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Haar wavelets and subdivision algorithms on the plane

Tatyana Zaitseva[1*]

[1] *Department of Mechanics and Mathematics, Moscow State University, Moscow, Russia*

Abstract: This paper presents a classification of all two-digits Haar systems and tiles on the two-dimension plane up to affine similarity. We obtain three cases, only two of them (rectangular and Dragon tile) are well known. In all of them we compute the Hölder regularity in L_2 of corresponding Haar functions. The technique of calculating the Hölder regularity is well known for univariate wavelets and it was recently expended into multivariate wavelets. These values also give us the information about the rate of convergence of the corresponding subdivision algorithms and the rate of convergence of the cascade algorithm of the corresponding Haar decomposition.

Keywords: self-similar tilings, Hölder regularity, Haar wavelets, subdivision schemes

1. INTRODUCTION

The subdivision schemes have been studied in the literature in great detail due to many applications in the approximation algorithms and in the curve and surface design [6, 7, 11, 13]. These algorithms possess several remarkable properties: linearity, shift-invariance, computational simplicity. A subdivision algorithm extrapolates a function by its values on a regular grid. In some sense they are development of de Rham cutting corner algorithm. In many numerical problems, they are much more effective than spline algorithms, etc. [5, 8]. In this paper, we address the problem of construction of multivariate subdivision algorithms based on Haar functions in $\mathbb{R}^d$. The procedure of construction of Haar systems is well elaborated [2, 10]. Every Haar basis is generated by a characteristic function of a *tile*, which is a self-similar attractor in $\mathbb{R}^d$, whose translates form a partition of the entire space. Such an attractor is defined by an expanding integer matrix M and a system of integer points ("digits") $d_0, \ldots, d_{m-1}$, where $m = |\det M|$. We classify all two-digits Haar systems and tiles on the two-dimension plane. It turns out, that there are exactly three types of two-digit flate tiles up to affine similarity. Then we compute the Hölder regularity in L_2 for all those Haar functions. The exponents of Hölder regularity are responsible for the rate of convergence of the corresponding subdivision algorithms.

In the next section we introduce all notation and formulate basic properties of tiles and of Haar functions in $\mathbb{R}^d$. Then, in Section 3, we formulate and prove our main result, the classification theorem. Finally, in Section 4, we compute the Hölder regularity of Haar functions and make conclusions on the rate of convergence of the corresponding subdivision algorithms.

*Corresponding author: zaitsevatanja@gmail.com

2. ATTRACTORS, TILINGS, AND HAAR FUNCTIONS

Let $M \in \mathbb{Z}^{d\times d}$ be an integer dilation matrix all of whose eigenvalues are larger than one in the absolute value. Let $m = |\det M|$. The matrix M splits the integer lattice $\mathbb{Z}^d$ into m equivalence (quotient) classes defined by relation $x \sim y \Leftrightarrow y - x \in M\mathbb{Z}^d$. Choosing one representative $d_i \in \mathbb{Z}^d$ from each equivalence class, we obtain a *set of digits* $D(M) = \{d_i : i = 0, \dots, m-1\}$. We always assume that $0 \in D(M)$.

We use the notation $0.d_1d_2\ldots = \sum\limits_{i=1}^{\infty} M^{-i}d_i, d_i \in D(M)$. Consider the following set

$$G = \left\{ \sum_{i=1}^{\infty} M^{-i}d_i : d_i \in D(M) \right\}.$$

We can construct this set even if $D(M)$ is not a correct set of digits and is just a set of m integer vectors including the zero vector (we will call it *attractor*).
By [1, 2], for every expansive integer matrix M and for an arbitrary set of digits D(M), the set G is a compact set with a nonempty interior and possesses the properties:

1. the Lebesgue measure $\mu(G) \in \mathbb{N}$;
2. $G = \bigcup\limits_{d\in D(M)} M^{-1}(G+d)$, the sets $M^{-1}(G+d)$ have intersections of zero measure;
3. the indicator function $\chi = \chi_G(x)$ of G satisfies the refinement equation
$$\chi(x) = \sum_{d\in D(M)} \chi(Mx - d), \quad x \in \mathbb{R}^d;$$
4. $\sum\limits_{k\in\mathbb{Z}^d} \chi(x+k) \equiv \mu(G)$, i.e. integer shifts of χ cover $\mathbb{R}^d$ with $\mu(G)$ layers;
5. $\mu(G) = 1$ if and only if the function system $\{\chi(\cdot + k)\}_{k\in\mathbb{Z}^d}$ is orthonormal.

