A New Modified Block Iterative Algorithm for a System of Equilibrium Problems and a Fixed Point Set of Uniformly Quasi-ϕ-Asymptotically Nonexpansive Mappings*

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Abstract *In this paper, we construct a new modified block hybrid projection algorithm for finding a common element of the set of common fixed points of an infinite family of closed and uniformly quasi -ϕ- asymptotically nonexpansive mappings, the set of the variational inequality for an α-inverse-strongly monotone operator, the set of solutions of a system of equilibrium problems. Moreover, we obtain a strong convergence theorem for the sequences generated by this process in the framework Banach spaces. The results presented in this paper improve and generalize some well-known results in the literature.*

Keywords *Modified block iterative algorithm Inverse-strongly monotone operator Variational inequality A system of equilibrium problem Uniformly quasi-ϕ-asymptotically nonexpansive mapping.*

1. Introduction

Let C be a nonempty closed convex subset of a real Banach space E with $\|\cdot\|$ and E^* the dual space of E and $A : C \to E^*$ be an operator. *The classical variational inequality problem* for an operator A is to find $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{1}$$

The set of solution of (1) is denote by $VI(A, C)$. Recall that let $A : C \to E^*$ be a mapping. Then A is called
(i) *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C,$$

(ii) $\alpha-$*inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C.$$

Such a problem is connected with the convex minimization problem, the complementary problem, the problem of finding a point $x^* \in E$ satisfying $Ax^* = 0$.

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Let $\{f_i\}_{i\in\Gamma} : C \times C \longrightarrow \mathbb{R}$ be a bifunction, $\{\varphi_i\}_{i\in\Gamma} : C \longrightarrow \mathbb{R}$ be a real-valued function, where Γ is an arbitrary index set. The *system of equilibrium problems*, is to find $x \in C$ such that

$$f_i(x, y) \geq 0, \quad i \in \Gamma, \quad \forall y \in C. \tag{2}$$

If Γ is a singleton, then problem (2) reduces to the *equilibrium problem*, is to find $x \in C$ such that

$$f(x, y) \geq 0, \quad \forall y \in C. \tag{3}$$

The above formulation (3) was shown in [5] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, variational inequality problems, vector equilibrium problems, Nash equilibria in noncooperative games. In addition, there are several other problems, for example, the complementarity problem, fixed point problem and optimization problem, which can also be written in the form of an $EP(f)$. In other words, the $EP(f)$ is an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc. In the last two decades, many papers have appeared in the literature on the existence of solutions of $EP(f)$; see, for example [5, 13] and references therein. Some solution methods have been proposed to solve the $EP(f)$; see, for example, [5, 13, 15, 16, 21, 24, 30, 29, 28, 35, 46] and references therein.

For each $p > 1$, the *generalized duality mapping* $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\}$$

for all $x \in E$. In particular, $J = J_2$ is called the *normalized duality mapping*. If E is a Hilbert space, then $J = I$, where I is the identity mapping. Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \tag{4}$$

As well know that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. It is obvious from the definition of function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \quad \forall x, y \in E. \tag{5}$$

If E is a Hilbert space, then $\phi(x, y) = \|x - y\|^2$, for all $x, y \in E$. On the author hand, the *generalized projection* (Alber [2]) $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \inf_{y\in C} \phi(y, x), \tag{6}$$

existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, for example, [1, 2, 12, 17, 37]).

Remark 1.1. If E is a reflexive, strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$ then $x = y$. From (4), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy \rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$; see [12, 37] for more details.

Let C be a closed convex subset of E, a mapping $T : C \to C$ is said to be *L-Lipschitz continuous* if $\|Tx - Ty\| \le L\|x - y\|, \forall x, y \in C$ and a mapping T is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|, \forall x, y \in C$. A point $x \in C$ is a *fixed point* of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. Recall that a point p in C is said to be an *asymptotic fixed point* of T [31] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F(T)}$.

