Monotone Hybrid Methods for a Finite Family of Nonexpasive Multi-valued Maps and Equilibrium Problems*

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Abstract *In this paper, we introduce a new monotone hybrid iterative scheme for finding a common element of the set of common fixed points of a finite family of nonexpansive multi-valued maps and the set of the solutions of the equilibrium problem in a Hilbert space. Moreover, we also introduce a new iterative scheme for finding a common fixed point of a finite family of nonexpansive multi-valued maps in a Banach space. Strong convergence theorem of the proposed iteration is established.*

Keywords *Nonexpansive multi-valued map Monotone hybrid method,*

1. Introduction

Let D be a nonempty convex subset of a Banach space E. Let f be a bifunction from $D\times D$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of all real number. The equilibrium problem for f is to find $x \in D$ such that $f(x, y) \geq 0$ for all $y \in D$. The set of such solutions is denoted by $EP(f)$. The set D is called *proximinal* if for each $x \in E$, there exists an element $y \in D$ such that $\|x-y\| = d(x, D)$, where $d(x, D) = \inf\{\|x - z\| : z \in D\}$. Let $CB(D)$, $K(D)$ and $P(D)$ denote the families of nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximinal bounded subsets of D, respectively. The *Hausdorff metric* on $CB(D)$ is defined by

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B),\ \sup_{y\in B} d(y, A)\right\}$$

for $A, B \in CB(D)$. A single-valued map $T : D \to D$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in D$. A multi-valued map $T : D \to CB(D)$ is said to be *nonexpansive* if $H(Tx, Ty) \leq \|x-y\|$ for all $x, y \in D$. An element $p \in D$ is called a fixed point of $T : D \to D$ (respectively, $T : D \to CB(D)$) if $p = Tp$ (respectively, $p \in Tp$). The set of fixed points of T is denoted by $F(T)$. The mapping $T : D \to CB(D)$ is called *quasi-nonexpansive*[18] if $F(T) \neq \emptyset$ and $H(Tx, Tp) \leq \|x - p\|$ for all $x \in D$ and all $p \in F(T)$. It is clear that every nonexpansive multi-valued map T with $F(T) \neq \emptyset$ is quasi-nonexpansive. But there exist quasi-nonexpansive mappings that are not nonexpansive, see [17].

The mapping $T : D \to CB(D)$ is called *hemicompact* if, for any sequence $\{x_n\}$ in D such that $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to$

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$p \in D$. We note that if D is compact, then every multi-valued mapping $T : D \to CB(D)$ is *hemicompact*.

A mapping $T : D \to CB(D)$ is said to satisfy *Condition (I)* if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0, \infty)$ such that

$$d(x, Tx) \geq f(d(x, F(T)))$$

for all $x \in D$.

A family $\{T_i : D \to CB(D), i = 1, 2, ..., N\}$ is said to satisfy *Condition (II)* if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0, \infty)$ such that

$$d(x, T_i x) \geq f(d(x, \bigcap_{i=1}^{N} F(T_i)))$$

for all $i = 1, 2, ..., N$ and $x \in D$.

In 1953, Mann [10] introduced the following iterative procedure to approximate a fixed point of a nonexpansive mapping T in a Hilbert space H:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \;\; \forall n \in \mathbb{N}, \tag{1}$$

where the initial point x_0 is taken in C arbitrarily and $\{\alpha_n\}$ is a sequence in [0,1].

However, we note that Mann's iteration process (1) has only weak convergence, in general; for instance, see [1, 7, 15].

In 2003, Nakajo and Takahashi [12] introduced the method which is the so-called CQ method to modify the process (1) so that strong convergence is guaranteed have recently been made. They also proved a strong convergence theorem for a nonexpansive mapping in a Hilbert space.

Recently, Tada and Takahashi [20] proposed a new iteration for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping T in a Hilbert space H.

In 2005, Sastry and Babu [16] proved that the Mann and Ishikawa iteration schemes for multi-valued map T with a fixed point p converge to a fixed point q of T under certain conditions. They also claimed that the fixed point q may be different from p. More precisely, they proved the following result for nonexpansive multi-valued map with compact domain.

In 2007, Panyanak [13] extended the above result of Sastry and Babu [16] to uniformly convex Banach spaces but the domain of T remains compact.

Later, Song and Wang [19] noted that there was a gap in the proofs of Theorem 3.1(see [13]) and Theorem 5 (see [16]). They further solved/revised the gap and also gave the affirmative answer to Panyanak [13] question using the following Ishikawa iteration scheme. In the main results, domain of T is still compact, which is a strong condition (see [19], Theorem 1) and T satisfies condition(I) (see [19], Theorem 1).

