System Compatibility: Price of Anarchy and Control Mechanisms in the Models of Concordance of Private and Public Interests

Olga I. Gorbaneva and Guennady A. Ougolnitsky
Southern Federal University, Russia

Abstract
The problem of system compatibility is considered. Its solution ensures maximization of the social welfare by consideration of individual interests of the agents. The conditions of system compatibility and respective control mechanisms are analyzed for the models of concordance of common and private interests in the agents' resource allocation.
Keywords Concordance of interests; Control mechanisms; Hierarchical game theory; System compatibility

1 Introduction

A problem of concordance of interests in the active systems may be considered in two aspects. First, it is well known that an egoistic behavior of independent active agents often implies a less social welfare than in the case of their coordinated actions. The quantitative side of this problem is named "inefficiency of equilibria"[1] and can be characterized by the price of anarchy index introduced by Papadimitriou [2].Second, the agents can allocate their resources between private and public interests. In the seminal paper by Germeier and Vatel [3],it is shown that if payoff functions of all agents are convolutions by minimum of the functions of public and private interests then in the respective game there is a Pareto-optimal Nash equilibrium (i.e. the price of anarchy is equal to the ideal value of one). We continue to investigate models of that type in literature [4,5] and in the present paper.

Mathematical methods of solution of the static problems of concordance of interests of active agents are developed in the theory of incentives [6], the information theory of hierarchical systems [7,8], the theory of control in organizations [9,10], mechanism design [1]. It should be noticed that the mechanism design investigates another setting of the problem: how to motivate agents to report the true information about their type (the problem of strategy-proofness, or incentive compatibility).

In the present paper a notion of system compatibility is introduced. The system compatibility means that individually optimal controls of agents form the globally optimal vector of controls for a social welfare function. This setting is close to the problem of meta-game synthesis in the theory of active systems [11]. Conditions of the system compatibility are studied for the models of concordance of private and public interests (CPPI-models). As the conditions are quite restrictive, the control mechanisms directed to ensure the system compatibility are proposed. In the framework of the authors' concept [12], the administrative and economic mechanisms with/without a feedback are constructed.

The rest of the paper is organized as follows. In the section 2 the principal notions such as price of anarchy, system compatibility, control mechanisms, and CPPI-models are introduced. Economic mechanisms without feedback and with it as well as algorithms of their implementation are considered in the sections 3 and 4 respectively. Administrative control mechanisms are discussed in the section 5. Section 6 concludes.

2 Principal Notions.

Let's consider a set $N = \{1, 2, ..., n\}$ of active agents maximizing their payoff functions

$$g_i(u_1, ..., u_n) \to \max \tag{1}$$

s.t.

$$u_i \in U_i, \quad i \in N \tag{2}$$

Let the solution of the game (1)-(2) be a Nash equilibrium $u^{NE} \in NE$,$u = (u_1, ..., u_n)$. Introduce a utilitarian social welfare function. $g_0(u) = \sum\limits_{j \in N} g_j(u)$. Let $u^{\max}$ be a solution of the maximization problem

$$g_0(u) \to \max, \quad u \in U = U_1 \times ... \times U_n \tag{3}$$

Denote $g_0^{\max} = g_0(u^{\max})$.

Definition 1. The model(1)-(3)is system compatible if $\forall u^{NE} \in NE \; g_0(u^{NE}) = g_0^{\max}$.

A quantitative measure of the system compatibility is the price of anarchy [1-2]

$$PA = \frac{\min\limits_{u^{NE} \in NE} g_0(u^{NE})}{g_0^{\max}} \tag{4}$$

It is evident that a model is system compatible if $PA = 1$. The system compatibility is a rare phenomenon, and it is worthwhile to use control mechanisms for its achievement[9].

Suppose that maximization of the social welfare (3) is the objective of a specific agent (Center, principal, mechanism designer and so on) who has an ability of impact on the sets of feasible controls and/or payoff functions of the other agents to provide this objective. Denote the first possibility as $U_i = U_i(q_i)$, and the second one $g_i = g_i(p_i, u_i)$, where q, p are vectors of the principal's administrative and economic controls respectively. In the context of literature[12] we can differentiate the following control mechanisms (methods of control).

Table 1 Control mechanisms

Principal's impact	Without a feedback (Γ_1)	With a feedback (Γ_2)
On the sets of feasible controls of the agents(administrative one,or compulsion)	$q_i = const$	$q_i = q_i(u)$
On the agents' payoff functions(economic one,or impulsion)	$p_i = const$	$p_i = p_i(u)$

Thus, the principal can exert influence on the sets of feasible controls of the agents (administrative control mechanism, or compulsion) or on the agents' payoff functions (economic control mechanism, or impulsion). Both mechanisms can include not or include a feedback on control. In the first case a hierarchical game of the type Γ_1(Stackelberg game) holds, meanwhile the second case generates a hierarchical game of the type Γ_2(Germeier game). So, four types of control mechanisms are possible (Table 1). Notice that now or in dependence of the used control mechanism.