A mapping T from C into itself is said to be *relatively nonexpansive* [25, 36, 45] if $\widetilde{F(T)} = F(T)$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [6, 7, 8]. T is said to be *ϕ-nonexpansive*, if $\phi(Tx, Ty) \le \phi(x, y)$ for $x, y \in C$. T is said to be *relatively quasi-nonexpansive* if $F(T) \ne \emptyset$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. T is said to be *quasi-ϕ-asymptotically nonexpansive* if $F(T) \ne \emptyset$ and there exists a real sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that $\phi(p, T^n x) \le k_n \phi(p, x)$ for all $n \ge 1$ $x \in C$ and $p \in F(T)$.

We note that the class of relatively quasi-nonexpansive mappings is more general than the class of relatively nonexpansive mappings [6, 7, 8, 23, 33] which requires the strong restriction: $F(T) = \widetilde{F(T)}$. A mapping T is said to be *closed* if for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $Tx_n \to y$, then $Tx = y$. It is easy to know that each relatively nonexpansive mapping is closed.

Definition 1.2. ([9]) (1) Let $\{T_i\}_{i=1}^{\infty} : C \to C$ be a sequence of mapping. $\{T_i\}_{i=1}^{\infty}$ is said to be *a family of uniformly quasi-ϕ-asymptotically nonexpansive mappings*, if $\cap_{i=1}^{\infty} F(T_i) \ne \emptyset$, and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that for each $i \ge 1$

$$\phi(p, T_i^n x) \quad \le \quad k_n \phi(p, x),\ \forall p \in \cap_{i=1}^{\infty} F(T_i),\ x \in C, \quad \forall n \ge 1. \tag{7}$$

(2) A mapping $T : C \to C$ is said to be *uniformly L-Lipschitz continuous*, if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \le L\|x - y\|, \quad \forall x, y \in C. \tag{8}$$

Remark 1.3. It is easy to see that an $\alpha-$inverse-strongly monotone is monotone and $\frac{1}{\alpha}$-Lipschitz continuous.

In 2004, Matsushita and Takahashi [22] introduced the following iteration: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T x_n), \tag{9}$$

where the initial guess element $x_0 \in C$ is arbitrary, $\{\alpha_n\}$ is a real sequence in $[0, 1]$, T is a relatively nonexpansive mapping and Π_C denotes the generalized projection from E onto a closed convex subset C of E. They proved that the sequence $\{x_n\}$ converges weakly to a fixed point of T.

In 2005, Matsushita and Takahashi [23] proposed the following hybrid iteration method (it is also called the CQ method) with generalized projection for relatively nonexpansive mapping

T in a Banach space E:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ C_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0. \end{cases} \tag{10}$$

They proved that $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$. In 2008, Iiduka and Takahashi [14] introduced the following iterative scheme for finding a solution of the variational inequality problem for an inverse-strongly monotone operator A in a 2-uniformly convex and uniformly smooth Banach space E : $x_1 = x \in C$ and

$$x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \tag{11}$$

for every $n = 1, 2, 3, \ldots$, where Π_C is the generalized metric projection from E onto C, J is the duality mapping from E into E^* and $\{\lambda_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ generated by (11) converges weakly to some element of $VI(A, C)$. Takahashi and Zembayashi [39, 40], studied the problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem in the framework of Banach spaces.

In 2009, Wattanawitoon and Kumam [41] using the idea of Takahashi and Zembayashi [39] extend the notion from relatively nonexpansive mappings or ϕ-nonexpansive mappings to two relatively quasi-nonexpansive mappings and also proved some strong convergence theorems to approximate a common fixed point of relatively quasi-nonexpansive mappings and the set of solutions of an equilibrium problen in the framework of Banach spaces. Cholamjiak [10], proved the following iteration:

$$\begin{cases} z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ y_n = J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSz_n), \\ u_n \in C \textit{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \ge 0, \qquad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \le \phi(z, x_n), \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{12}$$

where J is the duality mapping on E. Assume that α_n, β_n and γ_n are sequence in $[0, 1]$. Then $\{x_n\}$ converges strongly to $q = \Pi_F x_0$, where $F := F(T) \cap F(S) \cap EP(f) \cap VI(A, C)$.