In 2009, Shahzad and Zegeye [17] extended and improved the results of Panyanak [13], Sastry and Babu [16] and Song and Wang [19] to quasi-nonexpansive multi-valued maps. They also relaxed compactness of the domain of T and constructed an iteration scheme which removes the restriction of T namely $Tp = \{p\}$ for any $p \in F(T)$. The results provided an affirmative answer to Panyanak [13] question in a more general setting. In the main results, T satisfies condition(I)(see [17], Theorem 2.3) and T is hemicompact and continuous (see [17], Theorem 2.5).

Question: How can we modify iteration process for a nonexpansive multi-valued map T which the domain of T is not necessary to be compact to obtain strong convergence theorems for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of T ?

In the recent years, the problem of finding a common element of the set of solutions of equilibrium problems and the set of fixed points in the framework of Hilbert spaces and Banach spaces have been intensively studied by many authors, for instance, see [2, 3, 4, 5, 6, 8, 14, 20] and the references cited theorem.

In this paper, we introduce a monotone hybrid iterative scheme for finding a common element of the set of a common fixed points of a finite family of nonexpasive multi-valued maps and the set of solutions of an equilibrium problem in a Hilbert space. Let D be nonempty, closed and convex subset of a Hilbert space H and $\alpha_n^i \in (0,1)$ for all $i = 0, 1, ..., m$ with $\sum_{i=0}^m \alpha_n^i = 1$, $\forall n \geq 0$ and $r_n \in (0, \infty)$. For an initial point $x_0 \in D = C_0$, compute the sequence $\{x_n\}$ by the iterative process

$$\begin{cases} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \ \forall y \in D, \\ y_n = \sum_{i=0}^m \alpha_n^i z_n^i, \ \ z_n^i \in T_i u_n, \ \forall i = 1, 2, ..., m, \ z_n^0 = u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ \ n \geq 0, \end{cases} \tag{2}$$

where T_i is a nonexpansive multi-valued map for all $i = 1, 2, ..., m$.

2. Preliminaries

The following lemmas give some characterizations and a useful property of the metric projection P_D in a Hilbert space.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let D be a closed and convex subset of H. For every point $x \in H$, there exists a unique nearest point in D, denoted by $P_D x$, such that

$$\|x - P_D x\| \leq \|x - y\|, \ \ \forall y \in D.$$

P_D is called the *metric projection* of H onto D. We know that P_D is a nonexpansive mapping of H onto D.

Lemma 2.1. [11] *Let D be a closed and convex subset of a real Hilbert space H and let P_D be the metric projection from H onto D. Given $x \in H$ and $z \in D$. Then $z = P_D x$ if and only if the following holds:*

$$\langle x - z, y - z \rangle \leq 0, \quad \forall y \in D.$$

Lemma 2.2. [12] *Let D be a nonempty, closed and convex subset of a real Hilbert space H and $P_D : H \to D$ be the metric projection from H onto D. Then the following inequality holds:*

$$\|y - P_D x\|^2 + \|x - P_D x\|^2 \leq \|x - y\|^2, \ \ \forall x \in H, \ \forall y \in D.$$

Lemma 2.3. [11] *Let H be a real Hilbert space. Then the following equations hold:* (i) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle,\ \forall x, y \in H$;

(ii) $\|tx+(1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2,\ \forall t \in [0,1]$ *and* $x, y \in H$.

By using Lemma 2.3, we obtain the following lemma.

Lemma 2.4. *Let H be a real Hilbert space. Then for each $m \in \mathbb{N}$*

$$\|\sum_{i=1}^{m} t_i x_i\|^2 = \sum_{i=1}^{m} t_i \|x_i\|^2 - \sum_{i=1, i\neq j}^{m} t_i t_j \|x_i - x_j\|^2,$$

$x_i \in H$ *and* $t_i, t_j \in [0,1]$ *for all* $i, j = 1, 2, ..., m$ *with* $\sum_{i=1}^{m} t_i = 1$.

Lemma 2.5. [9]*Let D be a nonempty, closed and convex subset of a real Hilbert space H. Given $x, y, z \in H$ and also given $a \in \mathbb{R}$, the set*

$$\{v \in D: \ \|y-v\|^2 \leq \|x-v\|^2 + \langle z, v\rangle + a\}$$

is convex and closed.