Definition 2. A control mechanism $q(p)$ in the model (1)-(3) is *system compatible* if $g_0^{\max} = g_0(u^{NE}(q))$ or $g_0^{\max} = g_0(u^{NE}(p))$ respectively.

For definiteness let's specify the model (1)-(3) in the form

$$g_i(u) = p_i(r_i - u_i) + s_i c(u) \to \max,\ 0 \le u_i \le r_i,\ i \in N \tag{5}$$

$$g_0(u) = \sum_{j\in N} p_j(r_j - u_j) + c(u) \to \max,\ \sum_{j\in N} s_j = \begin{cases} 1,\ \exists i : s_i > 0, \\ 0,\ otherwise. \end{cases} \tag{6}$$

Here, $r_i > 0$ is a resource of the i-th agent; u_i is a part of the resource assigned for production of the public payoff described by a function $c(u)$; s_i is a share of the i-th agent in the public payoff; $p_i(r_i - u_i)$ is a function of the i-th agent's private interest. The functions p_i, c are supposed to be continuously differentiable and concave in all arguments. So, each agent shares his resource between public and private interests according to the ratio u_i and $r_i - u_i$ respectively. Thus, the model (5)-(6) describes a concordance of the private and public interests in resource allocation. Our investigation of models of the type (5)-(6) (CPPI-models) develops the approach by Germeier and Vatel and Burkov and Opoitsev [3,11].

Economic control mechanisms in the model (5)-(6) are implemented by the principal's choice of the values s_i. To use administrative mechanisms one should suppose additionally that the principal can bound feasible controls of the agents:

$$\tilde{q}_i \le u_i \le \bar{q}_i, \quad i \in N \tag{7}$$

The control mechanisms from the Table 1 can be specified for the CPPI-model (5)-(7) as follows (Table 2).

Table 2 Mechanisms of system compatibility in CPPI-models

Principal's impact	Without a feedback (Γ_1)	With a feedback (Γ_2)
On the sets of feasible controls of the agents(administrative one,or compulsion)	$\tilde{q}_i \leq u_i \leq \bar{q}_i,\ \tilde{q}_i, \bar{q}_i = const$	$\tilde{q}_i(u) \leq u_i \leq \bar{q}_i(u)$
On the agents' payoff functions(economic one,or impulsion)	$s_i = const$	$s_i = s_i(u)$

3 Economic Control Mechanisms Without a Feedback.

Assume that in the model (5)-(6) $\forall i\ s_i = const$ .Using the first order conditions we get that the internal system compatibility in the model (5)-(6) holds only if

$$\frac{\partial c}{\partial u_i} = 0, \quad i \in N \tag{8}$$

Let's notice that it is true also for models with partly coincident interests in more general form [5]

$$g_i(u) = p_i(u_i) + s_i c(u) \to \max,\ u_i \in U_i,\ i \in N \tag{9}$$

So, the following statement is true.

Theorem 1. Suppose that $\exists i \in N:\ \partial c / \partial u_i \neq 0$. Then for system compatibility in the model (5)-(6) it is necessary that $\forall i \in N\ u_i = 0 \vee u_i = r_i$.

In other words, the system compatibility in the model (5)-(6) is possible only if all agents are pure individualists ($u_i = 0$) or pure collectivists ($u_i = r_i$) .

Example 1 (linear CPPI-model). Consider a linear specification of the model (5)-(6):

$$g_i(u) = k_i(r_i - u_i) + s_i K \sum_{j \in N} u_j \to \max, \quad 0 \leq u_i \leq r_i, \quad i \in N \tag{10}$$

$$g_0(u) = \sum_{j \in N} k_j(r_j - u_j) + K \sum_{j \in N} u_j \to \max; \quad S: 0 \leq s_i \leq 1, \sum_{j \in N} s_j = \begin{cases} 1, \exists i: s > 0, \\ 0,\ otherwise. \end{cases} \tag{11}$$

Here $k_i > 0, K > 0$ are given constants which characterize efficiencies of the functions of private and public payoffs respectively.

The first order conditions for the agents and the principal give respectively:

$$\frac{\partial g_i}{\partial u_i} = Ks_i - k_i \begin{cases} \geq 0, \ s_i \geq \frac{k_i}{K} \Rightarrow u_i^{NE} = r_i; \\ < 0, \ s_i < \frac{k_i}{K} \Rightarrow u_i^{NE} = 0; \end{cases}$$

$$\frac{\partial g_0}{\partial u_i} = K - k_i \begin{cases} \geq 0, \ k_i \leq K \Rightarrow u_i^{\max} = r_i; \\ < 0, \ k_i > K \Rightarrow u_i^{\max} = 0. \end{cases}$$

