Stability Results for an Inverse Parabolic Problem

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Abstract

In this paper, a one dimensional inverse parabolic problem in a quarter plane will be considered. The unknown function in a boundary is estimated from an over specified condition at a fixed location inside the region by solving an ill-posed integral equation. The Tikhonov Regularization method of the 1st order is applied in order to stabilize the solution of the ill-posed problem. The solution of the inverse problem is defined by minimization of the Tikhonov functional. Some analytical results for regularization parameter determination and stability of solution of inverse problem are derived.

Keywords Tikhonov Regularization Method, Inverse Parabolic Problem, Regularization Parameter, Stable Solution, Ill-Posed Problem

1 Introduction

Inverse parabolic problems play a crucial role in applied Mathematics, Physics and engineering science. They arise for example, in the study of heat conduction processes, diffusion, control theory [1-10]. In recent years, a lot of attention has been devoted to the study of inverse parabolic problems. Hence, the last 20 years have seen growing attention paid in the literature to the development, analysis, and implementation of accurate methods for the solution of inverse parabolic problems, i.e., the determination of unknown boundary condition $g(t)$ in the parabolic partial differential equation.

In this paper, we investigate an inverse parabolic problem in quarter plane to obtain an unknown function $g(t)$ from over specified data $p(t)$ at a fixed location inside the body. This inverse problem can be reduced to the operator integral equation $Ag = G$. It is well known that the operator equation $Ag = G$ is an ill-posed problem, when a solution is unstable with respect to small variations in input data. Since in the usual case where only a measured or computation approximation G^{δ} is available, some kind of regularization methods is required in order to obtain a reasonable stable approximation G^{δ} to g [6, 11-15]. In order to obtain stable solution for this ill-posed problem the Tikhonov regularization method is applied for operator equation $Ag = G$ for retrieving solution in a stable manner. Applying general result in the theory of Tikhonov regularization method for ill-posed inverse problem which consists in solving the unconstrained minimization problem, we can find stable solution. Since the regularization parameter play an important role in applying the Tikhonov regularization method to the operator equation, we obtain this parameter based on the error in input data directly, which can be one of the advantage of our method for selecting regularization parameter with respect to other methods [16-19].

The organization of this paper is as follows, In forcecoming section, mathematical formulation for this inverse parabolic problem in a quarter plane is introduced. In section 3, Thikhonov regularization method is stated and we use this method to construct a stable solution for this ill-posed problem. Some theoretical results will be proved about the solution of this ill-posed problem and the choice of regularization parameter. It will be shown that the solution of Thikhonov regularization method is stable under small errors in input data and the existence of this stable solution is shown. We conclude this article with a brief conclusive discussion in section 4. Finally, some references are given at the end of this paper.

2 Mathematical Formulation

In this section, we consider the following inverse parabolic problem in a quarter plane:

$$u_t = u_{xx}, \qquad 0 < x, 0 < t < T \tag{1}$$

$$u(x,0) = f(x), \qquad 0 < x \tag{2}$$

$$u(0,t) = g(t), \qquad 0 < t < T, \tag{3}$$

$$|u(x,t)| \leq M, \qquad 0 < x, 0 < t < T \tag{4}$$

where $f(x)$ and $g(t)$ are piecewise known continuous functions and M is a positive number.The problem consist of using an overspecified data $p(t)$, which is given by

$$p(t) = u(1,t), \qquad 0 < t < T \tag{5}$$

to determine the unknown function $g(t)$.

For the forward problem (1)-(4), the unique bounded solution $u(x,t)$ is given by [20],

$$\begin{aligned} u(x,t) = &\frac{x}{\sqrt{4\pi}} \int_0^t \frac{g(\tau)}{\sqrt{(t-\tau)^3}} e^{-\frac{x^2}{4(t-\tau)}} d\tau \\ &+ \frac{1}{\sqrt{4\pi t}} \int_0^\infty (e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}}) f(\xi) d\xi \end{aligned} \tag{6}$$

Due to the setting $x = 1$, and using (5) we obtain

$$\int_0^t \frac{e^{-\frac{1}{4(t-\tau)}}}{\sqrt{(t-\tau)^3}} g(\tau) d\tau = 2\sqrt{\pi} p(t) - \frac{1}{\sqrt{t}} \int_0^\infty (e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}}) f(\xi) d\xi \tag{7}$$

which is written in the form

$$\int_0^t h(t-\tau)g(\tau)d\tau = 2\sqrt{\pi}p(t) - \frac{1}{\sqrt{t}}\int_0^\infty (e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}})f(\xi)d\xi$$

where the kernel $h(t,\tau) = \frac{1}{\sqrt{(t-\tau)^3}}e^{-\frac{1}{4(t-\tau)}}$ is a continuous function on $[0,T] \times [0,T]$.

