Finding Unbiased Simultaneous Prediction Limits for Order Statistics of Future Samples with Applications

Nicholas A. Nechval[1], Konstantin N. Nechval[2], Gundars Berzins[1], Juris Krasts[1], Maris Purgailis[1], Uldis Rozevskis[1] and Vladimir F. Strelchonok[3]

[1] *Statistics Department, EVF Research Institute, University of Latvia, Raina Blvd 19, LV-1050 Riga, Latvia*

[2] *Applied Mathematics Department, Transport and Telecommunication Institute, Lomonosov Street 1, LV-1019 Riga, Latvia*

[3] *Informatics Department, Baltic International Academy, Lomonosov Street 4, LV-1019 Riga, Latvia*

Abstract

This paper provides procedures for finding unbiased simultaneous prediction limits on the observations or functions of observations of all of k future samples using the results of a previous sample from the same underlying distribution belonging to invariant family. The results have direct application in reliability theory, where the time until the first failure in a group of several items in service provides a measure of assurance regarding the operation of the items. The simultaneous prediction limits are required as specifications on future life for components, as warranty limits for the future performance of a specified number of systems with standby units, and in various other applications. Prediction limit is an important statistical tool in the area of quality control. The lower simultaneous prediction limits are often used as warranty criteria by manufacturers. The initial sample and k future samples are available, and the manufacturer wants to have a high assurance that all of the k future orders will be accepted. It is assumed throughout that k + 1 samples are obtained by taking random samples from the same population. In other words, the manufacturing process remains constant. The results in this paper are generalizations of the usual prediction limits on observations or functions of observations of only one future sample. In the paper, attention is restricted to invariant families of distributions. The technique used here emphasizes pivotal quantities relevant for obtaining ancillary statistics and is applicable whenever the statistical problem is invariant under a group of transformations that acts transitively on the parameter space. Applications of the proposed procedures are given for the two-parameter exponential and Weibull distributions. The exact prediction limits are found and illustrated with a numerical example.

Keywords Future samples, order statistics, simultaneous prediction limits

1 Introduction

Statistical intervals used by engineers and others include confidence intervals on a population parameter, such as the mean, and tolerance intervals. Confidence intervals give information about parameter of the population or a function of population parameters such as a percentile; tolerance intervals give information about a region which contains a specified proportion of a population.

Often one desires to construct from the results of a previous sample an interval which will have a high probability of containing the values of all of k future observations. For example, such an interval would be required in establishing limits on the values of some performance variable for a small shipment of equipment when the satisfactory performance of all units is to be guaranteed, or in setting acceptance limits on a specific lot of material, when acceptance requires the values of all items in a future sample to fall within the limits. An interval which contains the values of a specified number of future observations with a specified probability is known as a prediction interval. Such an interval need be distinguished both from a confidence interval on an unknown distribution parameter, and from a tolerance interval to contain the values of a specified proportion of the population. Research works on prediction intervals related to a single future statistic are abundant (see Hahn and Meeker [1], Patel [2], and references therein).

In many situations of interest, it is desirable to construct lower simultaneous prediction limits that are exceeded with probability γ by observations or functions of observations of all of k future samples, each consisting of m units. The prediction limits depend upon a previously available complete or type II censored sample from the same distribution. For instance, two situations where such limits are required are:

1. A customer has placed an order for a product which has an underlying time-to-failure distribution. The terms of his purchase call for k monthly shipments. From each shipment the customer will select a random sample of m units and accept the shipment only if the smallest time to failure for this sample exceeds a specified lower limit. The manufacturer wishes to use the results of a previous sample of n units to calculate this limit so that the probability is γ that all k shipments will be accepted. It is assumed that the n past units and the km future units are random samples from the same population.

2. A system consists of n identical components whose times to failure follow an underlying distribution. Initially one component is operating and the remaining n-1 components are in a standby mode; a new component goes into operation as soon as the preceding component has failed. The system is said to fail when all n components have failed. Thus, the system time to failure is the total of the failure times for the n components. A simultaneous lower prediction limit to be exceeded with probability γ by the system time to failure of all of k future systems is desired. This limit is to be calculated from the times to failure of n previously tested components. Similar problems also arise in various product maintenance and servicing problems.

Prediction limits can be of several forms. Hahn [3] dealt with simultaneous prediction limits on the standard deviations of all of the k future samples from a normal population. Hahn [4] considered the problem of obtaining simultaneous prediction limits on the means of all of k future samples from an exponential distribution. In addition, Hahn and Nelson [5] discussed such limits and their applications. Mann, Schafer, and Singpurwalla [6] gave an interval that contains, with probability γ, all m observations of a single future sample from the same population. Fertig and Mann [7] constructed prediction intervals to contain at least $m - k + 1$ out of m future observations from a normal distribution with probability $1 - \beta$. They considered life-test data, and the performance variate of interest is the failure time of an item. Their lower prediction limit constitutes a "warranty period".