For solving the equilibrium problem, we assume the bifunction $f : D \times D \to \mathbb{R}$ satisfies the following conditions:
(A1) $f(x,x) = 0$ for all $x \in D$;
(A2) f is monotone, i.e., $f(x,y) + f(y,x) \leq 0$ for all $x, y \in D$;
(A3) for each $x, y, z \in D$, $\limsup_{t\downarrow 0} f(tz + (1-t)x, y) \leq f(x,y)$;
(A4) $f(x, \cdot)$ is convex and lower semicontinuous for each $x \in D$.

Lemma 2.6. [2] *Let D be a nonempty, closed and convex subset of a real Hilbert space H. Let f be a bifunction from $D \times D$ to $\mathbb{R}$ satisfying* (A1)-(A4) and let $r > 0$ and $x \in H$. Then, there exists $z \in D$ such that

$$f(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad for\ all\ y \in D.$$

Lemma 2.7. [6] *For $r > 0$, $x \in H$, defined a mapping $T_r : H \to 2^D$ as follows:*

$$T_r(x) = \left\{ z \in D : f(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad for\ all\ y \in D \right\}.$$

Then the followings hold:

(1) T_r *is single-value;*

(2) T_r *is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x-y\rangle;$$

(3) $F(T_r) = EP(f)$;

(4) $EP(f)$ *is closed and convex.*

Lemma 2.8. *Let D be a closed and convex subset of a real Hilbert space H. Let $T : D \to CB(D)$ be a nonexpansive multi-valued map with $F(T) \neq \emptyset$ and $Tp = \{p\}$ for each $p \in F(T)$. Then $F(T)$ is a closed and convex subset of D.*

Proof. First, we will show that $F(T)$ is closed. Let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \to x$ as $n \to \infty$. We have

$$\begin{aligned} d(x,Tx) &\leq d(x,x_n) + d(x_n,Tx) \\ &\leq d(x,x_n) + H(Tx_n,Tx) \\ &\leq 2d(x,x_n). \end{aligned}$$

It follows that $d(x,Tx) = 0$, so $x \in F(T)$. Next, we show that $F(T)$ is convex. Let $p = tp_1 + (1-t)p_2$ where $p_1, p_2 \in F(T)$ and $t \in (0,1)$. Let $z \in Tp$, by Lemma 2.3, we have

$$\begin{aligned} \|p-z\|^2 &= \|t(z-p_1) + (1-t)(z-p_2)\|^2 \\ &= t\|z-p_1\|^2 + (1-t)\|z-p_2\|^2 - t(1-t)\|p_1-p_2\|^2 \\ &= td(z,Tp_1)^2 + (1-t)d(z,Tp_2)^2 - t(1-t)\|p_1-p_2\|^2 \\ &\leq tH(Tp,Tp_1)^2 + (1-t)H(Tp,Tp_2)^2 - t(1-t)\|p_1-p_2\|^2 \\ &\leq t\|p-p_1\|^2 + (1-t)\|p-p_2\|^2 - t(1-t)\|p_1-p_2\|^2 \\ &= t(1-t)^2\|p_1-p_2\|^2 + (1-t)t^2\|p_1-p_2\|^2 - t(1-t)\|p_1-p_2\|^2 \\ &= 0, \end{aligned}$$

hence $p = z$. Therefore $p \in F(T)$.

Lemma 2.9. [21] *Let $p > 1, r > 0$ be two fixed numbers. Then a Banach space E is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - \omega_p(\lambda)g(\|x-y\|),$$

for all $x, y \in B_r(0) = \{x \in E : \|x\| \leq r\}$ and $\lambda \in [0,1]$ where $\omega_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$.

By using Lemma 2.9, we can prove the following Lemma by induction.

Lemma 2.10. *Let E be a uniformly convex Banach space and $B_r(0) = \{ x \in E : \|x\| \leq r\}$ be a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\sum_{i=1}^{m} \alpha_i x_i\|^2 \leq \sum_{i=1}^{m} \alpha_i\|x_i\|^2 - \alpha_1\alpha_2 g(\|x_1 - x_2\|),$$

for all $m \in N, x_i \in B_r(0)$ and $\alpha_i \in [0,1], i = 1,2,...,m$ with $\sum_{i=1}^{m} \alpha_i = 1$.

By interchanging the roles of vectors x_i in Lemma 2.10 and summing the inequalities together we obtain the following lemma.

Lemma 2.11. *Let E be a uniformly convex Banach space and $B_r(0) = \{x \in E : \|x\| \leq r\}$ be a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that for each $j \in \{1, 2, ..., m\}$,*

$$\|\sum_{i=1}^{m} \alpha_i x_i\|^2 \leq \sum_{i=1}^{m} \alpha_i \|x_i\|^2 - \frac{\alpha_j}{m-1}\big(\sum_{i=1}^{m} \alpha_i g(\|x_j - x_i\|)\big),$$

for all $m \in N, x_i \in B_r(0)$ and $\alpha_i \in [0,1]$ for all $i = 1, 2, ..., m$ with $\sum_{i=1}^{m} \alpha_i = 1$.

