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Stability of Solution Maps for a η-Parameter Weak Vector Variational Inequality

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Abstract
In this paper, we introduce a new class of η-parameter weak vector variational inequality (for short, η-PWVVI) in Banach space, which extends the existing parameter weak vector variational inequality. We use the concepts of $\eta(y, x)$ function, invex set, η-hemicontinuous and η-strongly C pseudomonotone mapping to study (η-PWVVI) and we obtain new η-generalized linearization lemma. The stability of solution maps for (η-PWVVI) is obtained by this lemma. Finally, we present an example to illustrate our results.
Keywords: η-parameter weak vector variational inequality, η-generalized linearization lemma, invex set, η-hemicontinuous, η-strongly C pseudomonotone.

1 Introduction

As very powerful and important tools in the study of nonlinear sciences, variational inequalities and vector optimization have attracted so much attention. Over the last decades, variational inequality and vector optimization techniques have been applied extensively in such diverse fields as biology, chemistry, economics, engineering, game theory, management science and physics.

In 1980, F. Giannessi [1] introduced a well-known inequality in finite dimensional space, which was called vector variational inequality (for short,VVI). Thereafter, G. Y. Chen and G. M. Chen [2] discussed this problem in infinite dimensional space. G.Y. Chen [3] presented a new class of vector variational inequality problem in real Banach space. The new vector variational inequality problem not only extended the classical vector variational inequality problem, but also related to the existence of non-dominated solution for vector optimization problem. Inspired by G. Y. Chen's result, A. H. Siddiqi et al. [4] and K. L. Lin et al. [5] introduced and researched general vector variational inequality problem. L. N. Wang et al. [6] proved the stability of solution maps for a weak vector variational inequality in 2013.

It is worth pointing out that the convexity plays a significant role while studying the continuity of the solution for vector variational inequality. The concepts of invex set and generalized convexity have been given by S. R. Mohan, S. K. Neogy [7] and X. M. Yang [8] respectively.

Motivated by the work reported in [1]-[8], the aim of this paper is to introduce a new class of η-parameter weak vector variational inequality (for short, η-PWVVI) in Banach space, which extends the existing parameter weak vector variational inequality. Our results unify, generalize and complement various known comparable results from the current literature.

The rest of the paper is organized as follows. In Sect.2, we recall some basic definitions and notations which will be used in the sequel. In Sect.3, we use the concepts of $\eta(y;x)$ function, invex set, η-hemicontinuous and η-strongly C pseudomonotone mappings to study (η-PWVVI) and we obtain new η-generalized linearization lemma. As a consequence, the stability of solution maps for (η-PWVVI) is obtained in Theorem 3.3. Finally, we present an example to illustrate our results in Sect.4.

2 Preliminaries

Let X, Y and W (parameter space) are Banach s-paces, $C \subseteq Y$ is a non-empty closed convex cone with $int\ C \neq \emptyset$. $L(x;y)$ denotes the space which consist of all the continuous linear operators, define the value of linear operator $t \in L(x,y)$ at $x \in X$ by $\langle t, x\rangle$.

Throughout this paper, assume that $\eta(y,x) : X \times X \to X$ satisfies all the conditions as follows:

$$\begin{aligned}&(\mathrm{C1})\ \eta(x, x+\lambda\eta(y,x)) = -\lambda\eta(y,x);\\&(\mathrm{C2})\ \eta(x,y)+\eta(y,x)=\theta;\\&(\mathrm{C3})\ \eta(y,\cdot)\ \textit{is continuous}.\end{aligned}$$

Here, θ denotes the zero element of X.

Consider the following η-weak vector variational inequality problem (for short, η-WVVI) of finding $x \in K$ such that

$$\langle T(x), \eta(y,x)\rangle \notin -int\ C, \forall y \in K$$

where $K \subseteq X$ is non-empty, $T : X \to L(X,Y)$ is a vector value function.

When the operator T perturbed by the parameter μ with $\mu \in \Lambda \subseteq W$ and Λ is non-empty, for fixed μ, we deal with the following η-parameter weak vector variational inequality problem (η-PWVVI) of finding $x \in K$ such that

$$\langle T(x,\mu), \eta(y,x)\rangle \notin -int\ C, \forall y \in K$$

where $K \subseteq X$ is non-empty, $T : X \times \Lambda \to L(X,Y)$ is a vector value bifunction.