Thus, the system compatibility is possible and holds on the bounds of the segments of feasible controls. Two partitions of the set N can be defined:

$$N = I_0 \cup C_0, \ I_0 \cap C_0 = \emptyset :$$

$$I_0 = \{i \in N : \ k_i > K \Rightarrow u_i^{\max} = 0\}(\textit{immanent individualists});$$

$$C_0 = \{i \in N : \ k_i \leq K \Rightarrow u_i^{\max} = r_i\}(\textit{immanent collectivists}).$$

$$N = I(s) \cup C(s), \ I(s) \cap C(s) = \emptyset :$$

$$I(s) = \{i \in N : \ s_i < \frac{k_i}{K} \Rightarrow u_i^{NE} = 0\}(\textit{controlled individualists});$$

$$C(s) = \{i \in N : \ s_i \geq \frac{k_i}{K} \Rightarrow u_i^{NE} = r_i\}(\textit{controlled collectivists}).$$

The immanent partition is determined by the objective properties of the model (10)-(11), meanwhile the controlled partition results from the optimal reaction of the agents on the choice of a vector $s = (s_1, ..., s_n)$ by the principal.

Notice that $i \in I_0 \Rightarrow k_i/K > 1 \Rightarrow \forall s \in S \, s_i \leq k_i/K \Rightarrow i \in I(s)$, *i.e.* $\forall s \in S \, I_0 \subset I(s)$.

The inverse statement is wrong: let $k_i \leq K \Rightarrow i \in C_0$ but $s_i = 0 \Rightarrow i \in I(s)$. Similarly, it is simple to show that $\forall s \in S \ C(s) \subset C_0$. It is also clear that if $\exists s \in S : \ I(s) = I_0, C(s) = C_0$ then the model is system compatible.

To find her optimal control $s*$ the principal should solve a discrete optimization problem

$$g_0(s) = \sum_{j \in I(s)} k_j r_j + K \sum_{j \in C(s)} r_j \rightarrow \max \tag{12}$$

$$S : \quad 0 \leq s_i \leq 1, \quad i \in N, \sum_{j \in N} s_j = \begin{cases} 1, \ \exists i : s_i > 0, \\ 0, \ \textit{otherwise}. \end{cases} \tag{13}$$

if $C_0 = \emptyset$ then the model is system compatible for $\forall s \in S$ .In this case $N = I(s) = I_0, g_0^{\max} = g_0^I = \sum\limits_{j \in N} k_j r_j$ (individualistic society).

Suppose that $C_0 \neq \emptyset$ if $\sum\limits_{i \in C_0} k_i \leq K$ then let for $i \in C_0$, $s_i^* = \frac{k_i}{K} + \varepsilon_i$ so that $\sum\limits_{i \in C_0} s_i^* = 1$. Then $(s*) = C_0$, and the model is system compatible. Particularly, if $I_0 = \emptyset$ then $N = C_0 = C(s*)$, $\quad g_0^{\max} = g_0^C = K \sum\limits_{j \in C_0} r_j$ (collectivistic society).

In general case $I_0 \neq \emptyset$, $C_0 \neq \emptyset$, $\sum\limits_{j \in C_0} k_j > K$. Here $|C_0| > 1$. Subject to $\forall s \in S$, $I_0 \subset I(s)$ it is impossible to put any $i \in I_0$ into $C(s)$, thus the problem (12)-(13) is reduced to the construction of a set $C(s) \subset C_0$ such that

$$g_0(s) = \sum_{j \in I(s) = C_0 \backslash C(s)} k_j r_j + K \sum_{j \in C(s)} r_j \to \max, \quad \sum_{j \in C(s)} k_j \leq K, \quad s \in S$$

This problem is still being solved.

Example 2 (power-linear CPPI-model). Consider the following specification of (5)-(6):

$$g_i(u) = k_i \sqrt{r_i - u_i} + s_i K \sum_{j \in N} u_j \to \max, \quad 0 \leq u_i \leq r_i, \quad i \in N \tag{14}$$

$$g_0(u) = \sum_{j \in N} k_j \sqrt{r_j - u_j} + K \sum_{j \in N} u_j \to \max, \quad 0 \leq s_i \leq 1, \quad \sum_{j \in N} s_j = \begin{cases} 1, \exists i : s > 0, \\ 0, \ otherwise. \end{cases} \tag{15}$$

The first order conditions for the agents and the principal give respectively:

$$0 = \frac{\partial g_i}{\partial u_i} = K s_i - \frac{k_i}{2\sqrt{r_i - u_i}} \Rightarrow u_i^{NE} = \begin{cases} r_i - \frac{k_i^2}{4K^2 s_i^2}, \ s_i \geq \frac{k_i}{2K\sqrt{r_i}}; \\ 0, \ otherwise; \end{cases}$$

$$0 = \frac{\partial g_0}{\partial u_i} = K - \frac{k_i}{2\sqrt{r_i - u_i}} \Rightarrow u_i^{\max} = \begin{cases} r_i - \frac{k_i}{4K^2}, \ k_i \leq 2K\sqrt{r_i}; \\ 0, \ otherwise. \end{cases}$$

Thus, the system compatibility in the model (14)-(15) holds only if

$$\forall i \in N \ , k_i \geq 2K\sqrt{r_i} \tag{16}$$

If this condition holds, then $N = I(s) = I_0$, $g_0^{\max} = g_0^I = \sum\limits_{j \in N} k_j \sqrt{r_j}$ for any $s \in S$.