Proof. Let $j \in \{1, 2, ..., m\}$ be fixed. By Lemma 2.10, there is a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that

$$\begin{aligned}
\|\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + ... + \alpha_m x_m\|^2 &\leq \sum_{i=1}^{m} \alpha_i \|x_i\|^2 - \alpha_j \alpha_1 g(\|x_j - x_1\|) \\
\|\alpha_1 x_1 + \alpha_3 x_3 + \alpha_4 x_4 + ... + \alpha_2 x_2\|^2 &\leq \sum_{i=1}^{m} \alpha_i \|x_i\|^2 - \alpha_j \alpha_2 g(\|x_j - x_2\|) \\
&\vdots \\
\|\alpha_1 x_1 + \alpha_4 x_4 + \alpha_5 x_5 + ... + \alpha_3 x_3\|^2 &\leq \sum_{i=1}^{m} \alpha_i \|x_i\|^2 - \alpha_j \alpha_{j-1} g(\|x_j - x_{j-1}\|) \\
\|\alpha_1 x_1 + \alpha_4 x_4 + \alpha_5 x_5 + ... + \alpha_3 x_3\|^2 &\leq \sum_{i=1}^{m} \alpha_i \|x_i\|^2 - \alpha_j \alpha_{j+1} g(\|x_j - x_{j+1}\|) \\
&\vdots \\
\|\alpha_1 x_1 + \alpha_m x_m + \alpha_2 x_2 + ... + \alpha_{m-1} x_{m-1}\|^2 &\leq \sum_{i=1}^{m} \alpha_i \|x_i\|^2 - \alpha_j \alpha_m g(\|x_j - x_m\|).
\end{aligned}$$

By summing up above inequalities, we obtain

$$\|\sum_{i=1}^{m} \alpha_i x_i\|^2 \leq \sum_{i=1}^{m} \alpha_i \|x_i\|^2 - \frac{\alpha_j}{m-1}\big(\sum_{i=1}^{m} \alpha_i g(\|x_j - x_i\|)\big).$$

3. Main Result

First, we prove a strong convergence theorem for a finite family of nonexpansive multi-valued mappings which satisfies the condition (II) in a uniformly convex Banach space.

Theorem 3.1. *Let D be a nonempty, closed and convex subset of a uniformly convex Banach space E. Let $T_i : D \to CB(D)$ be a nonexpansive multi-valued map for all $i = 1, 2, ..., m$ with $\bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ and $T_i p = \{p\}$ for each $p \in \bigcap_{i=1}^{m} F(T_i)$. Assume that $\{T_i : i = 1, 2, ..., m\}$ satisfies the condition (II) for all $i = 1, 2, ..., m$ and $\alpha_n^i \in (0,1)$ with $0 < \liminf_{n\to\infty} \alpha_n^i \leq$*

$\limsup_{n\to\infty} \alpha_n^i < 1$ for all $i = 0, 1, 2, ..., m$. Let $x_0 \in D$ and let $\{x_n\}$ be the sequence in D generated by iteration process:

$$x_{n+1} = \sum_{i=0}^{m} \alpha_n^i z_n^i, \tag{3}$$

where $z_n^0 = x_n$, $z_n^i \in T_i x_n$ for all $i = 1, 2, ..., m$ and $\sum_{i=0}^{m} \alpha_n^i = 1$. Then $\{x_n\}$ converges strongly to a common fixed point of T_i, $i = 1, 2, ..., m$.

Proof. Let $p \in \bigcap_{i=1}^{m} F(T_i)$. By the nonexpansiveness of T_i, we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \sum_{i=0}^{m} \alpha_n^i \|z_n^i - p\| \\ &= \alpha_n^0 \|x_n - p\| + \sum_{i=1}^{m} \alpha_n^i d(z_n^i, T_i p) \\ &\leq \alpha_n^0 \|x_n - p\| + \sum_{i=1}^{m} \alpha_n^i H(T_i x_n, T_i p) \\ &\leq \|x_n - p\|, \end{aligned} \tag{4}$$