For any $\mu \in \Lambda, S_\eta(\mu)$ denotes the solution set of (η-PWVVI), that is,

$$S_\eta(\mu) = \{x \in K \mid \langle T(x,\mu), \eta(y,x)\rangle \notin -int\ C, \forall y \in K\}$$

In this paper, we assume that for any $\mu \in \Lambda,\ S_\eta(\mu)$ is non-empty.

Now, we give some basic definitions and some properties needed in the following sections.

Definition 2.1. *(see [7]) A set $K \subseteq X$ is said to be invex with respect to a given $\eta(y,x) : X \times X \to X$ if*

$$\forall x, y \in K, \lambda \in [0,1] \Rightarrow x + \lambda\eta(y,x) \in K.$$

Definition 2.2. *Let $K \subseteq X$ and K is invex with respect to $\eta(y,x)$, the operator $T : K \to L(X,Y)$ is said to be η-hemicontinuous if and only if for any $x, y \in K, \lambda \in [0,1]$, the mapping $\lambda \to \langle T(x + \lambda\eta(y,x)), \eta(y,x)\rangle$ is continuous at 0^+.*

Definition 2.3. *Let $K \subseteq X$ and K is invex with respect to $\eta(y,x)$, the operator $T : K \to L(X,Y)$ is said to be η-weakly C pseudomonotone on K if for any $x, y \in K, \langle T(x), \eta(y,x)\rangle \notin -intC$ implies $\langle T(y), \eta(y,x)\rangle \notin -intC$.*

Definition 2.4. *Let $K \subseteq X$ and K is invex with respect to $\eta(y,x)$, the operator $T : K \to L(X,Y)$ is said to be η-strongly C pseudomonotone on K if there exists $\lambda > 0$ such that for any $x, y \in K, \langle T(x), \eta(y,x)\rangle \notin -intC$ implies $\langle T(y), \eta(y,x)\langle + \lambda \parallel \eta(y,x) \parallel^2 B_Y \in C$, where B_Y denotes the unit, closed ball in Y.*

Remark 2.1 If we take $\eta(y,x) = y - x$ in Definition 2.1, then invex set run into convex set. Similarly, in Definition 2.2, η-hemicontinuous reduce to ν-hemicontinuous (see [6]) with $\eta(y,x) = y - x$.

Remark 2.2 It is evident from Definition 2.3 and Definition 2.4 that if an operator is η-strongly C pseudomonotone, then it is η-weakly C pseudomonotone.

3 Main results

The following Lemma 3.1 (η-generalized linearization lemma) extends the generalized linearization lemma (see [3]) through the concepts of $\eta(y,x)$ function and invex set.

Lemma 3.1. *(η-generalized linearization lemma) Let $K \subseteq X$ and K is invex with respect to $\eta(y,x)$. Moreover, assume the operator $T : K \to L(X,Y)$ is η-weakly C pseudomonotone and η-hemicontinuous, then the following two problems (i) and (ii) are equivalent:*

(i) *there exists $x \in K$, such that for any $y \in K$, $\langle T(x), \eta(y,x)\rangle \notin -intC$;*

(ii) *there exists $x \in K$, such that for any $y \in K$, $\langle T(y), \eta(y,x)\rangle \notin -intC$;*

Proof. In view of Definition 2.3, it is obvious that (i) implies (ii).

Now, assume that (ii) holds, then there exists $x \in K$, such that for any $y_0 \in K$,

$$\langle T(y_0), \eta(y_0,x)\rangle \notin -intC \tag{1}$$

Note that K is invex with respect to $\eta(y,x)$, thus for any $x, y \in K, \lambda \in [0,1]$,

$$x + \lambda\eta(y,x) \in K \tag{2}$$

By (2) we can take $y_0 = x + \lambda\eta(y, x) \in K$ and combine the result with (1), we have

$$\langle T(x + \lambda\eta(y, x)), \eta(x + \lambda\eta(y, x), x)\rangle \notin -intC \tag{3}$$

In view of the conditions (C1) and (C2) of $\eta(y, x)$, we obtain that

$$\eta(x + \lambda\eta(y, x), x) = -\eta(x, x + \lambda\eta(y, x)) = \lambda\eta(y, x)$$

Thus, we claim that (3) is equivalent to

$$\langle T(x + \lambda\eta(y, x)), \lambda\eta(y, x)\rangle \notin -intC$$

Dividing by λ, we get

$$\langle T(x + \lambda\eta(y, x)), \eta(y, x)\rangle \notin -intC$$

Let $\lambda \to 0^+$, take into account the η-hemicontinuity of T, we have

$$\langle T(x), \eta(y, x)\rangle \notin -intC$$

Therefore, (i) holds. The proof is complete.

