Systems Defined on Dynamic Sets and Analysis of Their Characteristics

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Abstract
(Lin, 1999) describes the static structures of general systems and investigates systems dynamics using the concept of time systems without clearing pointing out where systems attributes would come into play. To resolve this problem, in this paper, we introduce the concept of dynamic sets so that the evolutions and changing structures of systems can be adequately described by using object sets, relation sets, attribute sets, and environments of systems. On the basis of this background, we revisit some of the fundamental properties of systems, including systems emergence, stability, etc.
Keywords Dynamic set, system description frame, attribute set, system property

1 Introduction

Systems research represents such a science that it investigates the structure, evolution, and control of systems. Its goal is to study the interactions of the parts or relevant matters or events that constitute a whole and the consequent emerging holistic properties and behaviors of the whole. This whole that is made up of the parts, relevant matters or events is known as a system. In order to investigate different behaviors of systems, as of this writing, there has been a good amount of published literature, constituting a well developed theory, see (Lin, 1999)[1] and listed references there[2-4]. Zadeh (1962)[5] believes that the main goal of systems science is to study the organization and structure of systems, that is, how parts of a system are organized together to form the whole. From the basis of set theory, (Lin, 1999)[1] develops a multi-relation systems theory of rigor by establishing and analyzing the elementary concepts and fundamental characteristics of systems. That is, Lin establishes the basic concepts and properties of systems science on the solid foundation and rigor of mathematics.

The general systems theory maintains (Lin, 1999)[1] that each system stands for such a whole that it consists of a set of objects and a set of interactions between the objects. Symbolically, a system can be written as

$$S = (M, R) \tag{1.1}$$

where the set $M = \{m_i : i \in I\}$ contains all the elements that make up of the system S, known as the object set, I is an index set, R the set of all relations between the objects in M that describe the system S. For any relation $r \in R$, there is an ordinal number $n(r)$, which is a function of r, such that r is a subset of the Cartesian product of $n(r)$ copies of $M : r \subseteq M^{n(r)}$. Both the object and relation sets M and R together completely describe the given system S, its component parts, and the interactions between the parts. On this basis, leveled systems, time systems, etc., are effectively described and studied in general systems theory.

Because the objects of systems are highly abstract, the only difference between sets of objects will be their index sets. That is why such general theory can be powerfully employed to investigate systems with parts of identical properties. That is, the general system, as given in equ.(1.1), grasps the commonality of objects by ignoring their individual specifics. However, in terms of a realistic system, the objects' specifics might play important roles in the operations of the system. They influence not only the composite of the system, but also the structure and functionality of the system, producing the relevant dynamic behaviors of evolution of the system. For example, a system that is made up of pure oxygen gas or of pure hydrogen gas is fundamentally different of that consisting of both oxygen and hydrogen gases. In particular, the interactions of the objects of the former system are mainly the repulsive and attractive forces between the gas molecules, while in the latter system, other than the similar repulsive and attractive forces, there are also following chemical reactions when the environment provides the needed condition:

$$2H_2 + O_2 \to 2H_2O \tag{1.2}$$

The resultant system after the chemical reactions is different of the system that existed before the reactions in terms of the object set and also the relation set. What is more important is that the property of further chemical reactions in the system resulted from the first round of chemical reactions is essentially different from that of the system that existed before the first round of chemical reactions. If the system that existed before the chemical reactions is written as $S_0 = (M_0, R_0)$ and the system that existed after as $S_1 = (M_1, R_1)$, then it is obvious that $M_0 \neq M_1$ and $R_0 \neq R_1$. Systems S_0 and S_1 are very different due to that the objects in S_0 have undergone substantial change with $2H_2O$ molecules added. Additionally, the interactions between the objects of S_1 are totally different of those in S_0. Other than repulsive and attractive reactions between the gas molecules, the interactions of the objects of S_1 also include those between liquid molecules and gas molecules, and those between liquid molecules.

From this example, it follows that for certain circumstance, it is just not enough to employ only object and relation sets to describe the systems and their behaviors, especially if changes and evolutionary behaviors of the systems are concerned with. It is because in the current general systems theory, both H_2 and

O_2 are treated as abstract objects in S_0, while ignoring their differences in other aspects. When such differences do not affect the evolution of the systems much, the description of systems in equ.(1.1) will be most likely sufficient. However, the undeniable fact is that there are many such systems that the interactions between the objects of specific attributes great affect the structures, functionalities, and evolutions of the systems, just as the differences described by the two gases in equ.(1.2).

Therefore, we need to develop such a systems theory to deal with this situation. We have to further deliberate the description of the object and relation sets of systems so that the consequent evolutions of systems can be adequately investigated.

2 Attributes of Systems

Because the usage of rigorous mathematical language in the discussion of properties of general systems is very advantageous, we will continue to employ this approach. In this paper, we will not consider the case when a set is empty; and all sets are assumed to be well defined. That is, we do not consider such paradoxical situations as the set containing all sets as its elements. To this end, those readers who are interested in set theory and the rigorous treatment of general systems theory are advised to consult with (Lin, 1999)[1].

As discussed earlier, our main focus here is the difference between various parts of a system, that is, the differences between the system's objects and between the relations of the objects. We will employ the concept of attributes to describe such differences. By attribute, it means a particular property of the objects and their interactions and the overall behavior and evolutionary characteristic of the system related to the property.