which implies that $\lim_{n\to\infty} \|x_n - p\|$ exists. For each $i = 1, 2, ..., m$, we have $\|z_n^i - p\| = d(z_n^i, T_i p) \leq H(T_i x_n, T_i p) \leq \|x_n - p\|$. It follows that $\{\|z_n^i - p\|\}$ is bounded for all $i = 1, 2, ..., m$. Put $r = \max_{1\leq i\leq m}\{\sup_n \|z_n^i - p\|\}$. By Lemma 2.11, there is a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \sum_{i=0}^{m} \alpha_n^i \|z_n^i - p\|^2 - \frac{\alpha_n^0}{m} \sum_{i=1}^{m} \alpha_n^i g\big(\|z_n^i - x_n\|\big) \\ &= \alpha_n^0 \|x_n - p\|^2 + \sum_{i=1}^{m} \alpha_n^i d(z_n^i, T_i p)^2 - \frac{\alpha_n^0}{m} \sum_{i=1}^{m} \alpha_n^i g\big(\|z_n^i - x_n\|\big) \\ &\leq \alpha_n^0 \|x_n - p\|^2 + \sum_{i=1}^{m} \alpha_n^i H(T_i x_n, T_i p)^2 - \frac{\alpha_n^0}{m} \sum_{i=1}^{m} \alpha_n^i g\big(\|z_n^i - x_n\|\big) \\ &\leq \|x_n - p\|^2 - \frac{\alpha_n^0}{m} \sum_{i=1}^{m} \alpha_n^i g\big(\|z_n^i - x_n\|\big). \end{aligned}$$

It follows that

$$\frac{\alpha_n^0}{m} \sum_{i=1}^{m} \alpha_n^i g\big(\|z_n^i - x_n\|\big) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

This implies that $g\big(\|z_n^i - x_n\|\big) \to 0$ as $n \to \infty$ for all $i = 1, 2, ..., m$. Since g is continuous strictly increasing with $g(0) = 0$, we can conclude that $\|z_n^i - x_n\| \to 0$ as $n \to \infty$ for all $i = 1, 2, ..., m$. Also $d(x_n, T_i x_n) \leq \|z_n^i - x_n\| \to 0$ as $n \to \infty$ for all $i = 1, 2, ..., m$. Since that $\{T_i\}_{i=1}^{m}$ satisfies the condition (II), we have $d(x_n, \bigcap_{i=1}^{m} F(T_i)) \to 0$. Thus there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\|x_{n_k} - p_k\| < \frac{1}{2^k}$ for some $\{p_k\} \subset \bigcap_{i=1}^m F(T_i)$ and all k. From (4), we obtain

$$\|x_{n_{k+1}} - p_k\| \quad \leq \quad \|x_{n_k} - p_k\| < \frac{1}{2^k}.$$

Next, we shall show that $\{p_k\}$ is Cauchy sequence in D. Notice that

$$\begin{aligned} \|p_{k+1} - p_k\| &\leq \|p_{k+1} - x_{n_{k+1}}\| + \|x_{n_{k+1}} - p_k\| \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}. \end{aligned}$$

This implies that $\{p_k\}$ is Cauchy sequence in D and thus converges to $q \in D$. Since

$$d(p_k, T_i q) \leq H(T_i q, T_i p_k) \leq \|q - p_k\|$$

for all $i = 1, 2, ..., m$ and $p_k \to q$ as $n \to \infty$, it follows that $d(q, T_i q) = 0$ for all $i = 1, 2, ..., m$ and thus $q \in \bigcap_{i=1}^m F(T_i)$ and $\{x_{n_k}\}$ converges strongly to q. Since $\lim_{n\to\infty} \|x_n - q\|$ exists, it follows that $\{x_n\}$ converges strongly to q. This completes the proof.

Note that in Theorem 3.1 in order to have strong convergence of the iterative sequence $\{x_n\}$ defined by (3), we need to assume that $\{T_i\}_{i=1}^m$ satisfy the condition (II). In the following theorem, we introduce a new monotone hybrid iterative scheme (2) for finding a common element of the set of a common fixed points of a family of nonexpasive multi-valued maps and the set of solutions of an equilibrium problem in a Hilbert space, and we prove strong convergence of the sequence $\{x_n\}$ defined by (2) without the condition (II).

Theorem 3.2. *Let D be a nonempty, closed and convex subset of a real Hilbert space H. Let f be a bifunction from $D \times D$ to $\mathbb{R}$ satisfying* (A1)-(A4) *and let $T_i : D \to CB(D)$ be nonexpansive multi-valued maps for all $i = 1, 2, ..., m$ with $\bigcap_{i=1}^m F(T_i) \cap EP(f) \neq \emptyset$ and $T_i p = \{p\}$ for each $p \in \bigcap_{i=1}^m F(T_i)$. Assume that $\alpha_n^i \in (0, 1)$ with $0 < \liminf_{n\to\infty} \alpha_n^i \leq \limsup_{n\to\infty} \alpha_n^i < 1$ for all $i = 0, 1, 2, ..., m$ and $r_n \in (0, \infty)$ with $\liminf_{n\to\infty} r_n > 0$. Then the sequence $\{x_n\}$ generated by* (2) *converges strongly to $P_{\bigcap_{i=1}^m F(T_i) \cap EP(f)} x_0$.*

Proof. We split the proof into six steps.