If $\mu(G) = 1$, then G is called a *tile*. The integer shifts of a tile define a *tiling*. This is a partition of $\mathbb{R}^d$ to integer translates of the tile. This partition is with disjoint interia.

Every tile defines a system of Haar functions, $\psi^{(s)}_{jk}(x) = m^{\frac{j}{2}}\psi^{(s)}(Mj - k), j, k \in \mathbb{Z}^d$, where $s = 1, \dots m-1$; $\psi^{(s)}, s = 1, \dots m-1$, are generating function of the Haar system. Each of them is defined as $\psi^{(s)}(x) = \sqrt{m}(\chi(MG) - \chi(MG + d_s))$. This is a complete orthonormal system in $L_2(\mathbb{R}^d)$ [2]. Thus, to classify Haar wavelets on $\mathbb{R}^d$ one needs to classify all attractors. We will do that for the case $m = 2, d = 2$ in the next section, and then compute the smoothness of those functions. Let us remark, that the smoothness is equal to the rate of convergence of the corresponding subdivision algorithms, therefore computing the regularity give us tight bounds for the rate of convergence.

3. THE CLASSIFICATION THEOREM

Theorem 1:
For $m = 2, d = 2$, there exist three types of attractors up to affine similarity, those types are:
1) $M = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}, D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$
2) $M = \begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix}, D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$
3) $M = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

Thus, every two-digit attractor on the plain is affinely similar to one of those three. We illustrate these three tiles in Fig. 1, 2, 3 (the bounding square in all figures has vertices $(-1,-1),(-1,1),(1,1),(1,-1)$). In Fig. 4, 5, 6, we see the plain tilings by those attractors (the black attractor is initial). This theorem classifies not only all possible attractors but also all possible types of plain two-digit Haar wavelets and the corresponding subdivision schemes.

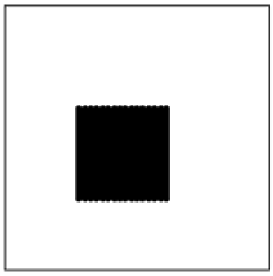

Fig.1: rectangular tile

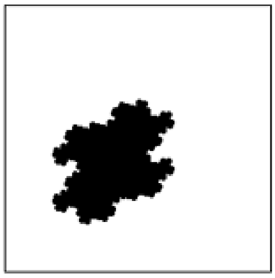

Fig.2: third type of tile

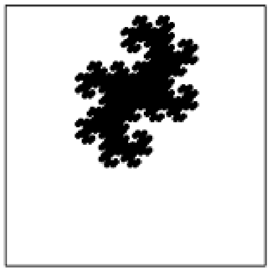

Fig.3: Dragon tile

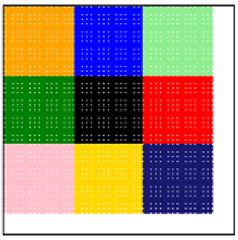

Fig.4: rectangular tile

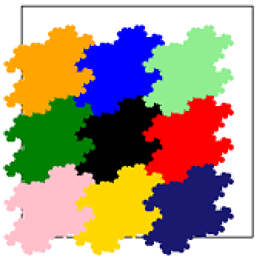

Fig.5: third type of tile

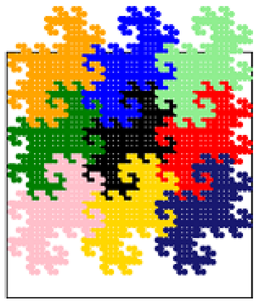

Fig.6: Dragon tile

The proof of this theorem will be splitted to several lemmas.