In 2010, Saewan et al. [33] introduced a new hybrid projection iterative scheme which is difference from the algorithm (12) of Cholamjiak in [10, Theorem 3.1] for two relatively quasi-nonexpansive mappings in a Banach space. Motivated by the results of Takahashi and Zembayashi [40], Cholumjiak and Suantai [11] proved the following strong convergence theorem by the hybrid iterative scheme for approximation of common fixed point of countable families of relatively quasi-nonexpansive mappings in a uniformly convex and uniformly smooth Banach space: $x_0 \in E$, $x_1 = \Pi_{C_1} x_0$, $C_1 = C$

$$\begin{cases} y_{n,i} = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n,) \\ u_{n,i} = T_{r_{m,n}}^{f_m} T_{r_{m-1,n}}^{f_{m-1}} \cdots T_{r_{1,n}}^{f_1} y_{n,i} \\ C_{n+1} = \{z \in C_n : \sup_{i>1} \phi(z, Ju_{n,i}) \le \phi(w, Jx_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, n \ge 1. \end{cases} \tag{13}$$

Then, they proved that under certain appropriate conditions imposed on $\{\alpha_n\}$, and $\{r_{n,i}\}$, the sequence $\{x_n\}$ converges strongly to $\Pi_{C_{n+1}} x_0$.

We note that the block iterative method is a method which often used by many authors to solve the convex feasibility problem (see, [18, 20], etc.). In 2008, Plubtieng and Ungchittrakool [27] established strong convergence theorems of block iterative methods for a finite family of relatively nonexpansive mappings in a Banach space by using the hybrid method in mathematical programming. Chang et al. [9] proposed the modified block iterative algorithm for solving the convex feasibility problems for an infinite family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mapping, they obtain the strong convergence theorems in a Banach space. In 2010, Saewan and Kumam [34] obtain the following result for the set of solutions of the generalized equilibrium problems and the set of common fixed points of an infinite family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings in a uniformly smooth and strictly convex Banach space E with Kadec-Klee property.

Very recently, Qin, Cho and Kang [28] purposed the problem of approximating a common fixed point of two asymptotically quasi-ϕ-nonexpansive mappings based on hybrid projection methods. Strong convergence theorems are established in a real Banach space. H. Zegeye, E. U. Ofoedu and N. Shahzad [46] introduced an iterative process which converges strongly to a common element of set of common fixed points of countably infinite family of closed relatively quasi- nonexpansive mappings, the solution set of generalized equilibrium problem and the solution set of the variational inequality problem for a α-inverse strongly monotone mapping in Banach spaces.

Motivated and inspired by the work of Chang et al. [9], Qin et al. [30], Takahashi and Zembayashi [39], Wattanawitoon and Kumam [41], Zegeye [44] and Saewan and Kumam [34], we introduce a new modified block hybrid projection algorithm for finding a common element of the set of the variational inequality for an α-inverse-strongly monotone operator, the set of solutions of the system of equilibrium problems and the set of common fixed points of an infinite family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings in a 2-uniformly convex and uniformly smooth Banach space. The results presented in this paper improve and generalize some well-known results in the literature.

2. Preliminaries

A Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then a Banach space E is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in U$. Let E be a Banach space. The *modulus of convexity* of E is the function $\delta : [0, 2] \to [0, 1]$ defined by

$$\delta(\varepsilon) = \inf\{1 - \|\frac{x+y}{2}\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon\}.$$

A Banach space E is *uniformly convex* if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. Let p be a fixed real number with $p \geq 2$. A Banach space E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in [0, 2]$; see [3, 38] for more details. Observe that every p-uniformly convex is uniformly convex. One should note that no a Banach space is p-uniformly convex for $1 < p < 2$. It is well known that a Hilbert space is *2-uniformly convex*, uniformly smooth. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

Remark 2.1. The following basic properties can be found in Cioranescu [12].
(i) If E is a uniformly smooth Banach space, then J is uniformly continuous on each bounded subset of E.
(ii) If E is a reflexive and strictly convex Banach space, then J^{-1} is norm-weak*-continuous.
(iii) If E is a smooth, strictly convex, and reflexive Banach space, then the normalized duality mapping $J : E \to 2^{E^*}$ is single-valued, one-to-one, and onto.
(iv) A Banach space E is uniformly smooth if and only if E^* is uniformly convex.
(v) Each uniformly convex Banach space E has the *Kadec-Klee property*, that is, for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.

