Some Results for Self-Stabilization of Stochastic Differential Equation *

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Abstract *A class of stochastic differential equation is studied about the self-stabilization in this paper. By constructing suitable hypothesis, sufficient criteria for the stochastic differential equation stabilized itself are established.*

Keywords *Self-stabilization; Stochastic neural networks; It's formula; Borel-Cantelli lemma*

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1. Introduction

The stability of solutions is very important in the theory of differential equations. Many authors have done very excellent work [1-3] on the fields of stability. However, the stability for stochastic differential equations are somewhat difficult than that of deterministic differential equations, related literature can be found in recent work. For stochastic differential equations, [4] discussed a criterion that determines the region of mean square stability for second-order weak numerical schemes, [5] had given some sufficient conditions corcerning stability of solutions of stochastic differential evolution equations with general decay rate and [6] considered the one-step approximations of solutions, respectively. Here we just mentioned the work for stability about Mao Xuerong, for example, he established stochastic versions of the well-known Lasalle stability theorem in [7] and investigated exponential stability of paths for a class of Hilbert space-valued non-linear stochastic evolutions in [8]. Our work is motivated by Mao [1], he had showed that the trivial solution of equation (1.1) in [1], namely,

$$dx(t) = f(x(t),t)dt + ug(x(t),t)dB(t) \qquad (1)$$

was almost surely exponentially stable for all sufficiently large u ,let $u > 0$ be the noise intensity parameter and $B(t)$ be an m -dimensional Brownian motion. Mao [1] had discussed if the intensity parameter u was replaced by $\int_0^t |r(s)x(s)|^p ds$, then the equation (1.1) becomes

$$dx(t) = f(x(t),t)dt + \left(\int_0^t |r(s)x(s)|^p ds\right) g(x(t),t)dB(t) \qquad (2)$$

Where $p > 0$ and $r(s)$ was a continuous $R^{n\times d}$-valued function defined

on R^+ satisfying $\|r(t)\| \le Me^{\gamma t}$ for all $t \ge 0$,which be called a convergence rate function.

The standing hypothesis (H1) and (H2) are imposed in [1] as follows:

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(H1.1) There exists a symmetric positive-definite $d \times d$ -matrix Q and three positive constants K, α, β with $2\beta > \alpha$, such that

$$\left| x^T Qf(x,t) \right| \le K \left| x \right|^2,$$
$$trace(g(x,t)^T Qg(x,t)) \le \alpha x^T Qx,$$
$$\left| x^T Qg(x,t) \right|^2 \ge \beta \left| x^T Qx \right|^2, \qquad \text{for all } t \ge 0 \text{ and } x \in R^d.$$

Now, we will prove that equation (4) stabilizes itself, that is a alternative theorem given by expressions (7)and (8). As far as the author's knowledge that there is no corresponding results.
To make the statement more clear, Mao [1] had stated the condition on the convergence rate function $r(t)$ as another hypothesis:

(H1.2) There exists a pair of constants $M > 0$ and $\gamma \ge 0$ such that

$$\left\| r(t) \right\| \le Me^{\gamma t} \qquad \text{for all } t \ge 0 \text{。}$$

Mao [1] had proved that if (H1.1) and (H1.2) hold, then for every $x_0 \in R^d$, the solution of equation (1.2), either

$$\int_0^\infty \left| r(t)x(t) \right|^p dt \le \sqrt{\frac{2K}{(2\beta - \alpha)\lambda_{\min}(Q)}} \tag{3}$$

or
$$\limsup_{t\to\infty} \frac{1}{t} \log\left(\left| x(t;x_0) \right| \right) < 0 \tag{4}$$

holds for almost all $\omega \in \Omega$.

If u is replaced by $\sup_{0\le s\le t} \left| r(s)x(s) \right|$, then equation (1.1) becomes

$$dx(t) = f(x(t),t)dt + (\sup_{0\le s\le t} \left| r(s)x(s) \right|) g(x(t),t)dB(t) \tag{5}$$

for general $t \ge t_0 = 0$ and denote $x(0) = x_0 \in R^d$.