Step 1. Show that $P_{C_{n+1}} x_0$ is well defined for every $x_0 \in D$.

By Lemma 2.8, we obtain that $\bigcap_{i=1}^m F(T_i)$ is a closed and convex subset of D. Since $EP(f)$ is also closed and convex, then $\bigcap_{i=1}^m F(T_i) \cap EP(f)$ is a closed and convex subset of D. From the definition of C_{n+1}, it follows from Lemma 2.5 that C_{n+1} is closed and convex for each $n \geq 0$. Let $v \in \bigcap_{i=1}^m F(T_i) \cap EP(f)$. From $u_n = T_{r_n} x_n$, we have

$$\|u_n - v\| = \|T_{r_n} x_n - T_{r_n} v\| \leq \|x_n - v\|, \tag{5}$$

for every $n \geq 0$. From this, we have

$$\begin{aligned} \|y_n - v\| &= \|\sum_{i=0}^m \alpha_n^i z_n^i - v\| \leq \sum_{i=0}^m \alpha_n^i \|z_n^i - v\| = \alpha_n^0 \|u_n - v\| + \sum_{i=1}^m \alpha_n^i d(z_n^i, T_i v) \\ &\leq \alpha_n^0 \|u_n - v\| + \sum_{i=1}^m \alpha_n^i H(T_i u_n, T_i v) \leq \|u_n - v\| \leq \|x_n - v\|. \end{aligned} \tag{6}$$

So, we have $v \in C_{n+1}$, thus $\bigcap_{i=1}^m F(T_i) \cap EP(f) \subset C_{n+1}$. Therefore $P_{C_{n+1}}x_0$ is well defined.

Step 2. Show that $\lim_{n\to\infty} \|x_n - x_0\|$ exists.

Since $\bigcap_{i=1}^m F(T_i) \cap EP(f)$ is a nonempty, closed and convex subset of H, there exists a unique $v \in \bigcap_{i=1}^m F(T_i) \cap EP(f)$ such that

$$v = P_{\bigcap_{i=1}^m F(T_i) \cap EP(f)} x_0.$$

From $x_n = P_{C_n}x_0$, $C_{n+1} \subset C_n$ and $x_{n+1} \in C_n$, $\forall n \geq 0$, we get

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\|, \ \ \forall n \geq 0.$$

On the other hand, as $\bigcap_{i=1}^m F(T_i) \cap EP(f) \subset C_n$, we obtain

$$\|x_n - x_0\| \leq \|v - x_0\|, \ \ \forall n \geq 0.$$

It follows that the sequence $\{x_n\}$ is bounded and nondecreasing. Therefore $\lim_{n\to\infty} \|x_n - x_0\|$ exists.

Step 3. Show that $x_n \to w \in D$ as $n \to \infty$.

For $m > n$, by the definition of C_n, we see that $x_m = P_{C_m}x_0 \in C_m \subset C_n$. By Lemma 2.2, we get

$$\|x_m - x_n\|^2 \leq \|x_m - x_0\|^2 - \|x_n - x_0\|^2.$$

From Step 2, we obtain that $\{x_n\}$ is Cauchy. Hence, there exists $w \in D$ such that $x_n \to w$ as $n \to \infty$.

Step 4. Show that $\|z_n^i - x_n\| \to 0$ as $n \to \infty$ for every $i = 1, 2, .., m$.

From $x_{n+1} \in C_{n+1}$, we have

$$\begin{aligned} \|x_n - y_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \\ &\leq 2\|x_n - x_{n+1}\| \to 0 \end{aligned} \tag{7}$$

as $n \to \infty$. For $v \in \bigcap_{i=1}^m F(T_i) \cap EP(f)$, by Lemma 2.4 we have