In this paper we give an expression for obtaining unbiased simultaneous prediction limits on order statistics of all of k future samples. In order to obtain the unbiased simultaneous prediction limits, attention is restricted to invariant families of distributions. In particular, the case is considered where a previously available complete or type II censored sample is from a continuous distribution with cumulative distribution function (cdf) $F((x-\mu)/\sigma)$ and probability density function (pdf) $1/\sigma f((x-\mu)/\sigma)$, where $F(.)$ is known but both the location (μ) and scale (σ) parameters are unknown. For such family of distributions the decision problem remains invariant under a group of transformations (a subgroup of the full affine group) which takes μ (the location parameter) and σ (the s-cale) into $c\mu + b$ and $c\sigma$, respectively, where b lies in the range of $\mu, c > 0$. This group acts transitively on the parameter space and, consequently, the risk of any equivariant estimator is a constant. Among the class of such estimators there is therefore a "best" one. The effect of imposing the principle of invariance, in this case, is to reduce the class of all possible estimators to one. In the present paper we investigate this question for the problem of constructing the unbiased simultaneous prediction limits on order statistics in future samples.

The technique used here emphasizes pivotal quantities relevant for obtaining ancillary statistics. It is a special case of the method of invariant embedding of sample statistics into a performance index [8-11] applicable whenever the statistical problem is invariant under a group of transformations which acts transitively on the parameter space (i.e., in problems where there is a unique best invariant procedure). The exact unbiased simultaneous prediction limits on order statistics of all of k future samples are obtained via the technique of invariant embedding and illustrated with numerical example.

2 Mathematical Preliminaries

The main theorem, which shows how to construct lower (upper) simultaneous prediction limit for the order statistics in all of k future samples when prediction limit for a single future sample is available, is given below.

Theorem 1 *(Lower (upper) simultaneous prediction limit under complete information).* Let $(Y_{1j}, ..., Y_{mj})$ be the jth random sample of m_j "future" observations from the cdf $F_\theta(.)$, where θ is the parameter (in general, vector), $j \in 1, ..., k$, and let $Y_{(r_j,m_j)}$ denote the r_jth order statistic in the jth sample of size m_j. Assume that all of k samples from the same cdf are independent. Then a lower simultaneous $(1-\alpha)$ prediction limit h on the r_jth order statistics $Y_{(r_j,m_j)}, j = 1, , k$, of all of k future samples may be obtained from

$$P_\theta\{Y_{(r_1,m_1)} \geq h, ..., Y_{(r_j,m_j)} \geq h, ..., Y_{(r_k,m_k)} \geq h\} = \sum_{i_1=0}^{r_1-1} ... \sum_{i_j=0}^{r_j-1} ... \sum_{i_k=0}^{r_k-1} \binom{m_1}{i_1} ... \binom{m_j}{i_j} ... \binom{m_k}{i_k} \times \frac{P_\theta\{Y_{(i_\Sigma+1,m_\Sigma)} \geq h\} - P_\theta\{Y_{(i_\Sigma,m_\Sigma)} \geq h\}}{\binom{m_\Sigma}{i_\Sigma}} = 1-\alpha \tag{1}$$

where

$$i_\Sigma = \sum_{j=1}^{k} i_j, m_\Sigma = \sum_{j=1}^{k} m_j \tag{2}$$

(Observe that an upper simultaneous α prediction limit h may be obtained from a lower simultaneous prediction limit by replacing $1-\alpha$ by α.)

Proof.

we have:

$$P_\theta\{Y_{(r_1,m_1)} \geq h, ..., Y_{(r_j,m_j)} \geq h, ..., Y_{(r_k,m_k)} \geq h\} = \prod_{j=1}^{k} P_\theta\{Y_{(r_j,m_j)} \geq h\}$$
$$= \prod_{j=1}^{k} \sum_{i_j=0}^{r_j-1} \binom{m_j}{i_j} [F_\theta(h)]^{i_j}[1-F_\theta(h)]^{m_j-i_j}$$
$$= \sum_{i_1=0}^{r_1-1} ... \sum_{i_j=0}^{r_j-1} ... \sum_{i_k=0}^{r_k-1} \binom{m_1}{i_1} ... \binom{m_j}{i_j} ... \binom{m_k}{i_k} [F_\theta(h)]^{i_\Sigma}[1-F_\theta(h)]^{m_\Sigma-i_\Sigma} \tag{3}$$

Since

$$[F_\theta(h)]^{i_\Sigma}[1-F_\theta(h)]^{m_\Sigma - i_\Sigma}$$
$$= \begin{pmatrix} m_\Sigma \\ i_\Sigma \end{pmatrix}^{-1} \left[\sum_{i=0}^{i_\Sigma} \begin{pmatrix} m_\Sigma \\ i \end{pmatrix} [F_\theta(h)]^i [1-F_\theta(h)]^{m_\Sigma - i} - \sum_{i=0}^{i_\Sigma - 1} \begin{pmatrix} m_\Sigma \\ i \end{pmatrix} [F_\theta(h)]^i [1-F_\theta(h)]^{m_\Sigma - i} \right]$$
$$= \frac{P_\theta\{Y_{(i_\Sigma+1, m_\Sigma)} \geq h\} - P_\theta\{Y_{(i_\Sigma, m_\Sigma)} \geq h\}}{\begin{pmatrix} m_\Sigma \\ i_\Sigma \end{pmatrix}} \tag{4}$$