If (16) doesn't hold, then the problem of system compatibility can be formulated in a weaker form of maximization the price of anarchy (4) by a mechanism of economic control. The following problem of discrete optimization arises

$$g_0(s) = \sum_{j \in I(s)} k_j \sqrt{r_j} + \sum_{j \in C'(s)} [K r_j + \frac{{k_j}^2}{2K s_j} - \frac{k_j^2}{4K s_j^2}] \to \max \tag{17}$$

s.t. (13), where

$$I(s)=\{i\in N:\ s_i\le \frac{k_i}{2K\sqrt{r_i}}\Rightarrow u_i^{NE}=0\},$$

$$C'(s)=\{i\in N:\ s_i> \frac{k_i}{2K\sqrt{r_i}}\Rightarrow u_i^{NE}=r_i-\frac{k_i^2}{4K^2s_i^2}\}$$

This problem is reduced to the following one:

$$\sum_{j\in C'(s)}(\frac{2k_j^2}{s_j}-\frac{k_j^2}{s_j^2})+\lambda(\sum_{j\in C'(s)}s_j-1)\to\max \tag{18}$$

The FOC gives: $-\frac{2k_j^2}{2s_j^2}+\frac{k_j^2}{2s_j^3}+\lambda=0$, s.t. $s\in S$.

Multiplication by s_j gives the system

$$\begin{cases} s_j^3-\frac{2k_j^2s_j}{\lambda}+\frac{2k_j^2}{\lambda}=0\\ \sum\limits_{i\in C'(s)}s_i=1\end{cases}$$

To find s_j and λ the following should be done: 1) to express s_j by λ from the first equation; 2) to substitute the expression into the second equation and find λ; 3) to substitute the value of λ back to the expression from 1) and find the respective s_i.

Given λ the first equation can be solved analytically by Cartan method (the solution is omitted due to its awkwardness) or numerically. As only real solutions such as $0\le s_i\le 1$ are feasible, the following conclusions can be received:

(1) a solution exists only if $\lambda<0$. Rewriting the equation as $s_j^3=\frac{2k_j^2}{\lambda}(s_j-1)$, we get $\lambda<0$ due to positive left part and negative terms in the numerator and in the brackets;

(2) when $\lambda<0$ the only real solution s_i of the equation exists, and $0\le s_i\le 1$.

Actually, lets find the derivative of the expression $f\left(s_j\right)=s_j^3-\frac{2k_j^2s_j}{\lambda}+\frac{2k_j^2}{\lambda}$:

$$f'\left(s_j\right)=3s_j^2-\frac{2k_j^2}{\lambda}>0.$$

Therefore, the function $f(s_i)$ increases in R , and if a root of the equation $f(s_i)=0$ exists then it is unique.Now let's prove that the root exists and satisfies $0\le s_i\le 1$.

$f\left(0\right)=\frac{2k_j^2}{\lambda}<0$, $f\left(1\right)=1>0$. Subject to continuousness of $f(s_i)$ the property is proved.

(3) s_j increases in λ. Let's find the derivative $\frac{\partial s_j}{\partial \lambda}$ of an implicit function: $3s_j^2 \cdot s_j' - \frac{2k_j^2 \lambda \cdot s_j' - 2k_j^2 s_j}{\lambda^2} - \frac{2k_j^2}{\lambda^2} = 0$, or $s_j' = \frac{2k_j^2(1-s_j)}{\lambda(3s_j^2\lambda - 2k_j)} < 0$ that implies.

(4) $\sum\limits_{i \in C'} s_i$ increases in λ.

Therefore, it is possible to choose such λ that $\sum\limits_{i \in C'(s)} s_i = 1$. Thus, a problem with two variables is reduced to a problem with one variable. The problem can be solved by the method of dichotomy in which the left bound of a segment is equal to $\lambda_1 : \sum\limits_{i \in C'(s)} s_i > 1$ (for example,$\lambda_1 = -\varepsilon\sqrt{b^2 - 4ac}$, and the right bound is a big enough $\lambda_2 : \sum\limits_{i \in C'(s)} s_i < 1$.

4 Economic Control Mechanisms with a Feedback.

Now assume that in the model (5)-(6) $s_i = s_i(u_i)$ or even $s_i = s_i(u)$. According to FOC, the internal system compatibility is possible only if

$$\frac{\partial s_i(u)}{\partial u_i} c(u) = [1 - s_i(u)] \frac{\partial c(u)}{\partial u_i}, \quad i \in N \tag{19}$$

This condition is less restrictive than (8) when $s_i = const$. Both empirical and theoretical approaches can be used in the following analysis. In the context of empirical approach widely spread in practical activity methods of resource allocated are investigated. For example, consider a natural method of proportional allocation

$$s_i(u) = \begin{cases} \frac{u_i}{\sum\limits_{j \in N} u_j}, \ \exists m : \ u_m > 0, \\ 0, \ otherwise. \end{cases} \tag{20}$$

In this case (19) takes the form

$$\sum_{j \neq i} u_j [\frac{\partial c(u)}{\partial u_i} \sum_{j \in N} u_j - c(u)] = 0, \quad i \in N$$

and therefore the following statement is evident.