Lemma 1:
If $m=2, d=2$, attractors G_1 and G_2 with the same dilation matrix M and different sets of digits $D_1(M)$ and $D_2(M)$ are affinely similar.

Proof
Since $m=2$, we can consider an attractor G_0 with matrix M and set of digits $D_0(M)=\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right\}$. It is sufficient to proof that G_1 and G_0 are affinely similar, then analogously G_2 and G_0 are affinely similar and then G_1 and G_2 are affinely similar.

Let $M=\begin{pmatrix}u & v\\ w & z\end{pmatrix}$, $D_1(M)=\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}a\\b\end{pmatrix}\right\}$. Let $X=\begin{pmatrix}a & x_1\\ b & x_2\end{pmatrix}$, $x_1\in\mathbb{R}, x_2\in\mathbb{R}$.
We will proof that we can choose such $x_1\in\mathbb{R}$ and $x_2\in\mathbb{R}$ that $XM=MX$.
It is equivalent to

$$\begin{pmatrix}a & x_1\\ b & x_2\end{pmatrix}\begin{pmatrix}u & v\\ w & z\end{pmatrix}=\begin{pmatrix}u & v\\ w & z\end{pmatrix}\begin{pmatrix}a & x_1\\ b & x_2\end{pmatrix}.$$

$$\begin{pmatrix}au+x_1w & av+x_1z\\ bu+x_2w & bv+x_2z\end{pmatrix}=\begin{pmatrix}au+bv & x_1u+x_2v\\ aw+bz & x_1w+x_2z\end{pmatrix}$$

$$\left\{\begin{array}{c} x_1 w = bv \\ bu + x_2 w = aw + bz \\ av + x_1 z = x_1 u + x_2 v \end{array}\right. \tag{3.1}$$

Suppose that $w = 0$. Then $|\det M| = uz = 2$.

Case 1: $\det M = 2$. Since $u \in \mathbb{Z}$ and $z \in \mathbb{Z}$, we have $\left[\begin{array}{l} u = 2 \\ z = 1 \end{array}\right.$ or $\left[\begin{array}{l} u = -2 \\ z = -1 \end{array}\right.$ or $\left[\begin{array}{l} u = 1 \\ z = 2 \end{array}\right.$ or $\left[\begin{array}{l} u = -1 \\ z = -2 \end{array}\right.$. Then $\operatorname{tr} M = \pm 3$, characteristic polynomial is $\lambda^2 \pm 3\lambda + 2 = 0$ and it has a root ∓ 1, then it is not a dilation matrix, the contradiction concludes the proof for this case.

Case 2: $\det M = -2$. Since $u \in \mathbb{Z}$ and $z \in \mathbb{Z}$, we have $\left[\begin{array}{c} u = -2 \\ z = 1 \end{array}\right.$ or $\left[\begin{array}{c} u = 2 \\ z = -1 \end{array}\right.$ or $\left[\begin{array}{c} u = -1 \\ z = 2 \end{array}\right.$ or $\left[\begin{array}{c} u = 1 \\ z = -2 \end{array}\right.$. Then $\operatorname{tr} M = \pm 1$, the characteristic polynomial is $\lambda^2 \pm \lambda - 2 = 0$ and it has root ± 1, then it is not a dilation matrix.

So $w \neq 0$ and $x_1 = \frac{bv}{w}$. From the second equality we have $x_2 = \frac{1}{w}(aw + bz - bu)$. It remains to verify the third equality if x_1, x_2 are chosen as above. Since $av + x_1 z = av + \frac{bvz}{w} = \frac{bvu}{w} + \frac{awv+bzv-buv}{w} = x_1 u + x_2 v$, we arrive at the third equality in (3.1).