Now, we formulate (7) in term of an operator integral equation :

$$(Ag)(t) = G(t), \qquad 0 < t < T \tag{8}$$

where the operator A is defined by

$$(Ag)(t) = \int_0^t \frac{e^{-\frac{1}{4(t-\tau)}}}{\sqrt{(t-\tau)^3}}g(\tau)d\tau, \quad 0 < t < T$$

and

$$G(t) = 2\sqrt{\pi}p(t) - \frac{1}{\sqrt{t}}\int_0^\infty (e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}})f(\xi)d\xi, \quad 0 < t < T$$

The above integral equation of the first kind (7) cannot be reduce into an integral equation of the second kind by differentiation and the problem is inherently ill-posed. For $1 \leq p < \infty$, A is a compact linear operator in $L_p[0,T]$. Zero is not an eigenvalue of A and is the only point in the spectrum of A, thus A^{-1} exist and is unbounded, so if $G(t)$ on $0 \leq t \leq T$ is in the range of A, $g(t)$ is uniquely determined from $g = A^{-1}G$. In practice, with $G(t)$ obtained from measurement, small error in $G(t)$ lead to enormous errors in $g(t)$ because A^{-1} is unbounded. In order to regularize the problem, we use Tikhonov regularization method for finding approximate stable solution for ill-posed integral equation (8). In the next section we describe Tikhonov regularization method.

3 Tikhonov Regularization Method for Ill-Posed Integral Equation

It is well known that the Tikhonov regularization method is one of the useful tools for solving an ill-posed problem of the form (7)[12-13, 19]. Since in practical purpose, the input data are non smooth, we apply Tikhonov regularization method to construct stable solution for solving ill-posed equation (8). Based on the main concept of Tikhonov regularization method, we introduce smoothing functional $M^\alpha[g,G]$ which is called also Thikhonov functional and stabilizing functional $\Omega(g)$ as follows:

$$M^\alpha[g,G] = \|Ag - G\|^2_{L^2[0,T]} + \alpha\Omega(g)$$

α is regularization parameter and $\Omega(g)$ is a stabilizer of the 1th order constant coefficient which is defined as:

$$\Omega(g) = \int_0^T (g^2(\tau) + g'^2(\tau))d\tau$$

We construct a regularize solution for the integral equation (8) by using the following minimization problem which is state by the following Theorem:
Theorem 1. For every function $G \in L_2[0,T]$, and any positive α, there exists an element $g_\alpha \in W_2^1$, such that smoothing functional $M^\alpha[g,G]$ attain it greatest lower bound.

$$inf M^\alpha[g,G] = M^\alpha[g_\alpha, G]$$

Proof. We have

$$M^\alpha[g,G] = \int_0^T (\int_0^t \frac{e^{-\frac{1}{4(t-\tau)}}}{\sqrt{(t-\tau)^3}} g(\tau)d\tau - G(t))^2 dt + \alpha \int_0^T (g^2(\tau) + g'^2(\tau))d\tau$$

A condition for a minimum of this functional is vanishing of its first variation, this is written in the form

$$\begin{aligned} \frac{1}{2}\frac{d}{d\varepsilon} M^\alpha[g+\varepsilon\eta, G]|_{\varepsilon=0} = & \int_0^T [\int_\tau^T \int_0^t h(t-s)h(t-\tau)g(s)dsdt - \int_\tau^T h(t-\tau)dt \\ & - \alpha(g''(\tau) - g(\tau))]\eta(\tau)d\tau + g'(\tau)\eta(\tau)|_0^T \\ = & 0 \end{aligned} \quad (9)$$

Here, $\eta(\tau)$ is an arbitrary variation of the function $g(\tau)$ such that both $g(\tau)$ and $g(\tau) + \varepsilon\eta(\tau)$ belong to the class of admissible function.

Condition (9) will be satisfied if

$$\int_\tau^T \int_0^t h(t-s)h(t-\tau)g(s)dsdt - \int_\tau^T h(t-\tau)dt = \alpha(g''(\tau) - g(\tau)) \quad (10)$$

and

$$g'(0) = g'(T) = 0 \quad (11)$$

The equation (10) is called Euler-Lagrange equation, therefore the minimizer $g_\alpha(t)$ for Tikhonov functional is determined by the solution of Euler-Lagrange equation corresponding to functional $M^\alpha[g,G]$. The solution of (10) with boundary condition (11) is unique by classical theorems in ODE.