If speaking in the abstract language of mathematics, the attributes of a system are a series of propositions regarding the system's objects, relations between the objects, and the system itself. Due to the differences widely existing between systems' objects, between systems' relations, and between systems themselves, these propositions can vary greatly. For example, a proposition might hold true for some objects, and become untrue for others. In this case, we say that these objects have the particular attribute, while the others do not. As another example, a proposition can be written as a mapping from the object set to the set of all real numbers so that different objects are mapped onto different real numbers. In this case, we say that the system's objects contain differences.

Additionally, the discussion above also indicates that each system evolves with time so that its objects, relations, and attributes should all change with time. So, when we consider the general description of systems, we must include the time factor. By doing so, the structure of a system at each fixed time moment is embodied in the object set, relation set, and attribute set, while the evolution of the system is shown in the changes of these sets with time.

Summarizing what is analyzed above, we can define the system of our concern as follows:

Definition 2.1. Assume that T is a connected subset of the interval $[0, +\infty)$, on which a system S exists. Then, for $t \in T$, the system S is defined as the following order triplet:

$$S_t = (M_t, R_t, Q_t) \tag{2.1}$$

where M_t stands for set of all objects of the system S at the time moment t, R_t the set of the relations between the objects in M_t, and Q_t the set of all attributes of the system S. For the sake of convenience of communication, T is referred to as the life span and S_t the momentary system of the system S.

What needs to be emphasized is that we study not only the evolution of systems with time, but also the evolution of the system along with continuous changes of some conditions, such as temperature, density, etc. In such cases, T will be understood as one of those external conditions.

The object set of the system of our concern is made up of the system's fundamental units. So, for each $t \in T$, the objects in M_t are fixed. Let us write

$$\widehat{M} = \{m_{t,a} : a \in I_t, t \in T\} = \bigcup_{t \in T} M_t \tag{2.2}$$

where $M_t = \{m_{t,a} : a \in I_t\}$ is the object set of the momentary system S_t, and I_t the index set of M_t as a function of time t. If for any $t \in T$, $M_t = M$, for some set M, then we say that the object set of the system S is fixed.

One of the most elementary relations between objects is binary, relating each pair of objects. Such a relation can be written by using the 2-dimensional Cartesian product of the system's object set:

$$R^0_{t,2} = \{(m_{t,a}, m_{t,b}) \in M_t \times M_t : \Phi^0_{t,2}(m_{t,a}, m_{t,b})\} \subseteq M_t^2 \tag{2.3}$$

for each $t \in T$, where $\Phi^0_{t,2}(,)$ is a proposition that defines $R^0_{t,2}$. Corresponding to different properties, the system S_t might contain different binary relations as subsets of the 2-dimensional Cartesian product M_t^2 of the object set. The set of all the binary relations in S_t is written as follows:

$$R_{t,2} = \{R^k_{t,2} : k \in K_2\} \tag{2.4}$$

where K_2 is the index set of all binary relations of S_t. Similarly, the system S_t might contain trinary relations:

$$R^0_{t,3} = \left\{(m_{t,a}, m_{t,b}, m_{t,c}) \in M_t^3 : \Phi^0_{t,3}\left(m_{t,a}, m_{t,b}, m_{t,c}\right)\right\} \subseteq M_t^3 \tag{2.5}$$

for each $t \in T$, where $\Phi^0_{t,3}(,,)$ is a proposition that defines $R^0_{t,3}$. The set of all trinary relations of S_t is written as follows:

$$R_{t,3} = \{R^k_{t,3} : k \in K_3\} \tag{2.6}$$

where K_3 is the index set of all trinary relations of S_t. Higher order relations of S_t can be introduced similarly. For convenience, let us define unitary relations of S_t as follows:

$$R^0_{t,1} = \{(m_{t,a}) \in M_t : \Phi^0_{t,1}(m_{t,a})\} \subseteq M_t \tag{2.7}$$

for each $t \in T$, where $\Phi^0_{t,1}()$ is a proposition that defines $R^0_{t,1}$. The set of all unitary relations of S_t is written as follows:

$$R_{t,1} = \{R^k_{t,1} : k \in K_1\} \tag{2.8}$$

where K_1 is the index set of all unitary relations of S_t. Now, the set R_t of relations of S_t can be written as follows:

$$R_t = \bigcup_{\alpha \in Ord} R_{i,\alpha} \tag{2.9}$$

where Ord stands for the set of all ordinal numbers. When Ord is taken to be N = the set of all natural numbers, equ. (2.9) stands for the set of all finite relations the momentary system S_t contains.

From the discussion above, it can be seen that all kinds of algebras and spaces studied in mathematics are special cases of equ. (2.9) with Ord replaced by N. Without loss of generality, let us assume that $M_t \in R_{t,1}$. With this convention, it can be seen that in the following when we talk about the attributes of a system S_t, we only need to mean the attributes of the relations of S_t, because the object set is also considered a relation, a unitary relation and the attributes of objects are now also those of relations.