the joint probability can be written as

$$P_\theta\{Y_{(r_1,m_1)} \geq h, ..., Y_{(r_j,m_j)} \geq h, ..., Y_{(r_k,m_k)} \geq h\}$$
$$= \sum_{i_1=0}^{r_1-1} ... \sum_{i_j=0}^{r_j-1} ... \sum_{i_k=0}^{r_k-1} \begin{pmatrix} m_1 \\ i_1 \end{pmatrix} ... \begin{pmatrix} m_j \\ i_j \end{pmatrix} ... \begin{pmatrix} m_k \\ i_k \end{pmatrix} \times \frac{P_\theta\{Y_{(i_\Sigma+1, m_\Sigma)} \geq h\} - P_\theta\{Y_{(i_\Sigma, m_\Sigma)} \geq h\}}{\begin{pmatrix} m_\Sigma \\ i_\Sigma \end{pmatrix}} \tag{5}$$

This ends the proof.
Corollary 1.1 If $r_j = 1, \forall j = 1(1)k$, then

$$P_\theta\{Y_{(1,m_1)} \geq h, ..., Y_{(1,m_j)} \geq h, ..., Y_{(1,m_k)} \geq h\} = P_\theta\{Y_{(1,m_\Sigma)} \geq h\} = 1 - \alpha \tag{6}$$

Theorem 2 *(Lower (upper) unbiased simultaneous prediction limit under parametric uncertainty).* Let $(X_1 \leq ... \leq X_r)$ be the r smallest observations in a random sample of size n from the cdf $F_\theta(.)$, where the θ is the parameter (in general, vector), and let $(Y_{1_j}, ..., Y_{m_j})$ be the jth random sample of m_j "future" observations from the same cdf, $j \in \{1, ..., k\}$. Assume that $(k+1)$ samples are independent and the parameter θ is unknown. Let $H = H(X_1, ..., X_r)$ be any statistic based on the preliminary sample and let $Y_{(r_j,m_j)}$ denote the r_jth order statistic in the jth sample of size m_j. Then an unbiased lower simultaneous $(1-\alpha)$ prediction limit H on the r_jth order statistics $Y_{(r_j,m_j)}, j = 1, , k$, of all of k future samples may be obtained from

$$E_\theta\left\{P_\theta\{Y_{(r_1,m_1)} \geq H, ..., Y_{(r_j,m_j)} \geq H, ..., Y_{(r_k,m_k)} \geq H\}\right\}$$
$$= \sum_{i_1=0}^{r_1-1} ... \sum_{i_j=0}^{r_j-1} ... \sum_{i_k=0}^{r_k-1} \binom{m_1}{i_1} ... \binom{m_j}{i_j} ... \binom{m_k}{i_k} \cdot \tag{7}$$
$$\frac{E_\theta\left\{P_\theta\{Y_{(i_\Sigma+1,m_\Sigma)} \geq H\}\right\} - E_\theta\left\{P_\theta\{Y_{(i_\Sigma,m_\Sigma)} \geq H\}\right\}}{\binom{m_\Sigma}{i_\Sigma}}$$

Proof. For the proof we refer to Theorem 1.
Corollary 2.1. If $r_j = 1, \forall j = 1(1)k$, then

$$E_\theta\left\{P_\theta\{Y_{(1,m_1)} \geq H, ..., Y_{(1,m_j)} \geq H, ..., Y_{(1,m_k)} \geq H\}\right\}$$
$$= E_\theta\left\{P_\theta\{Y_{(1,m_\Sigma)} \geq H\}\right\} = 1 - \alpha \tag{8}$$

Remark. In this paper, in order to find the unbiased lower simultaneous $(1-\alpha)$ prediction limit H on the r_jth order statistics $Y_{(r_j,m_j)}, j = 1, ..., k$, of all of k future samples, the technique of invariant embedding [8-11] is used.

2.1 Weibull Distribution

In this paper, the two-parameter Weibull distribution with the pdf

$$f_\theta(x) = \frac{\delta}{\beta}(\frac{x}{\beta})^{\delta-1} exp\left[-(\frac{x}{\beta})^\delta\right], x > 0, \beta > 0, \delta > 0 \tag{9}$$

indexed by scale and shape parameters β and δ is used as the underlying distribution of a random variable X in a sample of the lifetime data, where $\theta = (\beta, \delta)$. We consider both parameters β, δ to be unknown. Let $(X_1, ..., X_n)$ be a random sample from the two-parameter Weibull distribution (9), and let $\hat{\beta}, \hat{\delta}$ be maximum likelihood estimates of β, δ computed on the basis of $(X_1, ..., X_n)$. In terms of the Weibull variates, we have that

$$V_1 = (\frac{\hat{\beta}}{\beta})^\delta, V_2 = \frac{\delta}{\hat{\delta}}, V_3 = (\frac{\hat{\beta}}{\beta})^{\hat{\delta}} \tag{10}$$

are pivotal quantities. Further more, let

$$Z_i = (X_i/\hat{\beta})^{\hat{\delta}}, i = 1, ..., n \tag{11}$$

It is readily verified that any n-2 of the Z_i's, say $Z_i, ..., Z_{n-2}$ form a set of n-2 functionally independent ancillary statistics. The appropriate conditional approach, first suggested by Fisher [12], is to consider the distributions of V_1, V_2, V_3

conditional on the observed value of $\mathbf{Z}^{(n)} = (Z_i, ..., Z_n)$. (For purposes of symmetry of notation we include all of $(Z_i, ..., Z_n)$ in expressions stated here; it can be shown that Z_n, Z_{n-1}, can be determined as functions of $Z_i, ..., Z_{n-2}$ only.)