$$
\begin{aligned}
\|y_n - v\|^2 = \|\sum_{i=0}^m \alpha_n^i z_n^i - v\|^2 &\leq \sum_{i=0}^m \alpha_n^i \|z_n^i - v\|^2 - \sum_{i=1}^m \alpha_n^0 \alpha_n^i \|z_n^i - u_n\|^2 \\
&\leq \alpha_n^0 \|u_n - v\|^2 + \sum_{i=1}^m \alpha_n^i d(z_n^i, T_i v)^2 \\
&\quad - \sum_{i=1}^m \alpha_n^0 \alpha_n^i \|z_n^i - u_n\|^2 \\
&\leq \alpha_n^0 \|u_n - v\|^2 + \sum_{i=1}^m \alpha_n^i H(T_i u_n, T_i v)^2 \\
&\quad - \sum_{i=1}^m \alpha_n^0 \alpha_n^i \|z_n^i - u_n\|^2 \\
&\leq \|u_n - v\|^2 - \sum_{i=1}^m \alpha_n^0 \alpha_n^i \|z_n^i - u_n\|^2 \\
&\leq \|x_n - v\|^2 - \sum_{i=1}^m \alpha_n^0 \alpha_n^i \|z_n^i - u_n\|^2.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\sum_{i=1}^m \alpha_n^0 \alpha_n^i \|z_n^i - u_n\|^2 &\leq \|x_n - v\|^2 - \|y_n - v\|^2 \\
&\leq M\|x_n - y_n\|,
\end{aligned}
$$

where $M = \sup_{n\geq 0}\{\|x_n - v\| + \|y_n - v\|\}$. By our assumptions and (7), we obtain

$$
\|z_n^i - u_n\| \to 0 \;\; as\ n \to \infty, \;\; \forall i = 1, 2, ..., m. \tag{8}
$$

From Lemma 2.7, we obtain

$$
\begin{aligned}
\|u_n - v\|^2 = \|T_{r_n} x_n - T_{r_n} v\|^2 &\leq \langle T_{r_n} x_n - T_{r_n} v, x_n - v\rangle \\
&= \langle u_n - v, x_n - v\rangle \\
&= \frac{1}{2}\big\{\|u_n - v\|^2 + \|x_n - v\|^2 - \|x_n - u_n\|^2\big\},
\end{aligned}
$$

hence

$$
\|u_n - v\|^2 \leq \|x_n - v\|^2 - \|x_n - u_n\|^2.
$$

Therefore, by Lemma 2.4, we get

$$\begin{aligned}\|y_n - v\|^2 = \|\sum_{i=0}^{m} \alpha_n^i z_n^i - v\|^2 &\le \sum_{i=0}^{m} \alpha_n^i \|z_n^i - v\|^2 \\ &\le \alpha_n^0 \|u_n - v\|^2 + \sum_{i=1}^{m} \alpha_n^i d(z_n^i, T_i v)^2 \\ &\le \alpha_n^0 \|u_n - v\|^2 + \sum_{i=1}^{m} \alpha_n^i H(T_i u_n, T_i v)^2 \\ &\le \|u_n - v\|^2 \\ &\le \|x_n - v\|^2 - \|x_n - u_n\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_n - u_n\|^2 &\le \|x_n - v\|^2 - \|y_n - v\|^2 \\ &\le M\|x_n - y_n\|,\end{aligned}$$

where $M = \sup_{n\ge 0}\{\|x_n - v\| + \|y_n - v\|\}$. From (7), we obtain

$$\|x_n - u_n\| \to 0 \ \ as\ n \to \infty. \tag{9}$$

From (8) and (9), we have

$$\|x_n - z_n^i\| \le \|x_n - u_n\| + \|u_n - z_n^i\| \to 0 \ \ as\ n \to \infty. \tag{10}$$

Step 5. Show that $w \in \bigcap_{i=1}^m F(T_i) \cap EP(f)$.

From (9) and $\liminf_{n\to\infty} r_n > 0$, we get

$$\|\frac{x_n - u_n}{r_n}\| = \frac{1}{r_n}\|x_n - u_n\| \to 0 \ \ ,\ n \to \infty. \tag{11}$$

From $x_n \to w$ as $n \to \infty$ and (9), we obtain also that $u_n \to w$. We shall show that $w \in EP(f)$. By $u_n = T_{r_n} x_n$, we get

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \ \ \forall y \in D.$$

From the monotonicity of f, we have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge f(y, u_n), \ \ \forall y \in D,$$

hence

$$\langle y - u_n, \frac{u_n - x_n}{r_n}\rangle \ge f(y, u_n), \ \ \forall y \in D.$$