The attribute set Q_t of the momentary system S_t are a series of propositions about the relations. Symbolically, we can write:

$$Q_t = \{q_t(r_1, r_2, ..., r_\alpha, ...) : q_t(...) \textit{ is a proposition of } r_\alpha \in R_t, \alpha = 1, 2, 3...\} \tag{2.10}$$

Without any doubt, with this notation in place, these propositions in Q_t embody all aspects of the momentary system S_t. For example, the concept of mass of object in physics is a mapping:

$$\mu_t : R_t \to R^+ \tag{2.11}$$

which assigns each element in an unitary relation the mass of the element, and each element $\vec{x} = (x_1, x_2, ..., x_\alpha, ...)$ in an n-nary relation the sum of the masses of the objects contained in the element, if the sum exists. That is, for any $r \in R_t$, and any $\vec{x} = (x_1, x_2, ..., x_\alpha, ...) \in r$,

$$\mu_t(\vec{x}) = \mu_t(x_1, x_2, ..., x_\alpha, ...) = \sum\nolimits_\alpha \mu_t(x_\alpha) \tag{2.12}$$

assuming that the sum on the right hand side converges.

Let us look at the network model of systems as an example, where only unitary and binary relations are considered. In this model, all objects of the system of concern are treated as nodes; each binary relation is modeled as the set of edges of the network. By doing so, each binary relation of the system corresponds to a network or a graph; and different binary relations correspond to different sets of edges. Such correspondence can be seen as an attribute of the edges. Accordingly, different unitary relations of the system correspond to different attributes of nodes in the network, such as size of the nodes, flow intensities of the nodes, etc. If different types of edges are treated as identical, then the network can be expressed as $G = (V, E, Q)$, where V stands for the set of all nodes, E the set of edges, and Q some attributes of either the nodes or the edges or both, such as weights of the edges. The ordered pair (V, E) completely describes the topological structure of the network, which is sufficient for some applications. However, if the system we investigate is quite specific, for example, it is a network of railroads, a network of human relationships, etc., we may very well need to model multiple relations. In this case, the attribute set Q can be employed to describe the scales of the stations in the railroad network, the traffic conditions or transportation capabilities between stations, etc. If the system is a network of human relationships, then Q can be utilized to represent the social status of each individual person, the intensity of interaction between two chosen persons, etc. As a matter of fact, the concept of systems, as defined in equ.(2.1), is a generalization of that as defined in equ.(1.1) (Lin, 1999)[1]. In other words, we can rewrite equ.(1.1) in the format of equ.(2.1) as follows.

Let all object sets be static. So, for any $t \in T$, we have $M_t = M$ and $R_t = R$. And what is interesting is how an attribute is introduced. To this end, we can introduce a proposition q_0 on the Cartesian product $\widehat{M} = \sum_{\alpha \in Ord} M^\alpha$ of the object set M so that for any $r \in \widehat{M}$,

$$q_0(r) = \begin{cases} 1, if\ r \in R \\ 0, otherwise \end{cases} \tag{2.13}$$

Then, we take $Q = \{q_0\}$. That is, the attribute set is a singleton. Now, each system written in the format of equ. (1.1) is rewritten in the format of equ. (2.1). Here, the attribute q_0 describes if an arbitrarily chosen relation in the Cartesian product $\widehat{M}$ belongs to the system's relation set R or not. In essence, it restates the membership relation to the relation set R from the angle of attributes. In particular, because the relation set R of the system S is a subset of $\widehat{M}$; now the membership in the relation set R is determined by a proposition q_0, while such a description is an attribute of the system S. That is, the general systems theory developed on set theory (Lin, 1999)[1] has already implicitly introduced the concept of attributes. What we do here is to make this fact explicit. And because the systems we are interested in can have multiple attributes, our contribution to the general systems theory is to make the concept of attributes more general as a set Q of attributes, including more than just the particular attribute q_0.

3 Subsystems

Just like each set has its own subsets, every system has subsystems, which can be constructed from the object set, relation set, and the attribute set of the system. In short, a system s is a subsystem of the system S, provided that the object set, relation set, and attribute set of s are corresponding subsets of those of S so that the restrictions of the attributes of S on s agree with the attributes of s. Symbolically, we have

Definition 3.1. Let $s_t = (m_t, r_t, q_t)$ and $S_t = (M_t, R_t, Q_t)$ are two systems, for any $t \in T$. If the following hold true:

$$m_t \subseteq M_t, r_t \subseteq R_t|_{s_t}, and\ q_t \subseteq Q_t|_{s_t}, \forall t \in T \tag{3.1}$$

then s_t is known as a subsystem of S_t, denoted

$$s_t < S_t, \forall t \in T, or\ s < S \tag{3.2}$$

where $R_t|_{s_t}$ and $Q_t|_{s_t}$ represent respectively the restrictions of the relations and attributes in R_t and Q_t on the system S_t and are defined as follows:

$$R_t|_{s_t} = \{r|_{m_t} : r \in R_t\} \ \ and \ \ Q_t|_{s_t} = \{q|_{m_t \cup r_t} : q \in Q_t\}.$$