Now, we can assume that the regularized solution of the above minimization problem, g_α as an regularizing operator $R(G,\alpha)$ such that $g_\alpha = R(G,\alpha)$ where

$\alpha = \alpha(\delta, G_\alpha)$ in accordance with the error in the initial data G and δ, measuring the error in data, we select α in a suitable way such that $R(G, \alpha)$ is a regularizing operator for the equation (8), and $g_\alpha = R(G_\alpha, \alpha(\delta))$ can be taken as an approximate stable solution for ill-posed problem (8).

Theorem 2. If $g_T(t) \in C[0,T]$ be the exact solution of the original problem (8) associated with the exact right hand member $G = G_T$; that, $(Ag_T)(t) = G_T(t)$. Then, for every positive number ε, there exists a positive number $\delta(\varepsilon)$ such that for every $G_\delta \in L_2[0,T]$ the inequality

$$\|G_T(t) - G_\delta(t)\|_{L_2[0,T]} \le \delta < \delta(\varepsilon)$$

implies the inequality

$$\|g_{\alpha(\delta)}(t) - g_T(t)\|_{C[0,T]} \le \varepsilon$$

where $g_\alpha = R(G_{\alpha(\delta)}, \alpha(\delta))$ be the solution of (8) associated with perturbed data G_δ for all α satisfying $\alpha(\delta) = \delta^\gamma, 0 \le \gamma < 2$.

Proof. Since g_α is a minimizer of functional M^α, we have

$$M^{\alpha(\delta)}[g_\alpha, G_\delta] \le M^{\alpha(\delta)}[g_T, G_\delta]$$

therefore,

$$\begin{aligned}
&\|Ag_\alpha - G_\delta\|^2 \le M^{\alpha(\delta)}[g_\alpha, G_\delta] \\
&\le \int_0^T (A_{gT}(t) - G_\delta(t))^2 dt + \alpha(\delta) \int_0^T (g_T^2(\tau) + g_T'^2(\tau)) d\tau \\
&= \int_0^T (G_T(t) - G_\delta(t))^2 dt + \delta^\gamma \int_0^T (g_T^2(\tau) + g_T'^2(\tau)) d\tau \\
&\le \delta^2 + \delta^\gamma \int_0^T (g_T^2(\tau) + g_T'^2(\tau)) d\tau \\
&\le \delta^\gamma (1 + \int_0^T (g_T^2(\tau) + g_T'^2(\tau)) d\tau) \\
&= \delta^\gamma N
\end{aligned}$$

with $N = 1 + \int_0^T (g_T^2(\tau) + g_T'^2(\tau)) d\tau$.

Consequently, the elements $g_T(t)$ and $g_{\alpha(\delta)}(t)$ belong to the compact subset E of element g of $C[0,T]$ such that

$$E = \{g(t) | \ \|g\|_{W_2^1}^2 \le N\}$$

Since E is compact in $C[0,T]$, and the operator A is continuous, the mapping $A : E \to AE$ is continuous and one to one, therefore the inverse mapping $A^{-1} : AE \to E$ is also continuous. This means that,

$$\forall \varepsilon > 0, \exists \eta(\varepsilon), \|G_T - G_\alpha\| \le \eta(\varepsilon), Ag_T = G_T, Ag_\alpha = G_\alpha$$

then

$$\|g_T - g_\alpha\|_{C[0,T]} \leq \varepsilon$$

on the other hand we have

$$\|G_T - G_\alpha\|^2_{L^2} = \int_0^T (Ag_\alpha - G_T(t))^2 dt < \delta^2$$

an so,

$$\|g_T(t) - g_{\alpha(\delta)}(t)\|_{C[0,T]} = \|A^{-1}Ag_t - A^{-1}Ag_{\alpha(\delta)}\| \leq \|A^{-1}\|\|Ag_t - Ag_{\alpha(\delta)}\|$$

on the other hand

$$\begin{aligned}\|Ag_t - Ag_{\alpha(\delta)}\|_{L_2} &\leq \|Ag_T - G_\delta\| + \|Ag_{\alpha(\delta)} - G_\delta\| \\ &\leq \|G_T - G_\delta\| + \|Ag_{\alpha(\delta)} - G_\delta\| \\ &\leq \delta + \delta^{\frac{\gamma}{2}}\sqrt{N} \\ &\leq \delta^{\frac{\gamma}{2}}(1+\sqrt{N})\end{aligned}$$

Therefore,

$$\|g_T - g_{\alpha(\delta)}\| \leq \|A^{-1}\|\delta^{\frac{\gamma}{2}}(1+\sqrt{N})$$

The above results show that $\delta(\varepsilon)$ should be chosen in the form

$$\delta(\varepsilon) \leq [\frac{\varepsilon}{\|A^{-1}\|(1+\sqrt{N})}]^{\frac{2}{\gamma}}$$

such that the Theorem is satisfied.