Theorem 3. *(Joint pdf of the pivotal quantities V_1, V_2 from the two-parameter Weibull distribution)* Let $(X_1 \leq ... \leq X_r)$ be the first r ordered observations from a sample of size n from the two-parameter Weibull distribution (9). Then the joint pdf of the pivotal quantities

$$V_1 = (\frac{\hat{\beta}}{\beta})^{\delta}, V_2 = \frac{\delta}{\hat{\delta}} \tag{12}$$

conditional on fixed

$$\mathbf{Z}^{(r)} = (Z_i, ..., Z_r) \tag{13}$$

where

$$Z_i = (\frac{X_i}{\hat{\beta}})^{\hat{\delta}}, i = 1, ..., r \tag{14}$$

are ancillary statistics, any r-2 of which form a functionally independent set,$\hat{\beta}$ and $\hat{\delta}$ are the maximum likelihood estimates for β and δ based on the first r ordered observations $(X_1 \leq ... \leq X_r)$ from a sample of size n from the two-parameter Weibull distribution (9), which can be found from solution of

$$\hat{\beta} = \left(\left[\sum_{i=1}^{r} x_i^{\hat{\delta}} + (n-r) x_r^{\hat{\delta}} \right] /r \right)^{1/\hat{\delta}} \tag{15}$$

and

$$\hat{\delta} = \left[\left(\sum_{i=1}^{r} x_i^{\hat{\delta}} ln x_i + (n-r) x_r^{\hat{\delta}} ln x_r \right) \left(\sum_{i=1}^{r} x_i^{\hat{\delta}} + (n-r) x_r^{\hat{\delta}} \right)^{-1} - \frac{1}{r} \sum_{i=1}^{r} ln x_i \right]^{-1} \tag{16}$$

is given by

$$f(v_1, v_2|\mathbf{z}^{(r)}) = \vartheta^{\bullet}(\mathbf{z}^{(r)}) v_2^{r-2} \prod_{i=1}^{r} z_i^{v_2} v_1^{r-1} exp \left(-v_1 \left[\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2} \right] \right)$$
$$= f(v_2|\mathbf{z}^{(r)}) f(v_1|v_2, \mathbf{z}^{(r)}), v_1 \in (0, \infty), v_2 \in (0, \infty) \tag{17}$$

where

$$\vartheta^{\bullet}(\mathbf{z}^{(r)}) = \left[\int_0^{\infty} \Gamma(r) v_2^{r-2} \prod_{i=1}^{r} z_i^{v_2} \left(\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2} \right)^{-r} dv_2 \right]^{-1} \tag{18}$$

is the normalizing constant,

$$f(v_2|\mathbf{z}^{(r)}) = \vartheta(\mathbf{z}^{(r)}) v_2^{r-2} \prod_{i=1}^{r} z_i^{v_2} \left(\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2} \right)^{-r}, v_2 \in (0, \infty) \tag{19}$$

$$\vartheta(\mathbf{z}^{(r)}) = \left[\int_0^\infty v_2^{r-2} \prod_{i=1}^{r} z_i^{v_2} \left(\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2} \right)^{-r} dv_2 \right]^{-1} \tag{20}$$

$$\begin{aligned} f(v_1|v_2, \mathbf{z}^{(r)}) &= \frac{[\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2}]^r}{\Gamma(r)} v_1^{r-1} exp\left(-v_1 \left[\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2} \right] \right) \\ &= \frac{1}{\Gamma(r)} \left(v_1 \left[\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2} \right] \right)^{r-1} exp\left(-v_1 \left[\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2} \right] \right) \times \\ &\left[\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2} \right], v_1 \in (0, \infty) \end{aligned} \tag{21}$$

Proof. The joint density of $X_1 \leq ... \leq X_r$ is given by

$$f_\theta(x_1, ..., x_r) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} \frac{\delta}{\beta} (\frac{x_i}{\beta})^{\delta-1} exp(-(\frac{x_i}{\beta})^\delta) exp(-(n-r)(\frac{x_r}{\beta})^\delta) \tag{22}$$