In this definition, the notation of less than of mathematics is employed for the relationship of subsystems, because the relation of subsystems can be seen as a partial ordering on the collection of all systems. Similarly, when equ. (3.1) does not hold true, we say that s is not a subsystem of S, denoted $s_t \not< S_t$, $\forall t \in T$, or $s \not< S$. Let the set of all subsystems of S be $\mathbb{S}$, then we have

$$\mathbb{S} \equiv \{s : s < S\} = \{s_t : s_t < S_t, \forall t \in T\} \tag{3.3}$$

For any given system $S = (M, R, Q)$, where $M = \{M_t : t \in T\}$, $R = \{R_t : t \in T\}$, and $Q = \{Q_t : t \in T\}$, as defined by equ. (2.1), take a subset set of its object set

$A = \{A_t \subseteq M_t : t \in T\}$. Then by restricting the relation set R and the attribute set Q on A, we obtain the following subsystem of S induced by A:

$$S|_A = \{s_t = (A_t, R_t|_{A_t}, Q_t|_{(A_t, R_t|_{A_t})}) : t \in T\} \tag{3.4}$$

where the restrictions $R_t|_{A_t}$ and $Q_t|_{A_t}$ are assumed respectively to be

$$R_t|_{A_t} = \{r|_{A_t} : each\ element\ in\ r|_{A_t} has\ the\ same\ length, r \in R_t\} \tag{3.4a}$$

and

$$Q_t|_{(A_t,R_t|_{A_t})} = \{q|_{(A_t,R_t|_{A_t})} : q_t|_{(A_t,R_t|_{A_t})} is\ a\ well\ defined\ proposition\ on\ s_t, q \in Q_t\} \tag{3.4b}$$

Proposition 3.1. The induced subsystem $S|_A$ is the maximum subsystem induced by A.
Proof. Let $s = \{(A_t, r_{t,A}, q_{t,A}) : t \in T\} < S$ be an arbitrary subsystem with the entire A as its object set. According equ. (3.1) we have

$$r_{t,A} \subseteq R_t|_s, and\ q_{t,A} = Q_t|_s = Q_t|_{(A_t, r_{t,A})}, \forall t \in T$$

So, from equ. (3.4a), it follows that $s \leq S|_A$ only when $r_{t,A} = R_t|_s$, the equal sign holds true. QED.
Proposition 3.2. Assume that A and B are subsets of M satisfying that $B \subseteq A \subseteq M$, then $S|_B \leq S|_A$.
Proof. According equ. (3.4), we have $S|_B = \{s_t = (B_t, R_t|_{B_t}, Q_t|_{(B_t,R_t|_{B_t})}) : t \in T\}.B \subseteq A$ implies that $B_t \subseteq A_t, \forall t \in T$. So, it follows that $R_t|_{B_t} \subseteq R_t|_{A_t}$ and consequently $R_t|_{B_t} \subseteq (R_t|_{A_t})|_{B_t}$ and $Q_t|_{(B_t,R_t|_{B_t})} = (Q_t|_{(A_t,R_t|_{A_t})})|_{(B_t,R_t|_{B_t})}$. Therefore, $S|_B \leq S|_A$. QED.
Proposition 3.3. Let S be a system. Then the collection of all subsystems of S forms a partially ordered set by the subsystem relation "$<$".
Proof. This result is a straightforward consequence of Proposition 3.2. QED.

Assume that S^1, S^2, and S are systems with the same time span such that $S^1 < S$ and $S^2 < S$. Let $\mathbb{S}^i, i = 1, 2$, denote the set of all subsystems of S^i. Then each element in the set $\mathbb{S}^1 - \mathbb{S}^2 = \{s : s < S^1, s \nless S^2\}$ is a subsystem of S^1 but not a subsystem of S^2. Similarly, each element in the set $\mathbb{S}^2 - \mathbb{S}^1 = \{s : s < S^2, s \nless S^1\}$ is a subsystem of S^2 but not a subsystem of S^1. Let $\mathbb{S}^1 \Delta \mathbb{S}^2 = \{s : s < S^1, s \nless S^2\} \cup \{s : s < S^2, s \nless S^1\}$ be the union of the previous two sets of subsystems of either S^1 or S^2; and $\mathbb{S}^1 \cap \mathbb{S}^2 = \{s : s < S^1, s < S^2\}$ the set of all subsystems of both S^1 and S^2.
Proposition 3.4. Assume that $S^1 < S$ and $S^2 < S$. Then $\mathbb{S}^1 \cup \mathbb{S}^2 = \{s : s < S^1\} \cup \{s : s < S^2\}$ is a subset of $\mathbb{S}|_{M^1 \cup M^2}$.
Proof. $S|_{M^1 \cup M^2}$ is a maximal element in the partially ordered set $(\mathbb{S}|_{(M^1 \cup M^2)}, \subseteq)$, satisfying $\forall s \in S|_{M^1 \cup M^2}, s < S|_{M^1 \cup M^2}$. On the contrary,$\forall s < \mathbb{S}|_{M^1 \cup M^2}$, we

have $s \in \mathbb{S}|_{M^1 \cup M^2}$. So, $\forall s \in \mathbb{S}^1 \cup \mathbb{S}^2$, we have $s \in \mathbb{S}|_{M^1 \cup M^2}$. QED.

What this result indicates is that the union $\mathbb{S}^1 \cup \mathbb{S}^2$ of the sets of subsystems of two subsystems S^1 and S^2 is a subset of the $\mathbb{S}|_{M^1 \cup M^2}$ of the subsystems of the induced system on the union $M^1 \cup M^2$.