Theorem 2. The mechanism of proportional allocation (20) is system compatible in the CPPI-models with linear function of public payoff $c(u)$ and any functions of private payoffs.

Example 3. Suppose that $g_i(u) = k_i\sqrt{r_i - u_i} + s_i K \sum\limits_{j \in N} u_j$, where is determined by (20). Then $\frac{\partial g_i}{\partial u_i} = \frac{\partial g_0}{\partial u_i} = K - \frac{k_i}{2\sqrt{r_i - u_i}}$, $\quad u_i^{NE} = u_i^{\max} = \begin{cases} r_i - \frac{k_i^2}{4K^2}, & k_i \le 2K\sqrt{r_i}, \\ 0, & otherwise; \end{cases}$

$g_0^{\max} = \sum_{j\in I} k_j \sqrt{r_j} + K \sum_{j\in C'} r_j I = \{i \in N: \ k_i > 2K\sqrt{r_i}\}, C' = \{i \in N: \ k_i \le 2K\sqrt{r_i}\}.$

Theoretical approach to building of system compatible economic impulsion mechanisms is based on Germeier's theorem for games of the type Γ_2 (see Appendix). Let's apply this theorem to the linear model (5)-(6). We obtain s_i^D is arbitrary (g_0 does not depend on s), $s_i^P \equiv 0$;

$$L_i = k_i r_i; \quad E_i = \{u_i = 0\}; \quad D_i = \{(s_i, u_i): \ s_i > \frac{k_i u_i}{K \sum\limits_{j\in N} u_j}, \sum_{i=1}^{n} s_i = 1\};$$

$$K_2 = g_0^I = \sum_{j\in N} k_j r_j$$

To find K_1 we must solve an optimization problem

$$g_0(u) = \sum_{j\in I(s)} k_j (r_j - u_j) + K \sum_{j\in C(s)} u_j \to \max$$

s.t.

$$\frac{k_i u_i}{K \sum\limits_{j\in N} u_j} < s_i \le 1, \quad \sum_{j\in N} s_j = \begin{cases} 1, \ \exists i: s_i > 0, \\ 0, \ otherwise, \end{cases}, \quad 0 \le u_i \le r_i, \quad i \in N$$

From the first order condition

$$\frac{\partial g_0}{\partial u_i} = K - k_i \Rightarrow u_i^* = \begin{cases} r_i, & k_i \le K \quad (set \quad C_0), \\ 0, & otherwise \quad (set \quad I_0). \end{cases}$$

It is proved that it is possible to find s_i from the set D_i:

$$s_{i\in C_0} = \frac{k_i r_i}{K \sum\limits_{j\notin I_0} r_j} + \varepsilon_i, \sum_{i=1}^{n} \varepsilon_i = 1 - \frac{\sum\limits_{i\notin I_0} k_i r_i}{K \sum\limits_{j\notin I_0} r_j}; s_{i\in I_0} = 0.$$

Therefore, like in Γ_1 formulation (example 1), we obtain

$$g_0(u^*) = \sum_{j\in I_0} k_j r_j + K \sum_{j\in C_0} r_j$$

Three cases are possible:

1. $\forall i \in N \ , k_i > K \Rightarrow u_i^* = 0, \quad K_1 = g_0^{\max} = g_0^I = \sum_{j \in N} k_j r_j$, in this case $K_2 = K_1$, and $g_i = k_i r_i = L_i$.

2. $\forall i \in N \ , k_i \le K \Rightarrow u_i^* = r_i, \quad K_1 = g_0^{\max} = g_0^C = K \sum_{j \in N} r_j$, that is $K_2 \ge K_1$, and $g_i = K r_i > L_i$, hence, condition $s_i > \frac{k_i u_i}{K \sum_{j \in N} u_j}$ is satisfied. In this case $N = C_0 = C(s), I = \emptyset, \ g_0^{\max} = g_0^C = K \sum_{j \in N} r_j, \quad s^*$, is any allocation satisfying (13) ("collectivistic" society).

3. $\exists l, m : k_l > K, k_m \le K$. Here the solution of the problem (12)-(13) is given by the following algorithm: to assign $s_i^* = \begin{cases} \frac{k_i r_i}{K \sum_{j \notin I_0} r_j} + \varepsilon_i, \ k_i \le K; \\ 0, \ k_i > K. \end{cases}$ Then $u_i^* = \begin{cases} r_i, & k_i \le K, \\ 0, & k_i > K, \end{cases}$ and $g_0^{\max} = K \sum_{j \in I_0} r_j + \sum_{j \notin I_0} k_j r_j$.