We also need the following lemmas for the proof of our main results.

Lemma 2.2. (*Beauzamy* [4] and *Xu*[42]). *If E be a 2-uniformly convex Banach space. Then for all $x, y \in E$ we have*

$$\|x - y\| \leq \frac{2}{c^2}\|Jx - Jy\|,$$

where J is the normalized duality mapping of E and $0 < c \leq 1$.

The best constant $\frac{1}{c}$ in Lemma is called the *p-uniformly convex constant* of E.

Lemma 2.3. (*Beauzamy* [4] and Zalinescu [43]). *If E be a p-uniformly convex Banach space and let p be a given real number with $p \geq 2$. Then for all $x, y \in E$, $j_x \in J_p(x)$ and $j_y \in J_p(y)$*

$$\langle x - y, j_x - j_y \rangle \geq \frac{c^p}{2^{p-2}p}\|x - y\|^p,$$

where J_p is the generalized duality mapping of E and $\frac{1}{c}$ is the p-uniformly convexity constant of E.

Lemma 2.4. (*Kamimura and Takahashi* [17]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Lemma 2.5. (*Alber* [2]). *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.6. (*Alber* [2, Lemma 2.4]). *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Let E be a reflexive, strictly convex, smooth Banach space and J is the duality mapping from E into E^*. Then J^{-1} is also single value, one-to-one, surjective, and it is the duality mapping from E^* into E. We make use of the following mapping V studied in Alber [2]

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2, \tag{14}$$

for all $x \in E$ and $x^* \in E^*$, that is, $V(x, x^*) = \phi(x, J^{-1}(x^*))$.

Lemma 2.7. (*Alber* [2]). *Let E be a reflexive, strictly convex smooth Banach space and let V be as in* (14) . *Then*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^*\rangle \le V(x, x^* + y^*),$$

for all $x \in E$ and $x^*, y^* \in E^*$.

A set valued mapping $B : E \rightrightarrows E^*$ with graph $G(B) = \{(x, x^*) : x^* \in Bx\}$, domain $D(B) = \{x \in E : Bx \neq \emptyset\}$, and rang $R(B) = \cup\{Bx : x \in D(B)\}$. B is said to be *monotone* if $\langle x - y, x^* - y^*\rangle \ge 0$ whenever $(x, x^*) \in G(B), (y, y^*) \in G(B)$. We denote a set valued operator B form E to E^* by $B \subset E \times E^*$. A monotone B is said to be *maximal* if its graph is not property contained in the graph of any other monotone operator. If B is maximal monotone, then the solution set $B^{-1}0$ is closed and convex.

Let E be a reflexive, strictly convex and smooth Banach space, it is knows that B is a maximal monotone if and only if $R(J + rB) = E^*$ for all $r > 0$. Define the *resolvent* of B by $J_r x = x_r$. In other words, $J_r = (J + rB)^{-1}J$ for all $r > 0$. J_r is a single-valued mapping from E to $D(B)$. Also, $B^{-1}(0) = F(J_r)$ for all $r > 0$, where $F(J_r)$ is the set of all fixed points of J_r. Define, for $r > 0$, the *Yosida approximation* of B by $T_r x = (Jx - JJ_r x)/r$ for all $x \in C$. We know that $T_r x \in B(J_r x)$ for all $r > 0$ and $x \in E$.

Let A be an inverse-strongly monotone mapping of C into E^* which is said to be *hemicontinuous* if for all $x, y \in C$, the mapping F of $[0, 1]$ into E^*, defined by $F(t) = A(tx+(1-t)y)$, is continuous with respect to the weak* topology of E^*. We define by $N_C(v)$ *the normal cone* for C at a point $v \in C$, that is,

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^*\rangle \ge 0, \ \ \forall y \in C\}. \tag{15}$$

Lemma 2.8. (*Rockafellar* [32]). *Let C be a nonempty, closed convex subset of a Banach space E and A is a monotone, hemicontinuous operator of C into E^*. Let $B \subset E \times E^*$ be an operator defined as follows:*

$$Bv = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & otherwise. \end{cases} \tag{16}$$

Then B is maximal monotone and $B^{-1}0 = VI(A, C)$.