Thus, we find a proper matrix X. Since $XM = MX$, it follows that $XM^{-i} = M^{-i}X$ $\forall i \in \mathbb{Z}$. By construction $X\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}$.

$$G_1 = \left\{\sum_{i\in I} M^{-i}\begin{pmatrix}a\\b\end{pmatrix}, I \subset \mathbb{N}\right\} = \left\{\sum_{i\in I} M^{-i}X\begin{pmatrix}1\\0\end{pmatrix}, I \subset \mathbb{N}\right\} =$$

$$= \left\{\sum_{i\in I} XM^{-i}\begin{pmatrix}1\\0\end{pmatrix}, I \subset \mathbb{N}\right\} = X\left\{\sum_{i\in I} M^{-i}\begin{pmatrix}1\\0\end{pmatrix}, I \subset \mathbb{N}\right\} = XG_0.$$

□

Lemma 2:
Let M_1 and M_2 be integer dilation 2×2 matrices such that $\operatorname{tr} M_1 = \operatorname{tr} M_2, \det M_1 = \det M_2 = \pm 2$ and let D_1 and D_2 be arbitrary two-digit sets from $\mathbb{Z}^2$; then the attractors G_1 and G_2 produced by pairs (M_1, D_1) and (M_2, D_2) are affinely similar.

Proof
It is easy to prove that eigenvalues of M_1 are different (otherwise the discriminant $D = t^2 \pm 8$ of the characteristic polynomial ($x^2 - tx \pm 2$) has to be zero, which is impossible for integer t). Then we have a basis of eigenvectors for M_1 and M_2. Since the assumptions of the lemma are satisfied, there is X such that $XM_1 = M_2X$. This system with four integer coefficients is degenerate because it has a non-trivial solution X. Then it has a rational solution X_0, and therefore an integer solution X_*. $M_1 = X_*^{-1}M_2X_*$.

Using Lemma 1 we can choose any sets of digits. Let $D(M_1) = \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\}$, $\begin{pmatrix}a\\b\end{pmatrix} = X_*\begin{pmatrix}1\\0\end{pmatrix}, a, b \in \mathbb{Z}$, $D(M_2) = \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}a\\b\end{pmatrix}\right\}$.

$$G_1 = \left\{\sum_{i\in I} M_1^{-i}\begin{pmatrix}1\\0\end{pmatrix}, I \subset \mathbb{N}\right\} = \left\{\sum_{i\in I} X_*^{-1}M_2^{-i}X_*\begin{pmatrix}1\\0\end{pmatrix}, I \subset \mathbb{N}\right\} =$$

$$= X_*^{-1}\left\{\sum_{i\in I} M_2^{-i}\begin{pmatrix}a\\b\end{pmatrix}, I \subset \mathbb{N}\right\} = X_*^{-1}G_2.$$

□

Lemma 3:
If $m = 2, d = 2$, M *is a dilation integer matrix, there are six possible pairs of* $(tr\ M,\ det\ M)$.

Proof
Let $t = \operatorname{tr} M$. The characteristic polynomial is $x^2 - tx \pm 2 = 0$.
Let x_1, x_2 be eigenvalues of M. Then $|x_1| > 1, |x_2| > 1$, $|x_1 \cdot x_2| = 2$, and hence $|x_1| < 2, |x_2| < 2$. $|t| \le |x_1| + |x_2| < 4$. t is an integer.

Case 1: $\det M = 2$: $t \ne \pm 3$, because $x^2 \pm 3x + 2$ has root ∓ 1, but M is a dilation matrix, a contradiction. In this case $t = 0, \pm 1, \pm 2$, it is easy to construct examples for all these cases.

Case 2: $\det M = -2$: Since $x_1 \cdot x_2 = -2$, let $x_1 < 0, x_2 > 0$, then $-2 < x_1 < -1, 2 > x_2 > 1$, then $-1 < t = x_1 + x_2 < 1$, t is an integer, then only $t = 0$ is possible, there are easy examples of dilation M with $t = 0$. □

Now we are ready to prove Theorem 1.