In this case $C_0 = C(s) \ne \emptyset, \ I_0 = I(s) \ne \emptyset$ (≪mixed≪ society). In all cases 1-3 there is a system compatibility of the model (12)-(13) and economic impulsion mechanism $s*$.

$$g_i(u) = k_i \sqrt{r_i - u_i} + s_i K \sum_{j \in N} u_j \to \max, \quad 0 \le u_i \le r_i, \quad i \in N$$

$$g_0(u) = \sum_{j \in N} k_j \sqrt{r_j - u_j} + K \sum_{j \in N} u_j \to \max, \quad 0 \le s_i \le 1, \quad \sum_{j \in N} s_j = \begin{cases} 1, \exists i : s > 0, \\ 0, \ otherwise. \end{cases}$$

We obtain s_i^D is arbitrary (g_0 does not depend on s),$s_i^P \equiv 0$;

$$L_i = k_i \sqrt{r_i}; \quad E_i = \{u_i = 0\}; \quad D_i = \{(s_i, u_i) : \ s_i > \frac{k_i u_i}{K(\sqrt{r_i} + \sqrt{r_i - u_i}) \sum_{j \in N} u_j}\};$$

$$K_2 = g_0^I = \sum_{j \in N} k_j \sqrt{r_j}$$

. To find K_1 we must solve an optimization problem

$$g_0(u) = \sum_{j \in N} k_j \sqrt{r_j - u_j} + K \sum_{j \in N} u_j \to \max$$

$$\frac{k_i(\sqrt{r_i} - \sqrt{r_i - u_i})}{K \sum_{j \in N} u_j} < s_i \le 1, \quad \sum_{j \in N} s_j = \begin{cases} 1, \ \exists i : s_i > 0, \\ 0, \ otherwise, \end{cases} , \quad 0 \le u_i \le r_i, \quad i \in N$$

From the first order condition

$$u_i^* = \begin{cases} r_i - \frac{k_i{}^2}{4K^2},\ r_i \le 2K\sqrt{r_i}; \\ 0,\ otherwise. \end{cases}$$

It is proved that it is possible to find s_i from the set D_i:

$$s_i^* = \begin{cases} \frac{k_i(\sqrt{r_i} - \frac{k_i}{2K})}{K \sum\limits_{j \notin I_0} \left(r_j - \frac{k_j{}^2}{4K^2} \right)} + \varepsilon_i,\ u_i > 0; \\ 0,\ u_i = 0. \end{cases} \quad \text{where } \sum_{i=1}^{n} \varepsilon_i = 1 - \frac{\sum\limits_{i \notin I_0} k_i(\sqrt{r_i} - \frac{k_i}{2K})}{K \sum\limits_{j \notin I_0} \left(r_j - \frac{k_j{}^2}{4K^2} \right)};$$

Therefore $K_1 = \sum\limits_{j \in I_0} k_j \sqrt{r_j} + \sum\limits_{j \notin I_0} \left(K r_i + \frac{k_i^2}{4K} \right) > K_2$.

The following cases are possible.

1. $\forall i \in N\ , k_i > 2K\sqrt{r_i} \Rightarrow u_i^* = 0, \quad K_1 = g_0^{\max} = g_0^I = \sum\limits_{j \in N} k_j \sqrt{r_j}$, in this case $K_2 = K_1$, and $g_i = k_i \sqrt{r_i} = L_i$.

2. $\forall i \in N\ , k_i \le 2K\sqrt{r_i} \Rightarrow u_i^* = r_i, \quad K_1 = g_0^{\max} = \sum\limits_{j \in N} \left(K r_i + \frac{k_i^2}{4K} \right)$, i.e. $K_2 \le K_1, L_i = k_i \sqrt{r_i}; \quad E_i = \{u_i = 0\}; \quad D_i = \left\{ (s_i, u_i) :\ s_i > \frac{k_i u_i}{K(\sqrt{r_i} + \sqrt{r_i - u_i}) \sum\limits_{j \in N} u_j} \right\}$; and $g_i = K r_i + \frac{k_i^2}{4K} > L_i$, hence, condition $s_i > \frac{k_i u_i}{K(\sqrt{r_i} + \sqrt{r_i - u_i}) \sum\limits_{j \in N} u_j}$ is satisfied. In this case $N = C', I_0 = \emptyset, \ \sum\limits_{j \in N} \left(K r_i + \frac{k_i^2}{4K} \right), \quad s^*$ is any allocation satisfying (13) ("collectivistic" society).

3. $\exists l, m : k_l > 2K\sqrt{r_l}, k_m \le k_i > 2K\sqrt{r_m}$. Here the solution of the problem (12)-(13) is given by the following algorithm: to assign

$$s_i^* = \begin{cases} \frac{k_i(\sqrt{r_i} - \frac{k_i}{2K})}{K \sum\limits_{j \notin I_0} \left(r_j - \frac{k_j{}^2}{4K^2} \right)_i},\ k_i \le 2K\sqrt{r_i}; \\ 0,\ k_i > K. \end{cases} \quad \text{Then } u_i^* = \begin{cases} r_i - \frac{k_i{}^2}{4K^2},\ r_i \le 2K\sqrt{r_i}; \\ 0,\ otherwise. \end{cases}$$

and $g_0^{\max} = \sum\limits_{j \in I_0} k_j \sqrt{r_j} + \sum\limits_{j \notin I_0} \left(K r_i + \frac{k_i^2}{4K} \right)$.