Lemma 2.9. (*Chang et al.*[9]) . *Let E be a uniformly convex Banach space, $r > 0$ be a positive number and $B_r(0)$ be a closed ball of E. Then, for any given sequence $\{x_i\}_{i=1}^\infty \subset B_r(0)$ and for any given sequence $\{\lambda_i\}_{i=1}^\infty$ of positive number with $\sum_{n=1}^\infty \lambda_n = 1$, there exists a continuous, strictly increasing, and convex function $g : [0, 2r) \to [0, \infty)$ with $g(0) = 0$ such that, for any positive integer i, j with $i < j$,*

$$\|\sum_{n=1}^\infty \lambda_n x_n\|^2 \le \sum_{n=1}^\infty \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|). \tag{17}$$

Lemma 2.10. (*Chang et al.*[9]) . *Let E be a real uniformly smooth and strictly convex Banach space, and C be a nonempty closed convex subset of E. Let $T : C \to C$ be a closed and quasi-ϕ-asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$. Then $F(T)$ is a closed convex subset of C.*

For solving the equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$, let us assume that f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y);$$

(A4) for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semi-continuous.

For example, let A be a continuous and monotone operator of C into E^* and define

$$f(x, y) = \langle Ax, y - x \rangle, \forall x, y \in C.$$

Then, f satisfies (A1)-(A4). The following result is in Blum and Oettli [5].

Lemma 2.11. (*Blum and Oettli* [5]). *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)$-$(A4)$, and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.12. (*Takahashi and Zembayashi* [39]). *Let C be a closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E and let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying conditions $(A1)$-$(A4)$. For all $r > 0$ and $x \in E$, define a mapping $T_r^f : E \to C$ as follows:*

$$T_r^f x = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C\}.$$

Then the following hold:

(1) *T_r^f is single-valued;*

(2) *T_r^f is a firmly nonexpansive-type mapping [19], that is, for all $x, y \in E$,*

$$\langle T_r^f x - T_r^f y, JT_r^f x - JT_r^f y \rangle \leq \langle T_r^f x - T_r^f y, Jx - Jy \rangle;$$

(3) *$F(T_r^f) = EP(f)$;*

(4) *$EP(f)$ is closed and convex.*

Lemma 2.13. (*Takahashi and Zembayashi* [39]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)$-$(A4)$ and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r^f)$,*

$$\phi(q, T_r^f x) + \phi(T_r^f x, x) \leq \phi(q, x).$$

3. Main Results

In this section, we prove the new convergence theorems for finding the set of solutions of system of generalized mixed equilibrium problems, the common fixed point set of a family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings, and the solution set of variational inequalities for an α-inverse strongly monotone mapping in a 2-uniformly convex and uniformly smooth Banach space.

Theorem 3.1. *Let C be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. For each $j = 1, 2, ..., m$ let f_j be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)-(A4). Let A be an α-inverse-strongly monotone mapping of C into E^* satisfying $\|Ay\| \leq \|Ay - Au\|, \quad \forall y \in C$ and $u \in VI(A, C) \neq \emptyset$. Let $\{S_i\}_{i=1}^{\infty} : C \to C$ be an infinite family of closed uniformly L_i-Lipschitz continuous and uniformly quasi-ϕ-asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ such that $F := (\cap_{i=1}^{\infty} F(S_i)) \cap (\cap_{j=1}^{m} EP(f_j))(\cap VI(A, C))$ is a nonempty and bounded subset in C. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} v_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = J^{-1}(\alpha_{n,0} Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i} JS_i^n v_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n) Jz_n), \\ u_n = T_{r_{m,n}}^{f_m} T_{r_{m-1,n}}^{f_{m-1}} ... T_{r_{2,n}}^{f_2} T_{r_{1,n}}^{f_1} y_n, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases} \tag{18}$$