Proposition 3.5. Given two arbitrary systems S^1 and S^2, there is always a system S^{12} such that $S^1 < S^{12}$ and $S^2 < S^{12}$.

Proof. Without loss of generality, assume that $S^1 = \{(M_t^1, R_t^1, Q_t^1) : t \in T^1\}$ and $S^2 = \{(M_t^2, R_t^2, Q_t^2) : t \in T^2\}$. To construct the system $S^{12} = \{(M_t^{12}, R_t^{12}, Q_t^{12}) : t \in T^{12}\}$, $T^{12} = T^1 \cup T^2$, we first assume that the object sets M_t^1 and M_t^2 are disjoint, that is, $M_t^1 \cap M_t^2 = \emptyset$, for any $t \in T^1 \cap T^2$. Then, the desired system S^{12} is defined as follows:
for $t \in T^1 \cap T^2$,

$$\begin{cases} M_t^{12} = M_t^1 \cup M_t^2 \\ R_t^{12} = R_t^1 \cup R_t^2 \\ Q_t^{12} = Q_t^1 \cup Q_t^2 \end{cases} \tag{3.5}$$

for $t \in T^1 - T^2$, define $M_t^{12} = M_t^1$, $R_t^{12} = R_t^1$, and $Q_t^{12} = Q_t^1$, and for $t \in T^2 - T^1$, define $M_t^{12} = M_t^2$, $R_t^{12} = R_t^2$, and $Q_t^{12} = Q_t^2$, and is denoted as $S^{12} = S^1 \oplus S^2$. Now, if $M_t^1 \cap M_t^2 \neq \emptyset$, for some $t \in T^1 \cap T^2$, we simply take two systems ${}^*S^1 = \{({}^*M_t^1, {}^*R_t^1, {}^*Q_t^1) : t \in T^1\}$ and ${}^*S^2 = \{({}^*M_t^2, {}^*R_t^2, {}^*Q_t^2) : t \in T^2\}$ with $M_t^1 \cap M_t^2 = \emptyset$, for any $t \in T^1 \cap T^2$, such that ${}^*S^i$ is similar to $S^i, i = 1, 2$, where similar systems are defined in the same fashion as in [1]. Then, we define $S^{12} = {}^* S^1 \oplus {}^* S^2$. Up to a similarity, the system S^{12} is uniquely defined. Therefore, it can be seen as well constructed such that $S^1 < S^{12}$ and $S^2 < S^{12}$, where the time spans T^1 and T^2 are seen as the same as T^{12} such that when $t \in T^1 - T^2$ (respectively, $t \in T^2 - T^1$), we treat S_t^2 (respectively,S_t^1) as a system with empty object set. QED.

From this proposition, it follows that when the interactions of some given systems are considered, these systems can always be seen as subsystems of a larger system. On the other hand, this proposition also shows that there is always some kind of interaction between two given systems, which is embodied in the fact that they are subsystems of a certain system.

4 Interactions between Systems

When there is an interaction between two objects of a system $S = (M, R, Q)$ (of time span T), where $M = \{M_t : t \in T\}$, $R = \{R_t : t \in T\}$, and $Q = \{Q_t : t \in T\}$, it can be described by using a binary relation of the system. We say that objects m_1 and $m_2 \in M$, which means either $m_i = (m_t)_{t \in T}$ such that $m_t \in M_t$, for each $t \in T$, or $m_i = m_t \in M_t$, for a particular $t \in T, i = 1, 2$, interacts with respect to an attribute $q \in Q$, it means that the proposition q holds true for a binary relation

$r \in R$ that contains either (m_1, m_2) or (m_2, m_1) or both. The idea of interactions between systems is a natural generalization of that between two objects. Based on the discussion of the previous section, we will discuss interactions of systems in the framework of subsystems. Assume that $S_i < S, i = 1, 2$. Now, let us look at how these subsystems could interact with each other.

Definition 4.1. The system S_1 is said to have a weak effect on the system S_2 with respect to an attribute $q \in Q$, provided that for any $r_2 \in R_2$ there is $r_1 \in R_1$ such that the proposition q holds true for the ordered pair (r_1, r_2). When no confusion is caused, we simply say that the system S_1 affects the system S_2 weakly without mentioning q. If the system S_2 also exerts a weak affect of S_1, then we say that these systems interact with each other weakly.

Definition 4.2. The system S_1 is said to have a strong effect on the system S_2 with respect to an attribute $q \in Q$, provided that for any $r_2 \in R_2$ and any $r_1 \in R_1$, the proposition q holds true for the ordered pair (r_1, r_2). When no confusion is caused, we simply say that the system S_1 affects the system S_2 strongly without mentioning q. If the system S_2 also exerts a strong affect of S_1 , then we say that these systems interact with each other strongly.

From these definitions, it follows that strong interaction requires interactions between every ordered pair of objects, which is a more rigorous requirement than that of weak interactions. Also, to maintain the intuition behind the concepts of interactions, in Definitions 4.1 and 3.2, we only look at two relations $r_i \in R_i, i = 1, 2$. In order to capture the general spirit, these individual relations should be replaced by subsets $\{r_i \in R_i : \Phi_i(r_i)\}$, where $\Phi_i()$ stands for the proposition that defines the set, for $i = 1, 2$. By doing so, what are discussed in Definitions 4.1 and 4.2 become special cases.

