \leq -(\mu_2+\gamma) + \frac{\beta^2}{2\sigma^2} = -0.3349 < 0 \ \text{ a.s.} \tag{5.62}$$

The corresponding simulations would be shown in Figure 5.2 to support the main results of Theorem 1.2 we got in the previous section.

Under large perturbation, say $\sigma = 0.9$ in this case, we find that the density of the susceptible behaves the similar dynamics. While the curve of the infected shows more sharper than that in Example 1, and still decays exponentially with a larger rate -0.3349 at early time scale 1×10^4 days. And the density of the recovered is close to zero at 2×10^4 days compared with the deterministic model.

Example 3

Let the intensity of the white noise be $\sigma = 0.1$, and the initial value and other parameters be the same as shown in Example 1. Here

$$\tilde{R}_0 = R_0 - \frac{\sigma^2\Lambda^2}{2\mu_1^2(\mu_2+\gamma)} = 1.1357 > 1, \ \ \sigma^2 = 0.01 \leq \frac{\beta\mu_1}{\Lambda} = 0.8. \tag{5.63}$$

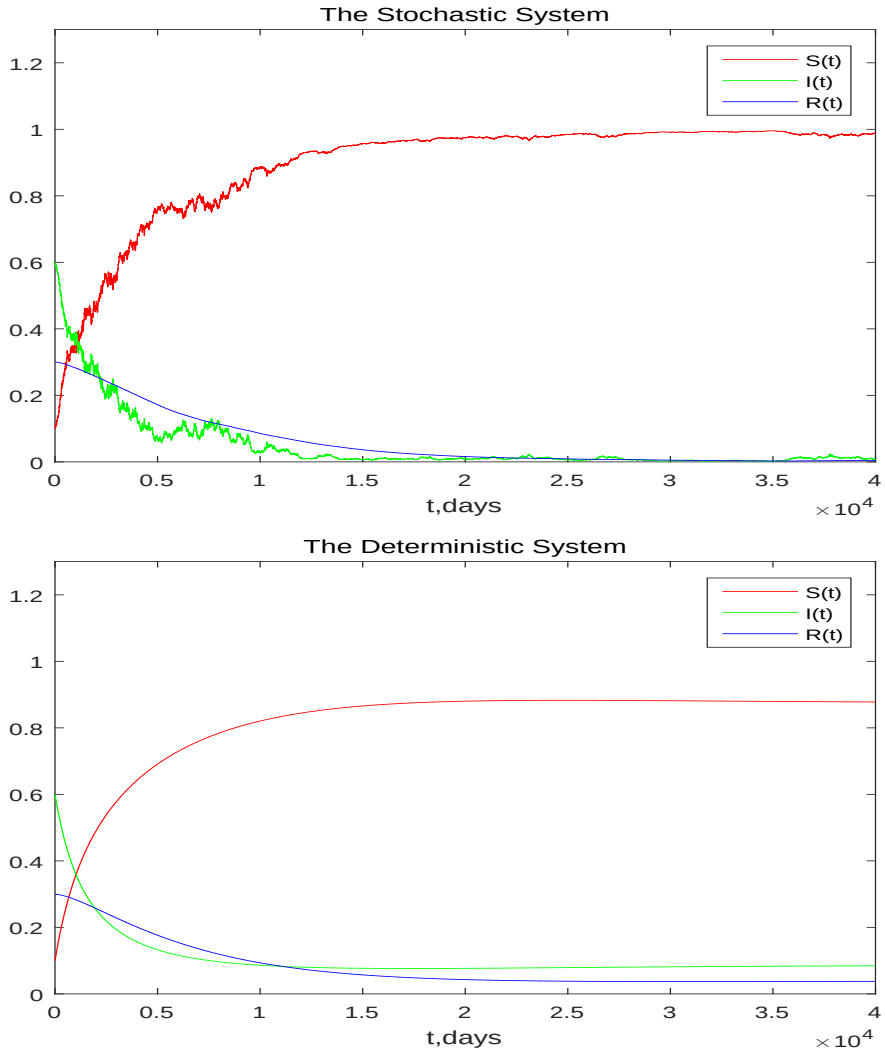

Fig. 5.1. Realizations for model (1.2) with $\tilde{R}_0 < 1$ and model (1.1) with $R_0 > 1$ respectively.