Proof
As we can see from these lemmas, there are six types of tiles with different characteristic polynomials. But it is easy to see that cases

1) tiles with $\operatorname{tr} = 0, \det = 2$ and $\operatorname{tr} = 0, \det = -2$ are affinely similar, for example, $\begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ are both rectangles.

2) tiles with $\operatorname{tr} = 1, \det = 2$ and $\operatorname{tr} = -1, \det = 2$ are affinely similar, for example, $\begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -2 \\ 1 & 0 \end{pmatrix}$.

3) tiles with $\operatorname{tr} = 2, \det = 2$ and $\operatorname{tr} = -2, \det = 2$ are affinely similar, for example, $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$.

So we can see that there are only three types. We can obtain the fact, that they are affinely different from their different smoothness, it is calculated in the next section. □

4. REGULARITY OF TWO-DIGIT ATTRACTORS

In this section we compute the exponent of regularity of the three types of the Haar wavelets, classified in Theorem 1. The regularity is important for estimating the rate of convergence of the cascade algorithm of the Haar decomposition and of the subdivision algorithms in approximation theory and surface design.

We fix again an expanding matrix M and the set of digits D. They, as we know, define a unique attractor set K. The characteristic function of this set can also be defined in the framework of refinement equations. A refinement equation is the linear functional difference equation with a contraction of the argument by the matrix M. Thus, general *refinement equation* has the form

$$\varphi(x) = \sum_{k \in \mathbb{Z}^d} c_k \varphi(Mx - k), \tag{4.2}$$

where c_k are prescribed coefficients. Refinement equations have been studied in the literature in great detail due to countless applications in wavelets, approximation theory, etc. (see [1, 2, 5, 7]). We are interested in a special case of refinement equations, where all the coefficients c_k are zeros and ones. This case will be referred to as *Haar case*. Refinement equations of the Haar case generate scaling function for Haar wavelets. In the univariate case, there is only one Haar scaling function $\varphi(x) = \chi_{[0,1]}$. In the multivariate case, an arbitrary tile generate Haar system by its characteristic function used as a scaling function.

Thus, we consider refinement equation (4.2) and suppose that $c_k = 1\ \forall k \in D$ and $c_k = 0$ if $k \notin D$, then all solutions of refinement equation are $\varphi = \lambda\chi_K$ [2].

Next part is devoted to calculating the exact *Hölder exponent (regularity)*

$$\alpha_\varphi = \sup\{\alpha \geq 0 : \|\varphi(\cdot + h) - \varphi\| \leq C|h|^\alpha, \forall h \in \mathbb{R}^d\}$$

of these solutions φ in all three cases. The Hölder exponent is responsible for the rate of convergence of wavelet expansions as well as of subdivision approximation. It is one of the most important characteristics of wavelet systems.

For univariate wavelets, it is well known that Hölder exponent can be expressed with spectral characteristic of special matrices. Recently, this technique was expended into multivariate wavelets. To attack the computation of Hölder regularity, we first construct these matrices. They will be matrices build by coefficients of refinement equation.

Let $D_0 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$. We construct the attractor G using M and D_0, it is a tile in these 3 cases.

Let $\Omega \subset \mathbb{Z}^d$ be a minimal subset of $\mathbb{Z}^d$ with the property $K \subset \Omega + G = \bigcup\limits_{k\in\Omega} (k + G)$. We denote $N = |\Omega|$.

We denote by T_d the transition $N \times N$ matrices $(T_d)_{ab} = c_{Ma-b+d}, a, b, \in \Omega\ \forall d \in D_0$. In our case these matrices are Boolean (consist in 0 and 1). There are two matrices, we will call them T_0, T_1.

It is easy to see that we can have at most one 1 in column: otherwise, for some $a_1, a_2, b \in \Omega,\ d \in D_0, d_1, d_2 \in D\ \ Ma_1 - b + d = d_1, Ma_2 - b + d = d_2,$ then $d_2 - d_1 = M(a_2 - a_1)$, which contradicts the fact that digits are from different classes of equivalence.