Definition 4.3. Given a subsystem $s = (m, r, q)$ of a system $S = (M, R, Q)$, the totality of all objects in $M - m$, each of which interacts weakly with at least one object in m, is known as the environment of the subsystem s in S, denoted E^S.

5 Systems Properties Based on Dynamic Set Theory

5.1 Basic Properties

For two given systems $S^1 = \{S_t^1 = (M_t^1, R_t^1, Q_t^1) : t \in T^1\}$ and $S^2 = \{S_t^2 = (M_t^2, R_t^2, Q_t^2) : t \in T^2\}$, let us consider

Definition 5.1. These systems S^1 and S^2 are equal, provided that

$$M_t^1 = M_t^2, R_t^1 = R_t^2, Q_t^1 = Q_t^2, and\ T^1 = T^2, \tag{5.1}$$

for each $t \in T^1 = T^2$.

The systems S^1 and S^2 are said to be identical on the time period $T \subseteq T^1 \cap T^2$, it means that

$$M_t^1 = M_t^2, R_t^1 = R_t^2, Q_t^1 = Q_t^2, for\ each\ t \in T. \tag{5.2}$$

Definition 5.2. The system S^1 is said to be homomorphically embeddable into the system S^2, provided that there is a non-decreasing mapping $f : T^1 \to T^2$ such that for any $t_1 \in T^1$, if $t_2 = f(t_1) \in T^2$, then $S^1_{t_1} = S^2_{t_2}$, or equivalently,

$$M^1_{t_1} = M^2_{t_2}, R^1_{t_1} = R^2_{t_2}, and\ Q^1_{t_1} = Q^2_{t_2} \tag{5.3}$$

The mapping f is referred to as an embedding mapping from S^1 into S^2. If the system S^1 can be homomorphically embeddable into S^2 and S^2 into S^1, then the systems S^1 and S^2 are said to be homophorhically equivalent. Evidently, equal systems are homomorphically equivalent with the identity mapping on the time set as the canonical embedding mapping.

Proposition 5.1. If the embedding mapping $f : T^1 \to T^2$ from the system S^1 into S^2 is bijective, then the systems are homomorphically equivalent.

Proof. It suffers to show that the inverse mapping $f^{-1} : T^2 \to T^1$ is an embedding mapping from S^2 into S^1.

Because f is bijective, it is strictly increasing from T^1 into T^2; and its inverse f^{-1} is also a strictly increasing mapping from T^2 into T^1 satisfying for any $t_2 \in T^2$, if $t_1 = f^{-1}(t_2) \in T^1$, then $M^2_{t_2} = M^1_{t_1}, R^2_{t_2} = R^1_{t_1}$, and $Q^2_{t_2} = Q^1_{t_1}$. Therefore, $f^{-1} : T^2 \to T^1$ is an embedding mapping from S^2 into S^1. QED.

Definition 5.3. A system $S = \{S_t = (M_t, R_t, Q_t) : t \in T\}$ is said to be cyclic or periodic, provided that there is time $t_c > 0$ such that $S_{t+t_c} = S_t$, for any $t \in T$. Evidently, in this case, for any natural number $n \in N$, nt_c is also a period of the system S. The minimum period is named the period of S, denoted T_c.

5.2 Systemic Emergence

By systemic emergence, it means the properties of the whole system that parts of the system do not have. Such a property might suddenly appear at a particular time moment. In terms of the system, the holistic emergence is mainly created and excited by the system's specific organization of its parts and how these parts interact, supplement, and constrain on each other. It is a kind of effect of relevance, the organizational effect, and the structural effect.

To be specific, let P represent such a property. It is defined on the entire system S. Because of the system's dynamic characteristics, the interactions between the system's objects, relations, attributes, and the environment change constantly. So, the value of P also varies accordingly. Define

$$P(S_t) = \begin{cases} 1, if\ S\ has\ this\ property \\ 0, otherwise \end{cases} ,\forall t \in T \tag{5.4}$$

It satisfies the following properties:

$$P(s_t) = 1 \to P(S_t) = 1, \forall s < S, \forall t \in T \tag{5.5}$$

That is, as long as a subsystem s has this property, the overall system S also has the property. That is another way to say that the whole is greater than the sum of parts. Of course, there are also such properties that some subsystems have, while the overall system does not have. For instance, let P be a property the system S does not have. Define $P^{-1}(1) = \{s < S : s \neq S\}$ be the collection of all proper subsystems of S, where the superscript (-1) stands for the inverse operation. Then, for any $s \in P^{-1}(1)$, $s < S$ and $P(s) = 1$. However, $P(S) = 0$ does not satisfy equ. (5.5).

When a property P is said to be system S's holistic emergence, provided that for any subsystem $s < S$, $P(s_t) = 0$, $\forall t \in T$, and $P(S_{t_0}) = 1$ for at least one $t_0 \in T$.