In this case $C_0 = C(s) \ne \emptyset,\ I_0 = I(s) \ne \emptyset$ (≪mixed≫ society). In all cases 1-3 there is a system compatibility of the model (12)-(13) and economical impulsion mechanism $s*$.

5 Administrative control mechanisms.

Suppose that the principal can bound the agents' sets of feasible controls. Consider the case of administrative control without a feedback. Then the model (5)-(6) takes the form

$$g_i(\tilde{q}_i, \bar{q}_i, u) = p_i(r_i - u_i) + s_i c(u) \to \max, \quad \tilde{q}_i \le u_i \le \bar{q}_i, \quad s_i \in [0, 1]; \tag{21}$$

$$g_0(\tilde{q}, \bar{q}, u) = \sum_{j\in N} p_j(r_j - u_j) + c(u) \to \max, \quad 0 \le \tilde{q}_i \le \bar{q}_i \le r_i, \quad i \in N. \quad (22)$$

It is clear that if there are no restrictions then the problem (22) has a trivial solution $\tilde{q}_i = \bar{q}_i = u_i^{\max}$, $i \in N$. Therefore the principals control cost should be considered. Then (22) takes the form

$$g_0(\tilde{q}, \bar{q}, u) = \sum_{j\in N} p_j(r_j - u_j) + c(u) - C(\tilde{q}, \bar{q}) \to \max, \quad 0 \le \tilde{q}_i \le \bar{q}_i \le r_i, \quad i \in N,$$

Where $C(\tilde{q}, \bar{q})$ is a continuously differentiable and convex in all arguments compulsion cost function. Let's give a simple example.

Example 4. Assume that

$$g_i(\tilde{q}_i, \bar{q}_i, u) = k_i(r_i - u_i) + s_i K \sum_{j\in N} u_j \to \max, \quad \tilde{q}_i \le u_i \le \bar{q}_i, \quad s_i \in [0,1];$$

$$g_0(\tilde{q}, \bar{q}, u) = \sum_{j\in N} k_j(r_j - u_j) + K \sum_{j\in N} u_j - \sum_{j\in N} (\tilde{m}_j \tilde{q}_j + \bar{m}_j \bar{q}_j) \to \max,$$

$$0 \le \tilde{q}_i \le \bar{q}_i \le r_i, \quad i \in N$$

where $K, k_i, \tilde{m}_i, \bar{m}_i$ are known positive constants. The FOC give

$$\frac{\partial g_i}{\partial u_i} = s_i K - k_i \Rightarrow u_i^{NE} = \begin{cases} r_i, \ k_i \le s_i K, \\ 0, \ k_i > s_i K; \end{cases}$$

$$\frac{\partial g_0}{\partial u_i} = K - k_i \Rightarrow u_i^{\max} = \begin{cases} r_i, \ k_i \le K, \\ 0, \ k_i > K; \end{cases}$$

$$\frac{\partial g_0}{\partial \tilde{q}_i} = -\tilde{m}_i, \ \frac{\partial g_0}{\partial \bar{q}_i} = -\bar{m}_i \Rightarrow \tilde{q}_i^{\max} = \bar{q}_i^{\max} = 0$$

Notice that if $k_i > K$ then $k_i > s_i K$, therefore $u_i^{\max} = u_i^{NE} = 0$, and compulsion is not required (the model is system compatible). Otherwise two cases should be differentiated:

(a) $k_i \le s_i K \Rightarrow u_i^{\max} = u_i^{NE} = r_i$, and compulsion is not required again;

(b) $s_i K < k_i \le K \Rightarrow u_i^{NE} = 0, \ u_i^{\max} = r_i$. In this case the principal's payoffs with and without compulsion should be compared. Denote by M the set of agents for whom $s_i K < k_i \le K$. If compulsion holds we have $g_0(r, 0, r) = K \sum_{j\in M} r_j - \sum_{j\in M} \tilde{m}_j r_j$, and if not then $g_0(0,0,0) = \sum_{j\in M} k_j r_j$. Thus, compulsion is rational if $\sum_{j\in M} (K - \tilde{m}_j - k_j) r_j > 0$ holds.

6 Conclusion.

In the present paper the problem of system compatibility was analyzed. Its solution ensures maximization of the social welfare by considering individual interests of the agents. This setting is close to the problem of mechanism design but distinct because in the latter problem it is required to motivate agents to report the true information about their types instead of the direct maximization of the social welfare.

The conditions of system compatibility are quite restrictive, therefore to provide them it is worthwhile to construct control mechanisms. In the framework of authors' concept, the mechanisms are classified by two attributes: direction of impact (sets of feasible controls of the agents or their payoff functions) and presence or absence of feedback in the control system. The first attribute differentiates administrative and economic control mechanisms (methods of compulsion and impulsion respectively), while the second one leads to hierarchical games of the types Γ_1 and Γ_2.