By Theorem 4.1, the following property of model (1.2) holds:

$$0.0817 \leq \liminf_{t\to\infty}\langle I(t)\rangle \leq \limsup_{t\to\infty}\langle I(t)\rangle \leq 0.0827 \ \text{ a.s.} \tag{5.64}$$

If we keep the initial value and other parameters same as shown in Example 1 except for $\mu_2 = 0.3, \gamma = 0.4$. We easily check that Examples 1 and 2 still keep the same conclusions. While the persistence level of model (1.2) is lower than that in (5.64) by Theorem 4.1. That is,

$$0.0638 \leq \liminf_{t\to\infty}\langle I(t)\rangle \leq \limsup_{t\to\infty}\langle I(t)\rangle \leq 0.0646 \ \text{ a.s.} \tag{5.65}$$

Figure 5.3 reveals that the prevalence of the disease takes place under small perturbation of the white noise. The properties of the solutions for the stochastic model (1.2) and the deterministic model (1.1) demonstrate the similar behaviors when set a small perturbation.

We would like to conclude that the density levels for the susceptible, the infected and the recovered are all alike when given a small perturbation environment. Especially, the higher the recovery rate for the infected individuals, the more the infected, such as, the density

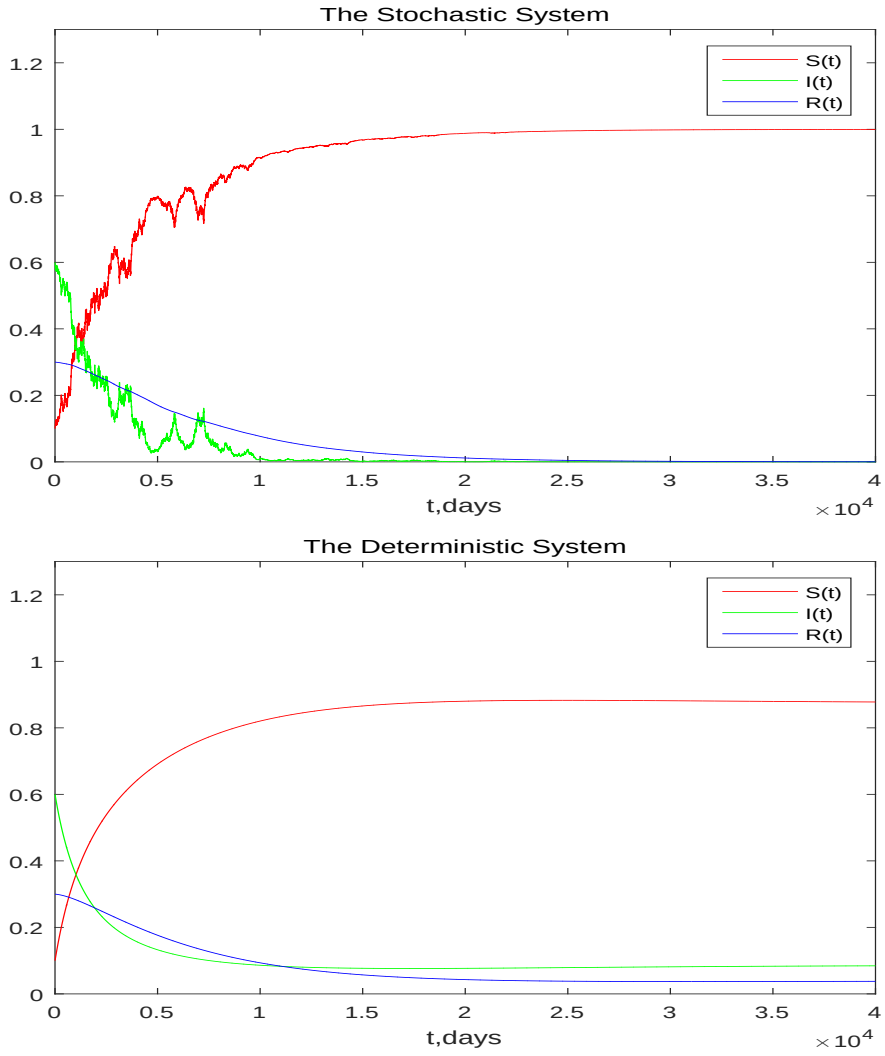

Fig. 5.2. Realizations for model (1.2) with $\sigma = 0.9$ and model (1.1) with $R_0 > 1$ respectively.

of the infected varies from $[0.0817, 0.0827]$ when $\gamma = 0.5$ (the green line in Figure 5.4) to $[0.0638, 0.0646]$ when $\gamma = 0.4$ (the blue line in Figure 5.4).

6. CONCLUSIONS

In this paper, we work on the susceptible-infected-recovered model, where the individuals stayed in the recovered compartment finally lost temporary immunity returned to the susceptible compartment.

The research results of this paper demonstrate that the existence and uniqueness of the global positive solution of model (1.2) has nothing to do with the temporary immunity due to the construction of Lyapunov function (2.10) therewith. While no matter how big the temporary immunity period is, the sufficient condition of the extinction of the diseases merely depends on the parameters of model (1.2), say condition (3.18) or (3.19) is $\Lambda, \mu_1, \mu_2, \beta, \gamma, \sigma$-dependent, and τ-dependent instead in Theorem 2.