In the traditional static description of systems, there is another definition of systemic emergence. In particular, consider a monotonically increasing sequence of subsystems $\emptyset \neq s^1 < s^2 < ... < s^k < ... < S$, there is a k_0 such that

$$P(s^k) = 0, \forall k < k_0, P(s^{k_0}) = 1 \tag{5.6}$$

In this case, the system S is said to have the emergent property P, while its parts $s^k, k < k_0$, do not share this property until they reach the whole s^{k_0} of certain scale. From a detailed analysis, it follows readily that what is just presented is a special case of the systemic emergence of dynamic systems, where we can surely treat the collection of all the superscript k as a subset of the time index T so that the sequence $\{s^k : k = 1, 2, 3, ...\}$ of subsystems a subsystem of a dynamic system by letting $S_t = s^t$, for $t = 1, 2, 3, ...$ From equ. (5.6), it follows that

$$P(s_t) = 0, for\ t = 1, 2, 3, ... < k_0, and\ P(s_{k_0}) = 1$$

That is, the property P emerges at the time moment k_0.

5.3 Stability of Systems

By stability, it means the maintenance or continuity of a certain measure of the dynamic system on a certain time scale. That is, under small disturbances, the measure does not undergo noticeable changes. Let us look at the trajectory system of a single point, where the focus is how the point moves under the influence of an external force. If a disturbance is given to a portion of the trajectory that is not at a threshold point, then the disturbed trajectory will not differ from the original trajectory much. However, if the same disturbance is given to the trajectory at an extremely unstable extreme point, then a minor change in the disturbed value could cause major deviations in the following portion of the trajectory. Thus, only the trajectory of motion at stable critical extrema is stable, while the trajectory systems with instable critical points, such as saddle point, are instable.

In general, assume that an attribute $q \in Q$ of the system S satisfies that q is a real-valued function defined for each momentary system S_t, for any $t \in T$. Let $t_0 \in T$. If for any $\varepsilon > 0$, there is a $\delta_{t_0,\varepsilon} > 0$ such that

$$|q(s_{t_0}) - q(s_t)| < \varepsilon, \forall t \in (t_0 - \delta_{t_0,\varepsilon}, t_0 + \delta_{t_0,\varepsilon}) \tag{5.7}$$

then the attribute q of the system S is regionally stable over time at $t_0 \in T$.

If for any $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) = \delta_\varepsilon > 0$ such that

$$|q(s_{t_1}) - q(s_{t_2})| < \varepsilon, \forall t_1, t_2 \in T \text{ such that } |t_1 - t_2| < \delta_\varepsilon \tag{5.8}$$

then the attribute q is said to be holistically stable over time or uniformly stable over time. When no confusion can be caused, the previous concepts of stability of the attribute q are respectively referred to as that the system S is regionally stable at $t_0 \in T$, or uniformly stable over T.

In addition, the structural stability of systems can also be defined. In particular, Let $d \in Q$ be such that $d : \mathbb{S}_t \to R^+$ is a positively real-valued function defined for each subsystem of the momentary system $S_t, t \in T$, satisfying

(1)$d(\emptyset) = 0$,where $\emptyset$ stands for the the subsystems with the empty set as their object set;

(2)$\forall S^1, S^2 < S, if\ S^1 < S^2, then\ d(S^1) \leq d(S^2)$;

(3)$\forall S^1, S^2 < S, if\ S^1 \triangle S^2 = \emptyset, then\ d(S^1 \cup S^2) = d(S^1) + d(S^2)$.

It can be readily seen that such an attribute d can be employed to measure the difference between subsystems of S; and for any chosen $S^1 < S$, and for any $S^2 \in \mathbb{S}$, the greater the attribute value $d(S^1 \triangle S^2)$ is, the more different the systems S^1 and S^2 are. By making use of such a $d \in Q$, which satisfies the previous properties, for any chosen $s \in S$ and any real number $\delta > 0$, we can define the δ -neighborhood $Nbrd_d(s, \delta)$ of s as follows:

$$Nbrd_d(s, \delta) = \{s' \in \mathbb{S} : d(s' \triangle s) < \delta\} \tag{5.9}$$

The attribute $q \in Q$ of the system S is said to be structurally stable in the neighborhood of a subsystem $s < S$ with respect to attribute $d \in Q$, provided that for any $\varepsilon > 0$, there is $\delta = \delta_\varepsilon > 0$ such that

$$|q(s) - q(s')| < \varepsilon, \forall s' \in Nbrd_d(s, \delta_\varepsilon) \tag{5.10}$$

Similarly, the concepts of regionally structural stability and uniformly structural stability of a system S at all of its subsystems can be defined.

If a system S is both uniformly stable over time and uniformly structural stable, then the system is referred to as a uniformly stable system.

5.4 Evolution of Systems

When systems' evolution is concerned with, the focus is how the system develops over time. Speaking rigorously, each system can be described by using the dynamic format in equ.(2.1). As for those systems, which do not seem to change with time, when seen from the angle of dynamics, they are either evolving with time extremely slowly or considered within a very short period of time. That is how they project a mistakenly incorrect sense of being static.