The conditions and mechanisms of system compatibility are analyzed in the class of models of concordance of the private and public interests in resource allocation (CPPI-models). Some preliminary results about system compatibility of the CPPI-models are obtained. Thus, system compatibility in CPPI-models when $s_i = const$ is reachable only if all agents are pure individualists (all resources are assigned for private interests) or pure collectivists (all resources are assigned for public interest). The exact dichotomous partition is built by specific algorithms of discrete optimization. Economic mechanisms with a feedback simplify the achievement of system compatibility. Administrative mechanisms of system compatibility are under development.

The research perspectives include:

- investigation of the system compatibility for more general classes of models;
- considering of corruption (an additional feedback on bribe);
- analysis of dynamic settings, including phase constraints (requirements of sustainable development), investigation of the conditions of time consistence.

Appendix *(Germeier theorem).* Assume that payoff functions of both players $M_1(x_1, x_2), M_2(x_1, x_2)$ are continuous on compact sets of feasible controls X_1, X_2. Introduce the punishment function $x_1^P(x_2)$ such that $M_2(x_1^P, x_2) = \min_{x_1 \in X_1} M_2(x_1, x_2)$, and the dominant strategy of the player 1 $x_1^D(x_2)$, which satisfies the condition $M_1(x_1^D, x_2) = \max_{x_1 \in X_1} M(x_1, x_2)$. Introduce also the following values and sets:

$$L_2 = \max_{x_2 \in X_2} M_2(x_1^P(x_2), x_2);\; E_2 = \{x_2 \in X_2 : M_2(x_1^P(x_2), x_2) = L_2\};$$

$$D_2 = \{(x_1, x_2) \in X_1 \times X_2 : M_2(x_1, x_2) > L_2\};$$

$$K_1 = \sup_{(x_1,x_2)\in D_2} M_1(x_1,x_2) \le M_1(x_1^\varepsilon, x_2^\varepsilon) + \varepsilon \quad (D_2 = \emptyset \Rightarrow K_1 = -\infty);$$

$$K_2 = \min_{x_2\in E_2} \max_{x_1\in X_1} M_1(x_1,x_2).$$

Then the guaranteed payoff of the player 1 (Leader) in the game Γ_2 (in which the first player knows the choice of the second player) is equal to $w_1 = \max(K_1, K_2)$, and the respective -optimal guaranteeing strategy has the form

$$\tilde{x}_1^\varepsilon(x_2) = \begin{cases} x_1^\varepsilon, & x_2 = x_2^\varepsilon,\ K_1 > K_2, \\ x_1^D(x_2), & x_2 \in D_2,\ K_1 \le K_2, \\ x_1^P(x_2), & \text{otherwise}, \end{cases}$$

where x_1^ε and x_2^ε are described above.

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References

[1] N. Nisan, T. Roughgarden, E. Tardos and V. Vazirani. (2007), *Algorithmic Game Theory*, Cambridge University Press.

[2] Papadimitriou C.H. (2001), "Algorithms, games, and the Internet", *Proc. 33th Symposium Theory of Computing*, pp.749-753.

[3] Germeier Yu.B. and Vatel I.A. (1975), "Equilibrium situations in games with a hierarchical structure of the vector of criteria" ,*Lecture Notes in Computer Science*, Vol.27, pp.460-465.

[4] Gorbaneva O.I. and Ougolnitsky G.A. (2013), "Purpose and non-purpose resource use models in two-level control systems", *Advances in Systems Science and Application* ,Vol.13,No.4,pp.378-390.

[5] Ougolnitsky G.A. (2011), "Games with differently directed interests", *Contributions to Game Theory and Management*, Vol.4,pp.327-338.[Collected papers presented on the Fourth International Conference Game Theory and Management / Editors L. Petrosyan, N. Zenkevich.]

[6] Laffont J.-J. and Martimort D. (2002), *The Theory of Incentives: The Principal-Agent Model*, Princeton University Press.

[7] Kononenko A.F. (1974), "Game-theory analysis of a two-level hierarchical control system", *USSR Computational Mathematics and Mathematical Physics*,Vol.14,No.5,pp.72-81.

[8] Kukushkin N.S. (1994), “A condition for existence of nash equilibrium in games with public and private objectives”, *Games and Economic Behavior*,Vol.7,pp.177-192.

[9] D. Novikov. (2013), *Mechanism Design and Management: Mathematical Methods for Smart Organizations*, Nova Science Publishers.

[10] Novikov D. (2013), *Theory of Control in Organizations*, Nova Science Publishers.

[11] Burkov V.N. and Opoitsev V.I. (1974), “Metagame approach to the control in hierarchical systems”, *Automation and Remote Control*, Vol 35, No.1, pp.93-103.

[12] Ougolnitsky G. (2011), *Sustainable Management*, Nova Science Publishers.

Corresponding author
Olga I. Gorbaneva, Ph.D. can be contacted at: gorbaneva@mail.ru
Guennady A. Ougolnitsky, Ph.D. can be contacted at:ougoln@gmail.com