Using the invariant embedding technique [8-11], we transform (22) to

$$\begin{aligned} &f_\theta(x_1, ..., x_r) d\hat{\beta} d\hat{\delta} \\ &= \frac{n!}{(n-r)!} \prod_{i=1}^{r} x_i^{-1} \delta^r \prod_{i=1}^{r} (\frac{x_i}{\beta})^\delta exp\left(-\sum_{i=1}^{r} (\frac{x_i}{\beta})^\delta - (n-r)(\frac{x_r}{\beta})^\delta \right) d\hat{\beta} d\hat{\delta} \\ &= \frac{n!}{(n-r)!} \hat{\beta} \hat{\delta}^r \prod_{i=1}^{r} x_i^{-1} (\frac{\delta}{\hat{\delta}})^{r-2} \prod_{i=1}^{r} (\frac{x_i}{\hat{\beta}})^{\hat{\delta}(\frac{\delta}{\hat{\delta}})} (\frac{\hat{\beta}}{\beta})^{\delta(r-1)} \times \\ &exp\left(-(\frac{\hat{\beta}}{\beta})^\delta \left[\sum_{i=1}^{r} (\frac{x_i}{\hat{\beta}})^{\hat{\delta}(\frac{\delta}{\hat{\delta}})} + (n-r)(\frac{x_r}{\hat{\beta}})^{\hat{\delta}(\frac{\delta}{\hat{\delta}})} \right] \right) \left(\frac{\delta}{\beta} (\frac{\hat{\beta}}{\beta})^{\delta-1} d\hat{\beta} \right) (-\frac{\delta}{\hat{\delta}^2} d\hat{\delta}) \\ &= \frac{n!}{(n-r)!} \hat{\beta} \hat{\delta}^r \prod_{i=1}^{r} x_i^{-1} v_2^{r-2} \prod_{i=1}^{r} z_i^{v_2} v_1^{r-1} exp\left(-v_1 \left[\sum_{i=1}^{r} z_i^{v_2} + (n-r) z_r^{v_2} \right] \right) dv_1 dv_2 \end{aligned} \tag{23}$$

Normalizing (23), we obtain (17). This ends the proof.

Theorem 4. *(Lower (upper) unbiased prediction limit H for the lth order statistic Y_l in a new (future) sample of m observations from the two-parameter Weibull distribution on the basis of the preliminary data sample)* Let $X_1 \leq ... \leq X_r$ be the first r ordered observations from the preliminary sample of size n from the two-parameter Weibull distribution (9). Then a lower unbiased $(1-\alpha)$ prediction limit H on the lth order statistic Y_l from a set of m future ordered observations $Y_1 \leq ... \leq Y_m$ also from the distribution (9) is given by

$$H = arg[E_\theta\{P_\theta\{Y_1 \geq H\}|\mathbf{z}^{(r)}\} = 1-\alpha] = z_H^{1/\hat{\delta}}\hat{\beta} \tag{24}$$

where

$$E_\theta\{P_\theta\{Y_l \geq H\}|z^{(r)}\} = \frac{\int\limits_0^\infty v_2^{r-2}\prod\limits_{i=1}^r z_i^{v_2}\sum\limits_{k=0}^{l-1}\left(\begin{array}{c} m \\ k \end{array}\right)\sum\limits_{j=0}^{k}\left(\begin{array}{c} k \\ j \end{array}\right)(-1)^j\left((m-k+j)z_H^{v_2}+\sum\limits_{i=1}^r z_i^{v_2}+(n-r)z_r^{v_2}\right)^{-r}dv_2}{\int\limits_0^\infty v_2^{r-2}\prod\limits_{i=1}^r z_i^{v_2}\left(\sum\limits_{i=1}^r z_i^{v_2}+(n-r)z_r^{v_2}\right)^{-r}dv_2} \tag{25}$$

$$z_H = \left(\frac{H}{\hat{\beta}}\right)^{\hat{\delta}} \tag{26}$$

$Z_i = (X_i/\hat{\beta})^{\hat{\delta}}, i = 1, ..., r; \hat{\beta}$ and $\hat{\delta}$ are the maximum likelihood estimates for β and β based on the first r ordered observations $(X_1 \leq ... \leq X_r)$ from a sample of size n from the two-parameter Weibull distribution (9).
(Observe that an upper unbiased α prediction limit H on the lth order statistic Y_l from a set of m future ordered observations $Y_1 \leq ... \leq Y_m$ may be obtained from a lower unbiased $(1-\alpha)$ prediction limit by replacing $1-\alpha$ by α.)
Proof. If there is a random sample of m ordered observations $Y_1 \leq ... \leq Y_m$ from the two-parameter Weibull distribution (9) with the pdf $f_\theta(y)$ and cdf $F_\theta(y)$, then for the lth order statistic Y_l we have

$$\begin{aligned} P_\theta\{Y_l \geq H\} &= \sum_{k=0}^{l-1}\left(\begin{array}{c} m \\ k \end{array}\right)[F_\theta(H)]^k[1-F_\theta(H)]^{m-k} \\ &= \sum_{k=0}^{l-1}\left(\begin{array}{c} m \\ k \end{array}\right)\left[1-\exp\left(-\left(\frac{H}{\beta}\right)^\delta\right)\right]^k\left[\exp\left(-\left(\frac{H}{\beta}\right)^\delta\right)\right]^{m-k} \end{aligned} \tag{27}$$