Lemma 3.2. *Let $K \subseteq X$ and K is invex with respect to $\eta(y, x)$. If for fixed $\mu \in \Lambda, T(\cdot, \mu)$ is η-strongly C pseudomonotone, then the solution set of (η-PWVVI) is single valued, i.e., for fixed $\mu \in \Lambda, S_\eta(\mu)$ is a singleton.*
Proof. Suppose, to the contrary, that there exist $x_1, x_2 \in S_\eta(\mu)$, but $x_1 \neq x_2$.
By definition of $S_\eta(\mu)$,

$$\langle T(x_1, \mu), \eta(y, x_1)\rangle \in Y \backslash - intC, \forall y \in K \tag{4}$$
$$\langle T(x_2, \mu), \eta(y, x_2)\rangle \in Y \backslash - intC, \forall y \in K \tag{5}$$

In particular, take $y = x_2$ in (4) and $y = x_1$ in (5), respectively, we obtain

$$\langle T(x_1, \mu), \eta(x_2, x_1)\rangle \in Y \backslash - intC \tag{6}$$
$$\langle T(x_2, \mu), \eta(x_1, x_2)\rangle \in Y \backslash - intC \tag{7}$$

Since (6) holds and T is η-strongly C pseudomonotone, we claim that there exists $\lambda > 0$, such that

$$\langle T(x_2, \mu), \eta(x_2, x_1)\rangle + \lambda \parallel \eta(x_2, x_1) \parallel^2 B_Y \in C$$

Consider that $x_1 \neq x_2$, we have

$$\langle T(x_2,\mu), \eta(x_2,x_1)\rangle \in intC \tag{8}$$

In view of the condition (C2) of $\eta(y,x)$, (8) is equivalent to

$$\langle T(x_2,\mu), -\eta(x_1,x_2)\rangle \in intC$$

That is to say

$$\langle T(x_2,\mu), \eta(x_1,x_2)\rangle \in -intC$$

This is a contradiction to (7). Therefore, for fixed $\mu \in \Lambda, S_\eta(\mu)$ is a singleton.

Theorem 3.3. *Let K is a non-empty, compact subset of X. Assume K is invex with respect to $\eta(y,x)$. If the following conditions hold:*
(i) *for fixed $\mu \in \Lambda, T(\cdot,\mu)$ is η-hemicontinuous on K;*
(ii) *for fixed $\mu \in \Lambda, T(\cdot,\mu)$ is η-strongly C pseudomonotone on K;*
(iii) *for fixed $x \in K, T(x,\cdot)$ is continuous on Λ.*
Then, $S_\eta(\cdot)$ is continuous on Λ.

Proof. First of all, we invoke Lemma 3.2 to conclude that $S_\eta(\cdot)$ is single valued. Thus, we assume, without loss of generality, that $S_\eta(\mu) = x(\mu), \forall \mu \in \Lambda$. Take $\mu_0 \in \Lambda$, in order to obtain that $S_\eta(\cdot)$ is continuous at μ_0, we only need to show that $x(\mu) \to x(\mu_0)$ as $\mu \to \mu_0$. For any sequence $\{\mu_n\} \subseteq \Lambda$ that satisfies $\mu_n \to \mu_0$, we can find a solution set sequence $x(\mu_n) \in K$. Further, note that K is compact, there exists a convergent subsequence $\{x(\mu_{n_k})\}$ such that $x(\mu_{n_k}) \to \nu$. Next we prove that $\nu = x(\mu_0)$.

Note that $x(\mu_n)$ is the solution of (η-PWVVI), we obtain

$$\langle T(x(\mu_n),\mu_n), \eta(y,x(\mu_n))\rangle \in Y\backslash -intC, \forall y \in K \tag{9}$$

In view of Lemma 3.1, (9) is equivalent to

$$\langle T(y,\mu_n), \eta(y,x(\mu_n))\rangle \in Y\backslash -intC, \forall y \in K \tag{10}$$