The evolution of systems takes two main forms: One is the transition of the system from one structure or form to another structure or form, and the other is that a system appears from its earlier state of non-existence and grows from a immature state to a more mature state. Both of these two forms of evolution can be described and illustrated by using the language of the dynamic systems in equ.(2.1). Assume that the system of our concern is indeed described in the form of equ.(2.1). Then for the first form of evolution, the relation set R_t and the attribute set Q_t change with time, while for the second form of evolution, the elements in the object set M_t develop over time together of course with changes of the relation and attributes sets R_t and Q_t. They represent two specific cases of the evolution of dynamic systems.

5.5 Boundary of Systems

By boundary, it stands for the separation between a system and its environment. The boundary is a part of the system and interacts with the environment more closely than the interior of the system. By reviewing the definition of environments in Definition 4.3, we can define the boundary of a system as follows. Assume that $s < S$ is a subsystem of the system S over the time span T. Let us denote the subsystem s and the system S respectively as $s_t = (m_t^s, r_t^s, q_t^s)$ and $S = (M, R, Q)$, for $t \in T$. For the sake of convenience of communication, we write $s = (m_s, r_s, q_s)$ for a fixed time moment. Then, the (external) environment E^s of s within the system S is defined to be:

$$\begin{aligned} E^s &= \{(M - m_s, r|_{m^E}, q|_{m_s^E, r^E}) : r \in R(r|_{m_s} \notin r_s), q \in Q(q|_{m_s} \notin q_s)\} \\ &= (m^E, r^E, q^E) \end{aligned}$$

where $m^E = M - m, r^E = r|_{m^E}$, and $q^E = q|_{m^E, r^E}$, for $r \in R(r|_{m^E} \notin r_s)$ and $q \in Q(q|_{m^E} \notin q_s)$.

Assume that there is an attribute $\mu \in Q_t$ of the system S such that it assigns each relation to a positive real number $\mu(r) : R_t \to R^+$. Intuitively, this attribute μ is an index that measures the intensity of each relation of the objects of S. Now, let us fix a threshold value μ_0 for the relational intensity, then by combining with the concept of weak interactions between systems (Definition 4.1), we can obtain the set of all relations in S of intensity at least μ_0 that relate the subsystem s

and its environment E^s as follows:

$$r_b = \{r \in R : r|_{m_s} \in r_s, Supp(r) \cap m_s \neq \emptyset \neq Supp(r) \cap m^E, \mu(r) \geq \mu_0\} \quad (5.11)$$

where $Supp(r)$ stands for the support of the relation r, which is the set of all objects that appear in the relation r. The set of all objects of s that interact with the environment E^s of intensity of at least μ_0 is given by the following:

$$m_b = \bigcup_{r \in r_b} Supp(r) - m^E \quad (5.12)$$

Then the system

$$\partial s = (m_b, r_b|_{m_b}, q_b|_{m_b, r_b|_{m_b}}) \quad (5.13)$$

satisfies that $\partial s < s$, and is referred to as the boundary (system) of s within the system S. The intensity of its interaction with the environment is no less than μ_0, while any other subsystem of s interacts with the environment E^s with strictly less intensity than μ_0.

If in the evolution of the system, ∂s has good stability, then we say that the boundary of the system is clear. Otherwise, we say that the boundary of the system is fuzzy. Symbolically, let $q \in Q$ be an attribute and $t_0 \in T$chosen. If for any $\varepsilon > 0$, there is $\delta = \delta_{t_0,\varepsilon} > 0$ such that

$$|q(\partial s_{t_0}) - q(\partial s_t)| < \varepsilon, \forall t \in (t_0 - \delta_{t_0,\varepsilon}, t_0 + \delta_{t_0,\varepsilon}) \quad (5.14)$$

then we say that the boundary of the system s at time $t_0 \in T$ is definite. Otherwise, the boundary is said to be fuzzy at $t_0 \in T$.

It is not hard to see that the definiteness of a system's boundary is defined by using the stability of the boundary system ∂s. Therefore, the stability of the boundary system at one time moment corresponds to the definiteness of the system's boundary. Similarly, we can study the concepts of regional definiteness of boundaries over time and structural definiteness of boundaries.

6 Some Final Words

By using the traditional set theory, the static structures and relations of systems are described. Then, the dynamic changes of systems are studied by using the concept of time systems. In this paper, on the basis of the concept of systems, we describe the dynamic structure and change of a general system from a more delicate angle by determining the objects, relations, and attributes. By considering the affects of environment, we establish the basic concepts of systems using the idea of dynamic sets that are related to the variable of time. Then on top of these developed concepts, we revisit the description and analysis of some of the most fundamental properties of systems, while making the necessary comparisons with the studies of systems developed on the classical set theory.

As for the analysis of systems properties under the new setting, additional rigorous deduction of various results of systems is badly needed in order to enrich the relevant systems theory.

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References

[1] Lin. Y. (1999), *General Systems Theory: A Mathematical Approach*, New York: Kluwer Academic and Plenum Publishers.

[2] Mickens. R. E. (1990), *Mathematics and Science*, Singapore: World Scientific.

[3] Quastler. H. (1965), *General Principles of Systems Analysis*, In: T. H. Waterman and H.

[4] J. Morrowits (eds.), *Theoretical and Mathematical Biology*, New York: Blaisdell Publishing.

[5] Zadeh. L. (1962), "From circuit theory to systems theory", *Proc. IRE*, Vol.50, pp.856-865.

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