where J is the duality mapping on E, $\theta_n = \sup_{q \in F}(k_n - 1)\phi(q, x_n)$, for each $i \geq 0$, $\{\alpha_{n,i}\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$, $\{r_{j,n}\} \subset [d, \infty)$ for some $d > 0$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E. If $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ for all $n \geq 0$, $\liminf_{n \longrightarrow \infty} \beta_n(1 - \beta_n) > 0$ and $\liminf_{n \longrightarrow \infty} \alpha_{n,0}\alpha_{n,i} > 0$ for all $i \geq 1$, then $\{x_n\}$ converges strongly to $p \in F$, where $p = \Pi_F x_0$.

***Proof*.** We first show that C_{n+1} is closed and convex for each $n \geq 0$. Clearly $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for each $n \in \mathbb{N}$. Since for any $z \in C_n$, we known that

$$\phi(z, u_n) \leq \phi(z, x_n) + \theta_n$$

is equivalent to

$$2\langle z, Jx_n - Ju_n \rangle \leq \|x_n\|^2 - \|u_n\|^2 + \theta_n.$$

Hence, C_{n+1} is closed and convex.

Next, we show that $F \subset C_n$ for all $n \geq 0$. Since by the convexity of $\| \cdot \|^2$, property of ϕ, Lemma 2.9 and by uniformly quasi-ϕ-asymptotically nonexpansive of S_n for each $q \in F \subset C_n$, we have

$$\begin{aligned}
\phi(q,u_n) &= \phi(q, T^{f_m}_{r_{m,n}} T^{f_{m-1}}_{r_{m-1,n}} \ldots T^{f_2}_{r_{2,n}} T^{f_1}_{r_{1,n}} y_n) \\
&\leq \phi(q,y_n) \\
&= \phi(q, J^{-1}(\beta_n Jx_n + (1-\beta_n)Jz_n) \\
&= \|q\|^2 - 2\langle q, \beta_n Jx_n + (1-\beta_n)Jz_n\rangle + \|\beta_n Jx_n + (1-\beta_n)Jz_n\|^2 \\
&\leq \|q\|^2 - 2\beta_n\langle q, Jx_n\rangle - 2(1-\beta_n)\langle q, Jz_n\rangle + \beta_n\|x_n\|^2 + (1-\beta_n)\|z_n\|^2 \\
&= \beta_n\phi(q,x_n) + (1-\beta_n)\phi(q,z_n), \qquad (19)
\end{aligned}$$

and

$$\begin{aligned}
\phi(q,z_n) &= \phi(q, J^{-1}(\alpha_{n,0}Jx_n + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n v_n)) \\
&= \|q\|^2 - 2\langle q, \alpha_{n,0}Jx_n + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n v_n\rangle + \|\alpha_{n,0}Jx_n + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n v_n\|^2 \\
&= \|q\|^2 - 2\alpha_{n,0}\langle q, Jx_n\rangle - 2\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}\langle q, JS_i^n v_n\rangle + \|\alpha_{n,0}Jx_n + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}JS_i^n v_n\|^2 \\
&\leq \|q\|^2 - 2\alpha_{n,0}\langle q, Jx_n\rangle - 2\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}\langle q, JS_i^n v_n\rangle + \alpha_{n,0}\|Jx_n\|^2 + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}\|JS_i^n v_n\|^2 \\
&\quad -\alpha_{n,0}\alpha_{n,j}g\|Jv_n - JS_j^n v_n\| \\
&= \|q\|^2 - 2\alpha_{n,0}\langle q, Jx_n\rangle + \alpha_{n,0}\|Jx_n\|^2 - 2\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}\langle q, JS_i^n v_n\rangle \\
&\quad + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}\|JS_i^n v_n\|^2 - \alpha_{n,0}\alpha_{n,j}g\|Jv_n - JS_j^n v_n\| \\
&= \alpha_{n,0}\phi(q,x_n) + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}\phi(q, S_i^n v_n) - \alpha_{n,0}\alpha_{n,j}g\|Jv_n - JS_j^n v_n\| \\
&\leq \alpha_{n,0}\phi(q,x_n) + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n\phi(q,v_n) - \alpha_{n,0}\alpha_{n,j}g\|Jv_n - JS_j^n v_n\|. \qquad (20)
\end{aligned}$$