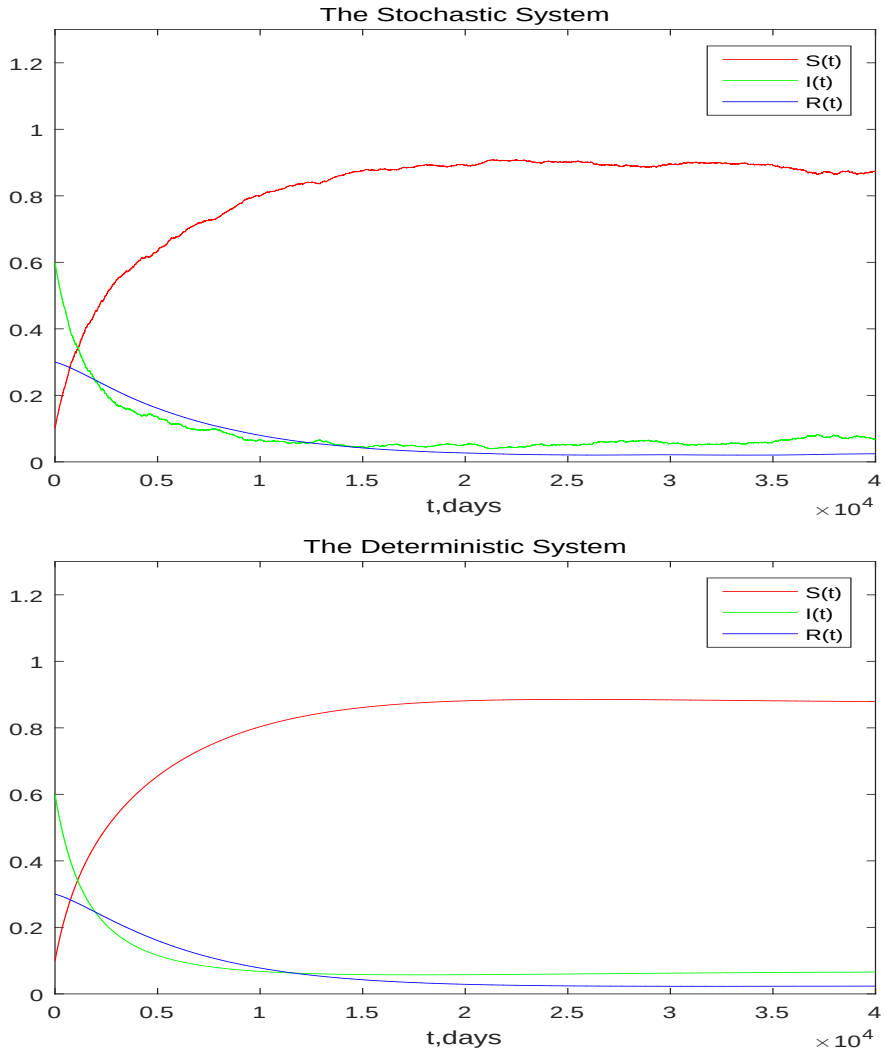

Fig. 5.3. Realizations for model (1.2) with $\sigma = 0.1$ and model (1.1) with $R_0 > 1$ respectively.

We therefore conclude that, in Theorem 3, the expressions of $\tilde{I}_*$ and $\tilde{I}^*$ are the inverse functions of factor $1 - e^{-\mu_3\tau}$, where $\tilde{R}_0$ is a τ-independent expression; and $\tilde{R}_*$ and $\tilde{R}^*$ are respectively the saturated functions