Mao [1] had also shown that if (H1) and (H2) hold, then for every 个 $x_0 \in R^d$, the solution of equation (5) has the property

$$\sup_{0\le t<\infty} \left| r(t)x(t) \right| < \infty \qquad \text{as.} \tag{6}$$

Furthermore,
(i) if $t \to \infty$ as $\lambda_{\min}(r(t)x(t)) \to \infty$, then

$$\lim_{t\to\infty} \left| x(t) \right| = 0 \qquad \text{as.} \tag{7}$$

(ii) if $\liminf_{t\to\infty} \log[\lambda_{\min}(r^T(t)r(t))]/t \ge \lambda > 0$， then

$$\limsup_{t\to\infty} \frac{1}{t} \log\left(\left| x(t) \right| \right) \le -\frac{\lambda}{2} \qquad \text{as.} \tag{8}$$

It is very interesting for us to analyze whether the system

$$dx(t) = f(x(t),t)dt + (\sup_{0\le s\le t} \left| r(s)x(s) \right|) g(x(t),t)dB(t) \tag{5}$$

has the altermnative theorem like the results given by expressions (3) and (4).

To the author's knowledge, there is no corresponding results. Now we will put our efforts on system (5), by use of exponential martingale formula, Lyapunov function and some special inequalities in our paper, the trivial solution of equation (5) is self-stabilization will been shown in the next section. Let us begin with our paper now.

2. Main Results

Lemma 2.1 Let hypothesis (H1) hold. Then the solution of equation (5) has the property that

$$p\{x(t;x_0)\neq 0 \text{ for all } t\geq 0\}=1$$

provided $x_0\neq 0$.

We consider the following problem of stochastic self-stabilization in this section. Suppose we are given a stochastic differential equation

$$dx(t)=f(x(t),t)dt+(\sup_{0\leq s\leq t}|r(s)x(s)|)g(x(t),t)dB(t) \tag{9}$$

on $t\geq t_0=0$ with initial value $x(0)=x_0\in R^d$ (it is just for convenience to set $t_0=0$ and the theory clearly works for general $t_0\geq 0$.

We have the following result.

Theorem 2.1 Let (H1) and (H2) hold. Then for every $x_0\in R^d$, the solution of equation (9), either

$$\sup_{0\leq s\leq\infty}|r(s)x(s)|\leq\sqrt{\frac{2K}{(2\beta-\alpha)\lambda_{\min}(Q)}} \tag{10}$$

or

$$\limsup_{t\to\infty}\frac{1}{t}\log\left(|x(t;x_0)|\right)<0 \tag{11}$$

Proof. Since hypothesis (1) guarantees $x(t;0)\equiv 0$, one only need to show the conclusions for all $x_0\neq 0$. Fix $x_0\neq 0$ arbitrarily, and write $x(t;x_0)=x(t)$, by lemma 2.1 $t\geq 0$ for all $x(t;x_0)\neq 0$ almost surely. Suppose (10) is false, then there exists some $x_0\neq 0$ for which $P(\bar{\Omega})>0$, where

$$\bar{\Omega}=\left\{\omega\in\Omega:\sup_{0\leq s\leq\infty}|r(s)x(s)|>\sqrt{\frac{2K}{(2\beta-\alpha)\lambda_{\min}(Q)}}\right\}.$$

Clearly, one only needs to show that (11) holds for almost all $\omega\in\Omega$, For each $i=1,2,\ldots,$

$$\bar{\Omega}_i=\left\{\omega\in\bar{\Omega}:\sup_{0\leq s\leq\infty}|r(s)x(s)|>(1+i^{-1})i^{\frac{1}{2}}\sqrt{\frac{2K}{(2\beta-\alpha)\lambda_{\min}(Q)}}\right\}.$$

define

Now, $\bar{\Omega}\in\bigcup_{i=1}^{\infty}\bar{\Omega}_i$, and hence one only needs to show that for each $i\geq 1$,（10）holds for almost $\omega\in\bar{\Omega}_i$. Fix any $i\geq 1$ let $V(x(t),t)=\log(x^T(t)Qx(t))$, one then derives that

$$V_t(x(t),t)=0, Vx(x(t),t)=\frac{2x^T(t)Q}{x^T(t)Qx(t)}, Vxx(x(t),t)=\frac{2Qx^T(t)Qx(t)-2Qx(t)\bullet 2x^T(t)Q}{(x^T(t)Qx(t))^2}.$$