It follows from Lemma 2.7, that

$$\begin{aligned}
\phi(q,v_n) &= \phi(q, \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)) \\
&\leq \phi(q, J^{-1}(Jx_n - \lambda_n Ax_n)) \\
&= V(q, Jx_n - \lambda_n Ax_n) \\
&\leq V(q, (Jx_n - \lambda_n Ax_n) + \lambda_n Ax_n) - 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - q, \lambda_n Ax_n\rangle \\
&= V(q, Jx_n) - 2\lambda_n\langle J^{-1}(Jx_n - \lambda_n Ax_n) - q, Ax_n\rangle \\
&= \phi(q,x_n) - 2\lambda_n\langle x_n - q, Ax_n\rangle + 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, -\lambda_n Ax_n\rangle. \qquad (21)
\end{aligned}$$

Since $q \in VI(A,C)$ and A is an α-inverse-strongly monotone mapping, we have

$$\begin{aligned}
-2\lambda_n\langle x_n - q, Ax_n\rangle &= -2\lambda_n\langle x_n - q, Ax_n - Aq\rangle - 2\lambda_n\langle x_n - q, Aq\rangle \\
&\leq -2\lambda_n\langle x_n - q, Ax_n - Aq\rangle \qquad (22) \\
&\leq -2\alpha\lambda_n\|Ax_n - Aq\|^2.
\end{aligned}$$

By Lemma 2.2 and $\|Ax_n\| \leq \|Ax_n - Aq\|, \ \forall q \in VI(A,C)$, we also have

$$\begin{aligned}
2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, -\lambda_n Ax_n\rangle &= 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}(Jx_n), -\lambda_n Ax_n\rangle \\
&\leq 2\|J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}(Jx_n)\|\|\lambda_n Ax_n\| \\
&\leq \tfrac{4}{c^2}\|JJ^{-1}(Jx_n - \lambda_n Ax_n) - JJ^{-1}(Jx_n)\|\|\lambda_n Ax_n\| \\
&= \tfrac{4}{c^2}\|Jx_n - \lambda_n Ax_n - Jx_n\|\|\lambda_n Ax_n\| \\
&= \tfrac{4}{c^2}\|\lambda_n Ax_n\|^2 \\
&= \tfrac{4}{c^2}\lambda_n^2\|Ax_n\|^2 \\
&\leq \tfrac{4}{c^2}\lambda_n^2\|Ax_n - Aq\|^2. \qquad (23)
\end{aligned}$$

Substituting (22) and (23) into (21), we have

$$\begin{aligned}\phi(q,v_n) &\leq \phi(q,x_n) - 2\alpha\lambda_n\|Ax_n - Aq\|^2 + \tfrac{4}{c^2}\lambda_n^2\|Ax_n - Aq\|^2 \\ &= \phi(q,x_n) + 2\lambda_n(\tfrac{2}{c^2}\lambda_n - \alpha)\|Ax_n - Aq\|^2 \\ &\leq \phi(q,x_n).\end{aligned} \tag{24}$$

Substituting (24) into (20), we also have

$$\begin{aligned}\phi(q,z_n) &\leq \alpha_{n,0}\phi(q,x_n) + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n\phi(q,x_n) - \alpha_{n,0}\alpha_{n,j}g\|Jv_n - JS_j^n v_n\| \\ &\leq \alpha_{n,0}k_n\phi(q,x_n) + \textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n\phi(q,x_n) - \alpha_{n,0}\alpha_{n,j}g\|Jv_n - JS_j^n v_n\| \\ &= k_n\phi(q,x_n) - \alpha_{n,0}\alpha_{n,j}g\|Jv_n - JS_j^n v_n\| \\ &\leq \phi(q,x_n) + \sup_{q\in F}(k_n - 1)\phi(q,x_n) - \alpha_{n,0}\alpha_{n,j}g\|Jv_n - JS_j^n v_n\| \\ &= \phi(q,x_n) + \theta_n - \alpha_{n,0}\alpha_{n,j}g\|Jv_n - JS_j^n v_n\| \\ &\leq \phi(q,x_n) + \theta_n.\end{aligned} \tag{25}$$