Writing (27) as

$$P_\theta\{Y_l \geq H\} = \sum_{k=0}^{l-1} \binom{m}{k} \left[1 - \exp\left(-\left(\frac{H}{\beta}\right)^{\delta}\right)\right]^k \exp\left(-(m-k)\left(\frac{H}{\beta}\right)^{\delta}\right)$$

$$= \sum_{k=0}^{l-1} \binom{m}{k} \left[1 - \exp\left(-\left(\frac{H}{\widehat{\beta}}\right)^{\widehat{\delta}\left(\frac{\delta}{\widehat{\delta}}\right)} \left(\frac{\widehat{\beta}}{\beta}\right)^{\delta}\right)\right]^k \exp\left(-(m-k)\left(\frac{H}{\widehat{\beta}}\right)^{\widehat{\delta}\left(\frac{\delta}{\widehat{\delta}}\right)} \left(\frac{\widehat{\beta}}{\beta}\right)^{\delta}\right)$$

$$= \sum_{k=0}^{l-1} \binom{m}{k} [1 - \exp(-z_H^{v_2} v_1)]^k \exp(-(m-k) z_H^{v_2} v_1)$$

$$= \sum_{k=0}^{l-1} \binom{m}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \exp[-v_1(m-k+j) z_H^{v_2}] = P\{Z_l > z_H | v_1, v_2\} \tag{28}$$

where

$$Z_l = \left(\frac{Y_l}{\widehat{\beta}}\right)^{\widehat{\delta}} \tag{29}$$

we have from (17) and (28) that

$$\begin{aligned} E_\theta\{P_\theta\{Y_l \geq H\} | z^{(r)}\} &= E\{P\{Z_l \geq z_H | v_1, v_2\} | z^{(r)}\} \\ &= \int_0^\infty \int_0^\infty P\{Z_l \geq z_H | v_1, v_2\} f(v_1, v_2 | z^{(r)}) dv_1 dv_2 \end{aligned} \tag{30}$$

Now v_1 can be integrated out of (30) in a straightforward way to give (25). This completes the proof.

Corollary 4.1. If $l = 1$, then

$$H = \arg\left[\frac{\int_0^\infty v_2^{r-2} \prod_{i=1}^r z_i^{v_2} \left(m\left[\left(\frac{H}{\widehat{\beta}}\right)^{\widehat{\delta}}\right]^{v_2} + \sum_{i=1}^r z_i^{v_2} + (n-r) z_r^{v_2}\right)^{-r} dv_2}{\int_0^\infty v_2^{r-2} \prod_{i=1}^r z_i^{v_2} \left(\sum_{i=1}^r z_i^{v_2} + (n-r) z_r^{v_2}\right)^{-r} dv_2} = 1 - \alpha\right] \tag{31}$$

Theorem 5 *((Lower (upper) unbiased prediction limit H for the lth order statistic Y_l in a new (future) sample of m observations from the left-truncated Weibull distribution on the basis of the preliminary data sample)* Let $X_1 \leq ... \leq X_r$ be the first r ordered observations from the preliminary sample of size n from the left-truncated Weibull distribution with the pdf

$$f_\theta(x) = \frac{\delta}{\sigma} x^{\delta-1} \exp[-(x^\delta - \mu)/\sigma], (x^\delta \geq \mu, \sigma, \delta > 0) \tag{32}$$

where $\theta = (\mu, \sigma, \delta), \delta$ is termed the shape parameter,δ is the scale parameter, and μ is the truncation parameter. It is assumed that the parameter δ is known. Then a lower unbiased $(1-\alpha)$ prediction limit H on the lth order statistic Y_l from a set of m future ordered observations $Y_1 \le ... \le Y_m$ also from the distribution (32) is given by

$$H = \left(X_1^{\delta} + w_H S\right)^{1/\delta} \tag{33}$$

where

$$w_H = \begin{cases} \arg\left(1 - \frac{m!(m+n-l)!}{(m-l)!(m+n)!}(1 - nw_H)^{-(r-1)} = 1-\alpha\right), \text{if } \alpha < \frac{m!(n+m-l)!}{(m-l)!(n+m)!}, \\ \arg\left(nl\begin{pmatrix} m \\ l \end{pmatrix} \sum_{i=0}^{l-1} \frac{\begin{pmatrix} l-1 \\ i \end{pmatrix}(-1)^i[1+w_H(m-l+i+1)]^{-(r-1)}}{(n+m-l+i+1)(m-l+i+1)} = 1-\alpha \right), \\ if \quad \alpha \ge \frac{m!(n+m-l)!}{(m-l)!(n+m)!}, \end{cases} \tag{34}$$

$$S = \sum_{i=1}^{r} (X_i^{\delta} - X_1^{\delta}) + (n-r)(X_r^{\delta} - X_1^{\delta}) \tag{35}$$

(Observe that an upper unbiased α prediction limit H on the lth order statistic Y_l may be obtained from a lower unbiased $(1-\alpha)$ prediction limit by replacing $1-\alpha$ by α.)