of $1 - e^{-\mu_3\tau}$. The illustrative examples have shown that the larger the perturbation, the sharper the infected individuals decline exponentially, the earlier the recovered individuals meet extinction.

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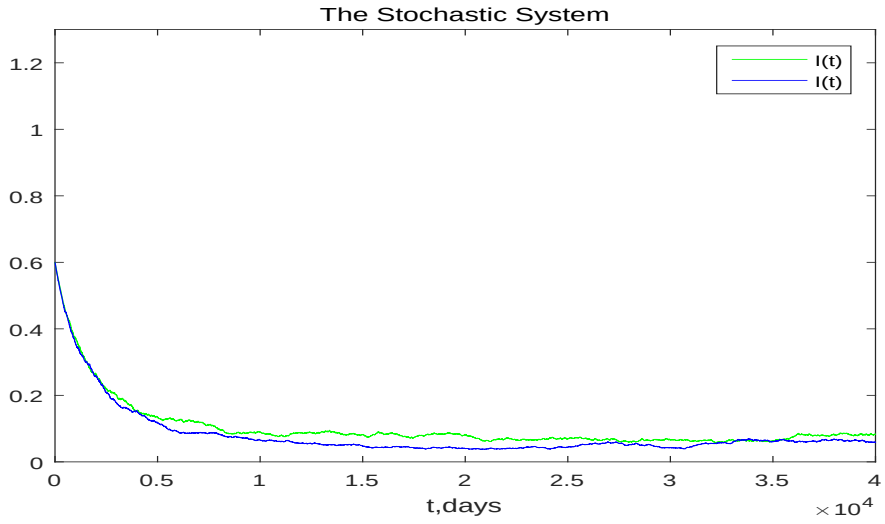

Fig. 5.4. Realizations for model (1.2) with $\gamma = 0.5$ (the green) and $\gamma = 0.4$ (the blue) respectively.