By It$\hat{o}$'s formula, we obtain, we obtain

$$\begin{aligned}
dV(x(t),t) &= V_t(x(t),t)dt+V_x(x(t),t)dx \\
&+\frac{1}{2}trace[g^T(x(t),t)(\sup_{0\le s\le t}|r(s)x(s)|)V_{xx}(x(t),t)(\sup_{0\le s\le t}|r(s)x(s)|)g(x(t),t)]dt \\
&= V_x(x(t),t)f(x(t),t)dt+(\sup_{0\le s\le t}|r(s)x(s)|)V_x(x(t),t)g(x(t),t)dB(t) \\
&+\frac{1}{2}(\sup_{0\le s\le t}|r(s)x(s)|)^2 trace(g^T(x(t),t)V_{xx}(x(t),t)g(x(t),t))dt \\
&= \frac{2x^T(t)Qf(x(t),t)}{x^T(t)Qx(t)}dt+2(\sup_{0\le s\le t}|r(s)x(s)|)\frac{x^T(t)Qg(x(t),t)}{x^T(t)Qx(t)}dB(t) \\
&+(\sup_{0\le s\le t}|r(s)x(s)|)^2\frac{1}{(x^T(t)Qx(t))^2}trace(g^T(x(t),t)Qx^T(t)Qx(t)g(x(t),t))dt \\
&-2(\sup_{0\le s\le t}|r(s)x(s)|)^2\frac{1}{(x^T(t)Qx(t))^2}trace(g^T(x(t),t)Qx(t)x^{\mathrm{T}}(t)Qg(x(t),t))dt,
\end{aligned}$$

this yields that

$$\begin{aligned}
&\log(x^T(t)Qx(t)) \\
&=\log(x_0{}^TQx_0)+\int_0^t\frac{2x^T(s)Qf(x(s),s)}{x^T(s)Qx(s)}ds+2\int_0^t(\sup_{0\le s\le t}|r(s)x(s)|)\frac{x^T(s)Qg(x(s),s)}{x^T(s)Qx(s)}dB(s) \\
&+\int_0^t(\sup_{0\le s\le t}|r(s)x(s)|)^2\frac{trace(g^T(x(s),s)Qg(x(s),s))}{x^T(s)Qx(s)}ds \\
&-2\int_0^t(\sup_{0\le s\le t}|r(s)x(s)|)^2\frac{|x^T(s)Qg(x(s),s)|^2}{(x^T(s)Qx(s))^2}ds.
\end{aligned}$$

By hypothesis (H1.1) and the fact that $\lambda_{\min}(Q)|x(t)|^2\le x^T(t)Qx(t)\le\lambda_{\max}(Q)|x(t)|^2$ for Q is a symmetric $d\times d$ matrix, one can show that for any $t\ge 0$

$$\begin{aligned}
&\log(x^T(t)Qx(t)) \\
&\le\log(x_0{}^TQx_0)+\frac{2Kt}{\lambda_{\min}(Q)}+M(t)+\alpha\int_0^t(\sup_{0\le v\le s}|r(v)x(v)|)^2ds \\
&-2\int_0^t(\sup_{0\le v\le s}|r(v)x(v)|)^2\frac{|x^T(s)Qg(x(s),s)|^2}{(x^T(s)Qx(s))^2}ds
\end{aligned}\tag{12}$$

Where $$M(t)=2\int_0^t(\sup_{0\le v\le s}|r(v)x(v)|)\frac{x^T(s)Qg(x(s),s)}{x^T(s)Qx(s)}dB(s)$$

is a continuous martingale vanishing at $t=0$, Let $k=1,2,\ldots$, Then by the exponential martingale inequality

$$p(\omega:\sup_{0\le t\le k}\left[M(t)-\frac{2\beta-\alpha}{4\beta(1+i^{-1})}\langle M(t),M(t)\rangle\right]>\frac{4\beta(1+i^{-1})\log k}{2\beta-\alpha})\le\frac{1}{k^2},$$

where $\langle M(t),M(t)\rangle=4\int_0^t(\sup_{0\le v\le s}|r(v)x(v)|)^2\frac{|x^T(s)Qg(x(s),s)|^2}{(x^T(s)Qx(s))^2}ds.$

whereHence the well-known Borel-Cantelli lemma yields that for almost all $\omega\in\Omega$ there exists a random integer $k_1(\omega)$ uch that for almost all $k\ge k_1$,

$$\sup_{0\le t\le k}\left[M(t)-\frac{2\beta-\alpha}{4\beta(1+i^{-1})}\langle M(t),M(t)\rangle\right]\le\frac{4\beta(1+i^{-1})\log k}{2\beta-\alpha}$$

that is, for $0\le t\le k$,

$$M(t)\le\frac{4\beta(1+i^{-1})\log k}{2\beta-\alpha}+\frac{2\beta-\alpha}{4\beta(1+i^{-1})}\langle M(t),M(t)\rangle$$

$$=\frac{4\beta(1+i^{-1})\log k}{2\beta-\alpha}+\frac{2\beta-\alpha}{\beta(1+i^{-1})}\int_0^t(\sup_{0\le v\le s}|r(v)x(v)|)^2\frac{|x^T(s)Qg(x(s),s)|^2}{(x^T(s)Qx(s))^2}ds. \quad (13)$$