The above theorem shows that when we construct regularizing operator by minimizing the smoothing functional $M^\alpha[g, G]$, the regularization parameter α can be obtain according to the error in the right hand member. By the next theorem we will show that G depends continuously on the initial data f, p.

Theorem 3. If the exact data $f_T(x)$ and $p_T(t)$ satisfy (8) and the approximate data $f_\delta(x)$ and $p_\delta(t)$ also satisfy (8), then inequalities $\|f_T(x) - f_\delta(x)\|_{L_2[0,\infty]} < \delta$ and $\|p_T(t) - p_\delta(t)\|_{L_2[0,T]} < \delta$ imply that

$$\|G_T(t) - G_\delta(t)\|_{L_2[0,T]} < \delta D, \qquad D = (9\sqrt{2\pi T} + 12\pi)^{\frac{1}{2}}$$

Proof. By using of Cauchy-Schwartz inequality, we have

$$
\begin{aligned}
&\|G_T(t)-G_\delta(t)\|^2\\
=&\|2\sqrt{\pi}(p_T(t)-p_\delta(t))-\frac{1}{\sqrt{t}}\int_0^\infty(e^{-\frac{(1-x)^2}{4t}}-e^{-\frac{(1+x)^2}{4t}})(f_T(x)-f_\delta(x))dx\|^2\\
\leq&3[\int_0^T 4\pi(p_T(t)-p_\delta(t))^2dt+\int_0^T\frac{1}{t}(\int_0^\infty e^{-\frac{(1-x)^2}{4t}}(f_T(x)-f_\delta(x))dx)^2dt\\
+&\int_0^T\frac{1}{t}(\int_0^\infty e^{-\frac{(1+x)^2}{4t}}(f_T(x)-f_\delta(x))dx)^2dt]\\
\leq&3(4\delta^2\pi+\int_0^T\frac{1}{t}(\int_0^\infty(e^{-\frac{(1-x)^2}{4t}})^2dx)(\int_0^\infty(f_T(x)-f_\delta(x))^2dx)dt\\
+&\int_0^T\frac{1}{t}(\int_0^\infty(e^{-\frac{(1+x)^2}{4t}})^2dx)(\int_0^\infty(f_T(x)-f_\delta(x))^2dx)dt)\\
\leq&3(4\delta^2\pi+\delta^2\int_0^T\frac{1}{t}\int_0^\infty e^{-\frac{(1-x)^2}{2t}}dxdt+\delta^2\int_0^T\frac{1}{t}\int_0^\infty e^{-\frac{x^2}{2t}}dxdt)\\
\leq&3(4\pi\delta^2+3\sqrt{2\pi T}\delta^2)\\
=&\delta^2(12\pi+9\sqrt{2\pi T})
\end{aligned}
$$

Theorem (3) and (4) show that by choosing α in such a way that the choice for α is consistent with the accuracy δ of the initial data, then the element $g_\alpha=R(G_\alpha,\alpha)$ obtained with the aid of the regularizing operator $R(G,\alpha)$, can be taken as approximate stable solution of equation (8).

We summarize the above results by the following Theorem:

Theorem 4. If $g_T(t)$ is the exact solution of equation (8) with exact data $f_T(x)$ and $p_T(t)$, then for every $\varepsilon>0$ and the approximate data $f_\delta(x)$ and $p_\delta(t)$ also satisfy (8), there exist $\delta(\varepsilon)$ and $\alpha(\delta)$ such that inequalities $\|f_T(x)-f_\delta(x)\|_{L_2[0,\infty]}<\delta$ and $\|p_T(t)-p_\delta(t)\|_{L_2[0,\infty]}<\delta$ imply that inequality

$$\|g_T(t)-G_\alpha(t)\|<\varepsilon$$

where $g_\alpha(\delta)=R(G_\delta,\alpha(\delta))$.

4 Conclusion

In this paper, we have introduced Tikhonov regularization method and we have shown why it is important to use it in order to solve ill-posed problems. We have shown that the Tikhonov regularization technique is introduced to treat the instability of obtaining a stable solution. The choice of regularization parameter and stability of solution are proved. It is very interesting to extend these results for higher dimensional problem and for nonstandard heat equation.

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