From (11) and condition (A4), we have

$$0 \ge f(y, w), \ \ \forall y \in D.$$

For t with $0 < t \le 1$ and $y \in D$, let $y_t = ty + (1-t)w$. Since $y, w \in D$ and D is convex, then $y_t \in D$ and hence $f(y_t, w) \le 0$. So, we have

$$0 = f(y_t, y_t) \le tf(y_t, y) + (1-t)f(y_t, w) \le tf(y_t, y).$$

Dividing by t, we obtain

$$f(y_t, y) \ge 0, \ \ \forall y \in D.$$

Letting $t \downarrow 0$ and from (A3), we get

$$f(w, y) \ge 0, \ \ \forall y \in D.$$

Therefore, we obtain $w \in EP(f)$. Next, we will show that $w \in \bigcap_{i=1}^m F(T_i)$. For each $i = 1, 2, ..., m$, we have

$$\begin{aligned} d(w, T_i w) &\le \|w - x_n\| + \|x_n - z_n^i\| + d(z_n^i, T_i w) \\ &\le \|w - x_n\| + \|x_n - z_n^i\| + H(T_i u_n, T_i w) \\ &\le \|w - x_n\| + \|x_n - z_n^i\| + \|u_n - w\|. \end{aligned}$$

It follows from Step 4 that $d(w, T_i w) = 0$ and thus $w \in F(T_i)$ for all $i = 1, 2, ..., m$.

Step 6. Show that $w = P_{\bigcap_{i=1}^m F(T_i) \cap EP(f)} x_0$.

Since $x_n = P_{C_n} x_0$, by Lemma 2.1, we have

$$\langle z - x_n, x_0 - x_n \rangle \le 0$$

for all $z \in C_n$. Since $w \in \bigcap_{i=1}^m F(T_i) \cap EP(f) \subset C_n$, we get

$$\langle z - w, x_0 - w \rangle \le 0$$

for all $z \in \bigcap_{i=1}^m F(T_i) \cap EP(f)$. Again by Lemma 2.1, we obtain that $w = P_{\bigcap_{i=1}^m F(T_i) \cap EP(f)} x_0$. This completes the proof.

Corollary 3.3. *Let D be a nonempty, closed and convex subset of a real Hilbert space H. Let f be a bifunction from $D \times D$ to $\mathbb{R}$ satisfying* (A1)-(A4) *and let $T : D \to CB(D)$ be a nonexpansive multi-valued map for all $i = 1, 2, ..., m$ with $F(T) \cap EP(f) \neq \emptyset$ and $Tp = \{p\}$ for each $p \in F(T)$. Assume that $\beta_n \in (0, 1)$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$ and $r_n \in (0, \infty)$ with $\liminf_{n\to\infty} r_n > 0$. For an initial point $x_0 \in D = C_0$, compute the sequence $\{x_n\}$ by the iterative process*

$$\begin{cases} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \ge 0, \ \ \forall y \in D, \\ y_n = \beta_n u_n + (1-\beta_n) z_n, \ \ z_n \in T u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ \ n \ge 0, \end{cases}$$

Then the sequence $\{x_n\}$ converges strongly to $P_{F(T) \cap EP(f)} x_0$.

Proof. Putting $T_1 = T$ and $T_i = I$ for $i = 2, 3, ..., m$, where $I : D \to CB(D)$ such that $Id = \{d\}$ for all $d \in D$ in Theorem 3.2, we obtain the desired result directly from Theorem 3.2.

Corollary 3.4. *Let D be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : D \to CB(D)$ be a nonexpansive multi-valued map for all $i = 1, 2, ..., m$ with $F(T) \cap EP(f) \neq \emptyset$ and $Tp = \{p\}$ for each $p \in F(T)$. Assume that $\beta_n \in (0, 1)$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. For an initial point $x_0 \in D = C_0$, compute the sequence $\{x_n\}$ by the iterative process*

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) z_n, \ \ z_n \in T x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ \ n \geq 0, \end{cases}$$

Then the sequence $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Proof. Putting $f(x, y) = 0$ for all $x, y \in D$ in Corollary 3.3, we obtain the desired result directly from Corollary 3.3.

The main result of this paper holds true under the assumption that $Tp = \{p\}$ for all $p \in F(T)$. This condition was introduced by Shahzad and Zegeye [17]. The following example gives an example of a nonexpansive multi-valued map T which satisfies the property that $Tp = \{p\}$ for all $p \in F(T)$ and Tx is not a singleton for all $x \notin F(T)$.

Example. Consider $D = [0, 1] \times [0, 1]$ with the usual norm. Define $T : D \to CB(D)$ by

$$T(x, y) = \begin{cases} \{(x, 0)\}, & x \neq 0, y = 0 \\ \{(0, y)\}, & x = 0, y \neq 0 \\ \{(x, 0), (0, y)\}, & x, y \neq 0 \\ \{(0, 0)\}, & x, y = 0. \end{cases}$$

Open problem: Can we drop the condition that $Tp = \{p\}$ for all $p \in F(T)$ in the main result of this paper ?

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