REFERENCES

1. Kyrychko Y. & Blyuss K. (2005) Global properties of a delayed SIR model with temporary immunity and nonlinear incidence rate. *Nonlinear Anal: Real World Appl.*, **6**, 495–507.
2. Xu R. & Ma Z. (2010) Global stability of a delayed SEIRS epidemic model with saturation incidence rate. *Nonlinear Anal: Real World Appl.*, **61**, 229–239.
3. Muroya Y., Enatsu Y. & Nakata Y. (2011) Global stability of a delayed SIRS epidemic model with a non-monotonic incidence rate. *J. Math. Anal. Appl.*, **377**, 1–14.
4. Lahrouz A. (2015) Dynamics of a delayed epidemic model with varying immunity period and nonlinear transmission. *Int. J. Biomath.*, **8**, 1550027.
5. Ma W., Song M. & Takeuchi Y. (2004) Global stability of an SIR epidemic model with time delay. *Appl. Math. Lett.*, **17**, 1141–1145.
6. Guo H. & Li M. (2006) Global dynamics of a staged progression model for infectious diseases. *Math. Biosci. Eng.*, **3** (3), 513–525.
7. Beretta E. & Kuang Y. (2001) Modeling and analysis of a marine bacteriophage infection with latency period. *Nonlinear Anal: Real World Appl.*, **2**, 35–74.
8. Beretta E. & Takeuchi Y. (1995) Global stability of an SIR epidemic model with time delays. *J. Math. Biol.*, **33**, 250–260.
9. Beretta E. & Takeuchi Y. (1997) Convergence results in SIR epidemic models with varying population sizes. *Nonlinear Anal: Theory Methods Appl.*, **28**, 1909–1921.
10. Wen L. & Yang X. Global stability of a delayed SIRS model with temporary immunity. *Chaos Solitons Fractals*, **38**, 221–226.
11. Melnichenko O.A. & Romanyukha A.A. (2008) A model of tuberculosis epidemiology: estimation of parameters and analysis of factors influencing the dynamics of an epidemic process. *Russian J. Numer. Anal. Math. Model.*, **23** (1), 63–75.
12. Melnichenko O.A. & Romanyukha A.A. (2009) A model of tuberculosis epidemiology: Data analysis and estimation of parameters, *Math. Models Comput. Simul.*, **1** (4), 428–444.
13. Liu J.M., Wei F.Y. Dynamics of stochastic SEIS epidemic model with varying population size. *Physica A*. 2016; **464**: pp. 241-250.
14. Wei F.Y., Liu J.M. Long-time behavior of a stochastic epidemic model with varying population size. *Physica A*. 2017; **470**: pp. 146-153.

15. Chen L.H., Wei F.Y. Persistence and distribution of a stochastic susceptible-infected-recovered epidemic model with varying population size. *Physica A*. 2017; **483**: pp. 386-397.
16. Dalal N., Greenhalgh D., Mao X. A stochastic model of AIDS and condom use. *J. Math. Anal. Appl.* 2007; **325**: pp. 36-53.
17. Lahrouz A., Omari L., Kiouach D., Belmaati A. Complete global stability for an SIRS epidemic model with generalized non-linear incidence and vaccination. *Appl. Math. Comput.* 2012; **218** (11): pp. 6519-6525.
18. Zhang X., Jiang D., Alsaedi A., Hayat T. Stationary distribution of stochastic SIS epidemic model with vaccination under regime switching. *Appl. Math. Lett.* 2016; **59**: pp. 87-93.
19. Yu J., Jiang D., Shi N. Global stability of two-group SIR model with random perturbation. *J. Math. Anal. Appl.* 2009; **360**: pp. 235-244.
20. Xue R., Wei F.Y. Persistence and extinction of a stochastic SIS epidemic model with double epidemic hypothesis. *Ann. Appl. Math.* 2017; **33**(1): pp. 77-89.
21. Dalal N., Greenhalgh D., Mao X. A stochastic model of AIDS and condom use. *J. Math. Anal. Appl.* 2007; **325**: pp. 36-53.
22. Mao X., Marion G., Renshaw E. Environmental Brownian noise suppresses explosions in population dynamics. *Stoch. Process Appl.* 2002; **97**; pp. 95-110.
23. Mao X. *Stochastic Differential Equations and Applications* (2nd ed.), Horwood, Chichester, UK, 2007.