We consider the following affine subset of the space $\mathbb{R}^N$:

$$V = \left\{ \omega = \{\omega_1, \omega_2, \ldots, \omega_N\} \in \mathbb{R}^N : \sum_{j=1}^{N} \omega_j = 1 \right\}.$$

We denote the linear part of subspace V by

$$W = \left\{ \omega = \{\omega_1, \omega_2, \ldots, \omega_N\} \in \mathbb{R}^N : \sum_{j=1}^{N} \omega_j = 0 \right\}.$$

As we have seen above, all of T_d are invariant respectively to V and W.

Let $v(x) = (\varphi(x + k_1), \ldots, \varphi(x + k_N), k_i \in \Omega) \in \mathbb{R}^N$.

Then the refinement equation is equal to $v(x) = T_d v(Mx - d), \forall x \in M^{-1}(G + d), d \in D_0$.

Let $U = \operatorname{span}\{v(x_1) - v(x_2) \mid x_1, x_2 \in G\},\ n = \dim U$. Since $v(x) \in V\ \ \forall x \in G,$ it follows that $U \subset W$ and $n \leq N - 1$.

Since all of T_d are invariant respectively to V and W, we have they are invariant respectively to U. Then the restrictions $A_d = T_d|_U$ of the operators T_d are well-defined. We have only two operators in our case, we will call them A_0, A_1.

From [3, Theorem 7] it follows that $\alpha_\varphi = -\log_2(\rho_2(A_0, A_1))$, where $\rho_2(A_0, A_1) = \lim\limits_{m\to\infty} (2^{-m} \cdot \sum\limits_\sigma \|A_{\sigma(1)} \ldots A_{\sigma(m)}\|^2)^{\frac{1}{2m}},\ \sigma\colon \{1, 2, \ldots, m\} \to \{0, 1\}$. ρ_2 is called L_2 *joint spectral radius*. We will use the short notation L_2 *spectral radius*.

Let us see an example. Let $M = \begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix}$, it is the second type from Theorem 1. Let $D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$, it is a correct set of digits.

In this case we obtain that

$$\Omega = \left\{ \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}.$$

As we can see, $N = 9$.

Further there are illustrations of the process (the bounding square has vertices $(-1,-1)$, $(-1,1), (1,1), (1,-1)$):

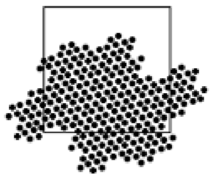

Fig.1: Attractor K

Fig.2: Tilings of G

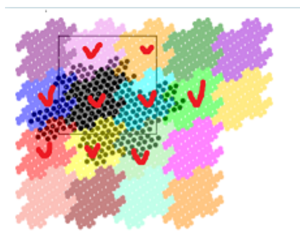

Fig.3: How to find Ω

Let T_0 be the matrix which corresponds to $d = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, let T_1 correspond to $d = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$T_0 = \begin{pmatrix} 0&0&0&0&0&1&0&0&0 \\ 1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&1 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&1&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \end{pmatrix} \qquad T_1 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 1&0&0&0&0&0&1&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0 \end{pmatrix}$$

Let v be the eigenvector of $\frac{T_0+T_1}{2}$ which corresponds to the value 1 (such an eigenvector exists because both T_0 and T_1 have a common left eigenvector $(1, 1, \dots, 1)$). Then we have $v \in V$ and $T_0 v \in V$, then it follows that $u_0 := v - T_0 v \in U$.

Using the algorithm from [3, section 3.2], we regard u_0 as the first basis vector in U. Then for all vectors v_1 in the current basis we try to complement to basis vectors $T_0 v$ and $T_1 v$ (evidently these vectors are in U). If we complement vector $T_0 v$ and vectors in the basis are still linear independent, then we keep this vector in the basis. The same is done with vector $T_1 v$. We keep doing this for all vectors v_1 until our system stays the same.