We claim, bear in mind that for fixed $x \in K, T(x,\cdot)$ is continuous on Λ and $\eta(y,\cdot)$ is continuous, that

$$\begin{aligned}
&\parallel \langle T(y,\mu_n), \eta(y,x(\mu_n))\rangle - \langle T(y,\mu_0), \eta(y,\nu)\rangle \parallel \\
\leq &\parallel \langle T(y,\mu_n), \eta(y,x(\mu_n))\rangle - \langle T(y,\mu_0), \eta(y,x(\mu_n))\rangle \parallel \\
+ &\parallel \langle T(y,\mu_0), \eta(y,x(\mu_n))\rangle - \langle T(y,\mu_0), \eta(y,\nu)\rangle \parallel \\
\leq &\parallel T(y,\mu_n) - T(y,\mu_0) \parallel \cdot \parallel \eta(y,x(\mu_n)) \parallel \\
+ &\parallel T(y,\mu_0) \parallel \cdot \parallel \eta(y,x(\mu_n)) - \eta(y,\nu) \parallel
\end{aligned}$$

Thus, let $n \to \infty$, we have

$$\langle T(y,\mu_n), \eta(y, x(\mu_n))\rangle \to \langle T(y,\mu_0), \eta(y,\nu)\rangle \tag{11}$$

Note that $Y\backslash - intC$ is closed and (11), we get

$$\langle T(y,\mu_0), \eta(y,\nu)\rangle \in Y\backslash - intC, \forall y \in K$$

Combine this with condition (ii) and use Lemma 3.1 again, we find

$$\langle T(\nu,\mu_0), \eta(y,\nu)\rangle \in Y\backslash - intC, \forall y \in K$$

Hence, $v \in S_\eta(\mu_0)$. Recall that, by Lemma 3.2, $S_\eta(\cdot)$ is single valued. Thus, $\nu = x(\mu_0)$. The proof is complete.

Remark 3.1 Note that in Theorem 3.3, T is required to be a η-hemicontinuous operator which extend ν-hemicontinuous in [6] and hence it weaken the continuous condition in [13].

4 Example

In this section, we present an example to show that there exists $\eta(y,x)$ function that satisfies the condition (C1)-(C3).

Let $K = R$, take the function

$$\eta(y,x) = \begin{cases} y - x, & x \le 0, y \le 0 \ and\ x \ge 0, y \ge 0, \\ x - y, & x \le 0, y \ge 0 \ and\ x \ge 0, y \le 0. \end{cases}$$

It is evident that K is invex with respect to $\eta(y,x)$. The function $\eta(y,x)$ above satisfies condition (C2) and (C3) is obvious. Next we verify that it also satisfies condition (C1).

(i) For $x \le 0,\ y \le 0$ and any $\lambda \in [0,1], x + \lambda(y-x) = (1-\lambda)x + \lambda y \le 0$,

$$\begin{aligned} \eta(x, x + \lambda\eta(y,x)) &= \eta(x, x + \lambda(y-x)) \\ &= \eta(x, (1-\lambda)x + \lambda y) \\ &= -\lambda(y-x) = -\lambda\eta(y,x). \end{aligned}$$

(ii) For $x \ge 0,\ y \ge 0$ and any $\lambda \in [0,1], x + \lambda(y-x) = (1-\lambda)x + \lambda y \ge 0$,

$$\begin{aligned} \eta(x, x + \lambda\eta(y,x)) &= \eta(x, x + \lambda(y-x)) \\ &= \eta(x, (1-\lambda)x + \lambda y) \\ &= -\lambda(y-x) = -\lambda\eta(y,x). \end{aligned}$$

(iii) For $x \leq 0,\ y \geq 0$ and any $\lambda \in [0,1], x+\lambda(x-y) = (1+\lambda)x - \lambda y \leq 0$,

$$\begin{aligned}\eta(x, x+\lambda\eta(y,x)) &= \eta(x, x+\lambda(x-y)) \\ &= -\lambda(x-y) = -\lambda\eta(y,x).\end{aligned}$$

(iv) For $x \geq 0,\ y \leq 0$ and any $\lambda \in [0,1], x+\lambda(x-y) = (1+\lambda)x - \lambda y \geq 0$,

$$\begin{aligned}\eta(x, x+\lambda\eta(y,x)) &= \eta(x, x+\lambda(x-y)) \\ &= -\lambda(x-y) = -\lambda\eta(y,x).\end{aligned}$$

The analysis above shows that the $\eta(y,x)$ function satisfies the condition (C1)-(C3) but $\eta(y,x) \neq y-x$. That is to say η-parameter weak vector variational inequality(η-PWVVI) extends the existing parameter weak vector variational inequality and the results we obtained is reasonable.

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