Proof. It can be justified by using the factorization theorem that (X_1^{δ}, S) is a sufficient statistic for (μ, δ). We wish, on the basis of the sufficient statistic (X_1^{δ}, S) for (μ, δ), to construct the predictive density function of the lth order statistic Y_l from a set of m future ordered observations $Y_1 \le ... \le Y_m$. By using the technique of invariant embedding [8-11] of (X_1^{δ}, S), if $X_1 \le Y_l$, or (Y_l^{δ}, S), if $X_1 \ge Y_l$, into a pivotal quantity $(Y_l^{\delta} - \mu)/\sigma$ or $(X_1^{\delta} - \mu)/\sigma$, respectively, we obtain an ancillary statistic

$$W_l = \left(Y_l^{\delta} - X_1^{\delta}\right)\Big/ S \tag{36}$$

It can be shown that the pdf of W_l is given by

$$f(w_l) = \begin{cases} n(r-1)l\begin{pmatrix} m \\ l \end{pmatrix} \sum_{i=0}^{l-1} \frac{\begin{pmatrix} l-1 \\ i \end{pmatrix}(-1)^i\left[1 + w_l(m-l+i+1)\right]^{-r}}{n+m-l+i+1}, if\ w_l \ge 0, \\ n(r-1)\frac{m!(n+m-l)!}{(m-l)!(n+m)!}(1 - nw_l)^{-r}, if\ w_l < 0. \end{cases} \tag{37}$$

It follows from (37) that

$$P(W_l > w_H) = \begin{cases} nl\begin{pmatrix} m \\ l \end{pmatrix}\sum_{i=0}^{l-1}\frac{\begin{pmatrix} l-1 \\ i \end{pmatrix}(-1)^i[1+w_H(m-l+i+1)]^{-(r-1)}}{(n+m-l+i+1)(m-l+i+1)}, \\ \quad if\ w_H \geq 0, \\ 1-\frac{m!(m+n-l)!}{(m-l)!(m+n)!}(1-nw_H)^{-(r-1)}, if\ w_H < 0. \end{cases} \tag{38}$$

where

$$w_H = \left(H^\delta - X_1^\delta\right)\Big/S \tag{39}$$

This ends the proof.

Corollary 5.1. If $l = 1$, then a lower $(1-\alpha)$ prediction limit H on the minimum Y_1 of a set of m future ordered observations $Y_1 \leq ... \leq Y_m$ is given by

$$H = \begin{cases} \left(X_1^\delta + \frac{S}{m}\left[\left(\frac{n}{(1-\alpha)(n+m)}\right)^{\frac{1}{r-1}} - 1\right]\right)^{1/\delta}, \text{if } \alpha \geq \frac{m}{n+m}, \\ \left(X_1^\delta - \frac{S}{n}\left[\left(\frac{m}{\alpha(n+m)}\right)^{\frac{1}{r-1}} - 1\right]\right)^{1/\delta}, \text{if } \alpha < \frac{m}{n+m}. \end{cases} \tag{40}$$

2.2 Two-parameter Exponential Distribution

Theorem 6 *((Lower (upper) unbiased prediction limit H for the lth order statistic Yl in a new (future) sample of m observations from the two-parameter exponential distribution on the basis of the preliminary data sample)* Let $X_1 \leq ... \leq X_r$ be the first r ordered observations from the preliminary sample of size n from the two-parameter exponential distribution with the pdf

$$f_\theta(x) = \frac{1}{\sigma}\exp[-(x-\mu)/\sigma], (x^\delta \geq \mu, \sigma > 0) \tag{41}$$

where $\theta = (\mu, \sigma), \sigma$ is the scale parameter, and μ is the shift parameter. It is assumed that these parameters are unknown. Then a lower unbiased $(1-\alpha)$ prediction limit H on the lth order statistic Y_l from a set of m future ordered observations $Y_1 \leq ... \leq Y_m$ also from the distribution (41) is given by

$$H = X_1 + w_H S \tag{42}$$

where

$$w_H = \begin{cases} \arg\left(1-\frac{m!(m+n-l)!}{(m-l)!(m+n)!}(1-nw_H)^{-(r-1)} = 1-\alpha\right), \text{if } \alpha < \frac{m!(n+m-l)!}{(m-l)!(n+m)!}, \\ \arg\left(nl\begin{pmatrix} m \\ l \end{pmatrix}\sum_{i=0}^{l-1}\frac{\begin{pmatrix} l-1 \\ i \end{pmatrix}(-1)^i[1+w_H(m-l+i+1)]^{-(r-1)}}{(n+m-l+i+1)(m-l+i+1)} = 1-\alpha\right), \\ if\ \alpha \geq \frac{m!(n+m-l)!}{(m-l)!(n+m)!} \end{cases} \tag{43}$$

$$S = \sum_{i=1}^{r} (X_i - X_1) + (n - r)(X_r - X_1) \tag{44}$$

(Observe that an upper unbiased α prediction limit H on the lth order statistic Y_l may be obtained from a lower unbiased $(1-\alpha)$ prediction limit by replacing $1-\alpha$ by α.)

Proof. For the proof we refer to Theorem 5.