In our case, $T_0|_U$ and $T_1|_U$ are 5×5 matrices (the coefficients are rounded to three decimal digits after the point):

$$A_0 = T_0|_U = \begin{pmatrix} 0.0 & 0.0 & -1.044 & -0.159 & -0.23 \\ 0.0 & 0.0 & 0.693 & 0.106 & 0.153 \\ 1.088 & 0.0 & 0.214 & -0.167 & 0.256 \\ 0.0 & 0.0 & -0.291 & -0.044 & -0.064 \\ 0.0 & 0.618 & -0.29 & -0.449 & -0.17 \end{pmatrix}$$

$$A_1 = T_1|_U = \begin{pmatrix} 0.0 & 0.0 & -0.174 & -0.425 & -0.843 \\ 0.817 & 0.0 & -0.354 & -0.888 & 0.56 \\ 0.0 & 0.0 & 0.282 & 0.142 & 0.938 \\ 0.0 & 1.0 & 0.463 & 0.34 & -0.235 \\ 0.0 & 0.0 & -0.36 & 0.558 & -0.622 \end{pmatrix}$$

Now we have A_0, A_1 and we can find the coefficient of smoothness. There are different ways to calculate $\rho_2(A_0, A_1)$.

Firstly, we can use formula $\rho_2 = \lambda_{max}(\frac{1}{2}(A_0 \otimes A_0 + A_1 \otimes A_1))$. Secondly, we can consider operator $\mathscr{A}X = \frac{1}{2}(A_0^T X A_0 + A_1^T X A_1)$ in the space of symmetric matrices and then find its spectral radius: $\rho_2 = \lambda_{max}(\mathscr{A})$ (these formulas can be found in [4, 12, 14].
But both ways led us to the same result. In our case $\rho_2 \approx$ *0.7607* and $\alpha_\varphi \approx$ *0.3946*.
We can calculate Hölder regularity of other types of tiles using the same method.

Let $M = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, it is the third type from Theorem 1. We suppose that in this case $D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$ too.
Here $N = 21$ and A_0,A_1 are 5×5 matrices. We obtain that $\rho_2 \approx$ *0.8478* and $\alpha_\varphi \approx$ *0.2382*.

In the first case from Theorem 1, we suppose $M = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$, $D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$.
Here $N = 18$ and A_0,A_1 are 9×9 matrices. We obtain that $\rho_2 \approx$ *0.7071* and $\alpha_\varphi =$ *0.5*.
This answer is natural because the Hölder L_2 regularity of a charasteristic function of a polygon is always 0.5.

Remark 1:
We can note, that in the case of Dragon type Hölder regularity is higher than in the case with $\mathrm{tr} = 1$, *this means that the convergence of subdivision schemes in Dragon case is worse although this case is better known.*

5. CONCLUSION

We have classified all two-digit tiles on the plain up to affine similarity. It turned out that there only three types, two of them (rectangular and Dragon tiles) are well known, the third one is much less studied. For all the types, we computed the Hölder regularity of the corresponding characteristic functions. The Hölder exponents are different, which proved that these classes are also different (0.5 for rectangular, ≈ 0.2382 for Dragon type, ≈ 0.3946 for third type). We find not only the smoothness of two-digit Haar functions, but also the rate of convergence of the corresponding subdivision algorithms. Thus, for two-digit plain Haar system the problem is completely solved. The generalization for bigger dimensions ($d \geq 3$) is possible for two-digit tiles, this is a subject of future research. On the other hand, the generalization to a bigger number of digits seems to be a hard problem, because already for three digits there are infinitely many types of flat tiles (but in case, when our digits are $d_0 = 0$, d_1, d_2 and $\det(d_1, d_2) = \pm 1$, there are 10 types of tiles). The method for computing the L_2 Hölder exponent (and correspondingly, the rate of convergence of the subdivision schemes) is universal and can be applied for every tile, with an arbitrary number of digits and arbitrary dimension.

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