Substituting (13）into (12) and then applying (H1) one obtains that for each $\omega\in\Omega-\hat{\Omega}$, with $\hat{\Omega}$ a P-null set, there exists a random integer $k_2(\omega)$, such that, for every 个 $\omega\in\bar{\Omega}_i$ there exists a random number $k_3(\omega)$ such that

$$\sup_{0\le s\le t}|r(s)x(s)|\ge(1+i^{-1})i^{\frac{1}{2}}\sqrt{\frac{2K}{(2\beta-\alpha)\lambda_{\min}(Q)}} \qquad \text{for almost all } t\ge k_3.$$

It then follows from that for almost all $\omega\in\bar{\Omega}_i-\hat{\Omega}$, if $k-1\le t\le k, k\ge k_2\vee(k_3+1)$

$$\log(x^T(t)Qx(t))$$

$$\le\log(x_0{}^TQx_0)+\frac{2Kt}{\lambda_{\min}(Q)}+\frac{4\beta(1+i^{-1})\log k}{2\beta-\alpha}-\frac{2\beta-\alpha}{(1+i^{-1})i}\int_{k_3}^t(\sup_{0\le v\le s}|r(v)x(v)|)^2ds$$

$$\le\log(x_0{}^TQx_0)+\frac{2Kt}{\lambda_{\min}(Q)}+\frac{4\beta(1+i^{-1})\log k}{2\beta-\alpha}-\frac{2K(1+i^{-1})}{i\lambda_{\min}(Q)}(k-1-k_3)$$

$$\le\log(x_0{}^TQx_0)+\frac{2K(k_3+1)}{\lambda_{\min}(Q)}+\frac{4\beta(1+i^{-1})\log k}{2\beta-\alpha}-\frac{2K}{i\lambda_{\min}(Q)}(k-1-k_3).$$

This implies that

$$\frac{1}{t}\log(x^T(t)Qx(t))\le\frac{1}{k-1}\left[\log(x_0{}^TQx_0)+\frac{2K(k_3+1)}{\lambda_{\min}(Q)}+\frac{4\beta(1+i^{-1})\log k}{2\beta-\alpha}-\frac{2K}{i\lambda_{\min}(Q)}(k-1-k_3)\right].$$

It then follows that

$$\limsup_{t\to\infty}\frac{1}{t}\log(x^T(t)Qx(t))\le-\frac{2K}{i\lambda_{\min}(Q)} \qquad \text{for almost all } \omega\in\bar{\Omega}_i-\hat{\Omega}. \quad (14)$$

Thus, for almost all $\omega \in \bar{\Omega}_i - \hat{\Omega}$, there exists a random number $k_4(\omega)$ and $\delta > 0$ arbitrarily such that

$$\frac{1}{t}\log(x^T(t)Qx(t)) \le -\frac{2K}{i\lambda_{\min}(Q)} + 2\delta \qquad \text{for almost all } t \ge k_4,$$

from the expression $\lambda_{\min}(Q)\left|x(t)\right|^2 \le x^T(t)Qx(t)$ one derives that

$$\left|x(t)\right| \le \frac{\exp\{(-\frac{K}{i\lambda_{\min}(Q)} + \delta)t\}}{\sqrt{\lambda_{\min}(Q)}} \qquad \text{for almost all } t \ge k_4.$$

Consequently,

$$\limsup_{t\to\infty}\frac{1}{t}\log(\left|x(t;x_0)\right|) \le -\frac{2K}{i\lambda_{\min}(Q)} + \delta \quad \text{for almost all } \omega \in \bar{\Omega}_i - \hat{\Omega}.$$

Since $\delta > 0$ is arbitrary, we must have that

$$\limsup_{t\to\infty}\frac{1}{t}\log(\left|x(t;x_0)\right|) < -\frac{2K}{i\lambda_{\min}(Q)} < 0 \qquad \text{for almost all}$$

$\omega \in \bar{\Omega}_i - \hat{\Omega}$.

The proof is now complete.

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