Corollary 6.1. If $l = 1$, then a lower $(1-\alpha)$ prediction limit H on the minimum Y_1 of a set of m future ordered observations $Y_1 \leq ... \leq Y_m$ is given by

$$H = \begin{cases} \left(X_1 + \frac{S}{m} \left[\left(\frac{n}{(1-\alpha)(n+m)} \right)^{\frac{1}{r-1}} - 1 \right] \right), \text{if } \alpha \geq \frac{m}{n+m}, \\ \left(X_1 - \frac{S}{n} \left[\left(\frac{m}{\alpha(n+m)} \right)^{\frac{1}{r-1}} - 1 \right] \right), \text{if } \alpha < \frac{m}{n+m}. \end{cases} \tag{45}$$

Remark 2. Let us assume that the parent distributions are the two-parameter exponential

$$F_\theta(x) = 1 - \exp\left(-\frac{x - \theta_2}{\theta_1} \right), x \geq \theta_2, \theta_1 > 0 \tag{46}$$

where $\theta = (\theta_1, \theta_2)$ and the Pareto distribution

$$F_\theta(x) = 1 - (\theta_2/x)^{1/\theta_1}, x \geq \theta_2 > 0, \theta_1 > 0 \tag{47}$$

Let X be a random variable with the Pareto distribution (47), and define $Y = lnX$. Then Y becomes a random variable with the exponential distribution (46), where θ_2 is replaced by $ln\theta_2$. Therefore it is enough to consider only the exponential distribution, because the results for the Pareto distribution are easily obtained from those for the exponential distribution.

3 Numerical Example

An industrial firm has the policy to replace a certain device, used at several locations in its plant, at the end of 24-month intervals. It doesn't want too many of these items to fail before being replaced. Shipments of a lot of devices are made to each of three firms. Each firm selects a random sample of 5 items and accepts his shipment if no failures occur before a specified lifetime has accumulated. The manufacturer wishes to take a random sample and to calculate the lower prediction limit so that all shipments will be accepted with a probability of 0.95. The resulting lifetimes (rounded off to the nearest month) of an initial sample of size 15 from a population of such devices are given in Table 1. *Goodness-of-fit*

Table 1 The resulting lifetimes

Statistical Results														
x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}
8	9	10	12	14	17	20	25	29	30	35	40	47	54	62
Lifetime (in number of month intervals)														

testing. It is assumed that

$$X_i \sim f_\theta(x) = \tfrac{\delta}{\sigma} x^{\delta-1} \exp[-(x^\delta - \mu)/\sigma], (x \geq \mu, \sigma, \delta > 0), i = 1(1)15 \tag{48}$$

where the parameters μ and δ are unknown; $(\delta = 0.87)$. Thus, for this example, $r = n = 15, k = 3, m = 5, 1 - \alpha = 0.95, X_1^\delta = 6.1$,and $S = 170.8$. It can be shown that the

$$U_j = 1 - \left(\frac{\sum\limits_{i=2}^{j+1} (n-i+1)(X_i^\delta - X_{i-1}^\delta)}{\sum\limits_{i=2}^{j+2} (n-i+1)(X_i^\delta - X_{i-1}^\delta)} \right)^j, j = 1(1)n-2 \tag{49}$$

are i.i.d. $U(0,1)$ rv's (Nechval *et al*. [13]). We assess the statistical significance of departures from the left-truncated Weibull model by performing the Kolmogorov-Smirnov goodness-of-fit test. We use the K statistic (Muller *et al*. [14]). The rejection region for the α level of significance is $K \geq K_{n;\alpha}$. The percentage points for $K_{n;\alpha}$ were given by Muller *et al*. [14]. For this example,

$$k = 0.220 < K_{n=13,\alpha=0.05} = 0.361 \tag{50}$$

Thus, there is not evidence to rule out the left-truncated Weibull model. It follows from (8) and (40), for

$$\alpha = 0.05 < \frac{km}{n+km} = 0.5 \tag{51}$$

that

$$H = \left(x_1^\delta - \tfrac{s}{n} \left[\left(\tfrac{km}{\alpha(n+km)} \right)^{\frac{1}{n-1}} - 1 \right] \right)^{\frac{1}{\delta}} = \left(6.1 - \tfrac{170.8}{15} \left[\left(\tfrac{15}{0.05(15+15)} \right)^{\frac{1}{14}} - 1 \right] \right)^{\frac{1}{0.87}} = 5 \tag{52}$$

Thus, the manufacturer has 95% assurance that no failures will occur in each shipment before $H = 5$ month intervals.

4 Conclusion and Future Work

In this paper we propose the technique of constructing unbiased simultaneous prediction limits on observations or functions of observations in all of k future samples under parametric uncertainty of the underlying distribution. These unbiased simultaneous prediction limits are based on a previously available complete or type II censored sample from the same distribution. We present an equation for this type of unbiased simultaneous prediction limits which holds for any distribution and any statistic from the previous sample when a prediction limit for a single future sample is available. The exact prediction limits are found and illustrated with a numerical example. The methodology described here can be extended in several different directions to handle various problems that arise in practice. We have illustrated the proposed methodology for the two-parameter exponential and Weibull distributions. Application to other distributions could follow directly.

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Corresponding author
Nicholas A. Nechval can be contacted at: nechval@junik.lv