and substituting (25) into (19), we obtain

$$\phi(q,u_n) \leq \phi(q,x_n) + \theta_n. \tag{26}$$

Thus, this show that $q \in C_{n+1}$ implies that $F \subset C_{n+1}$ and hence, $F \subset C_n$ for all $n \geq 0.$ This implies that the sequence $\{x_n\}$ is well defined. From definition of C_{n+1} that $x_n = \Pi_{C_n}x_0$ and $x_{n+1} = \Pi_{C_{n+1}}x_0, \in C_{n+1} \subset C_n$ we have

$$\phi(x_n,x_0) \leq \phi(x_{n+1},x_0), \quad \forall n \geq 0. \tag{27}$$

Form Lemma 2.6, it follows that

$$\begin{aligned}\phi(x_n,x_0) &= \phi(\Pi_{C_n}x_0,x_0) \\ &\leq \phi(q,x_0) - \phi(q,x_n) \\ &\leq \phi(q,x_0), \quad \forall q \in F.\end{aligned} \tag{28}$$

By (27) and (28), then $\{\phi(x_n,x_0)\}$ are nondecreasing and bounded. So, we obtain that $\lim\limits_{n\to\infty}\phi(x_n,x_0)$ exists. In particular, by (5), the sequence $\{(\|x_n\| - \|x_0\|)^2\}$ is bounded. This implies $\{x_n\}$ is also bounded. We denote

$$M := \sup_{n\geq 0}\{\|x_n\|\} < \infty. \tag{29}$$

Moreover, by the definition of θ_n and (29), it follows that

$$\theta_n \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{30}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in C. Since $x_m = \Pi_{C_m}x_0 \in C_m \subset C_n$, for $m > n$, by Lemma 2.6, we have

$$\begin{aligned}\phi(x_m,x_n) &= \phi(x_m,\Pi_{C_n}x_0) \\ &\leq \phi(x_m,x_0) - \phi(\Pi_{C_n}x_0,x_0) \\ &= \phi(x_m,x_0) - \phi(x_n,x_0).\end{aligned}$$

Since $\lim_{n\longrightarrow\infty}\phi(x_n,x_0)$ exists and we taking $m,n\to\infty$ then, we get $\phi(x_m,x_n)\to 0$. From Lemma 2.4, we have $\lim_{n\to\infty}\|x_m-x_n\|=0$. Thus $\{x_n\}$ is a Cauchy sequence and by the completeness of E and there exist a point $p\in C$ such that $x_n\to p$ as $n\to\infty$.

Now, we claim that $\|Ju_n-Jx_n\|\to 0$, as $n\to\infty$. By definition of $x_n=\Pi_{C_n}x_0$, we have

$$\begin{aligned}\phi(x_{n+1},x_n) &= \phi(x_{n+1},\Pi_{C_n}x_0)\\ &\leq \phi(x_{n+1},x_0)-\phi(\Pi_{C_n}x_0,x_0)\\ &= \phi(x_{n+1},x_0)-\phi(x_n,x_0).\end{aligned}$$

Since $\lim\limits_{n\to\infty}\phi(x_n,x_0)$ exists, we also have

$$\lim_{n\to\infty}\phi(x_{n+1},x_n)=0. \tag{31}$$

Again form Lemma 2.4, that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{32}$$

From J is uniformly norm-to-norm continuous on bounded subsets of E, we obtain

$$\lim_{n\to\infty}\|Jx_{n+1}-Jx_n\|=0. \tag{33}$$

Since $x_{n+1}=\Pi_{C_{n+1}}x_0\in C_{n+1}\subset C_n$ and the definition of C_{n+1}, we have

$$\phi(x_{n+1},u_n)\leq\phi(x_{n+1},x_n)+\theta_n.$$

By (30) and (31), that

$$\lim_{n\to\infty}\phi(x_{n+1},u_n)=0. \tag{34}$$

Applying Lemma 2.4, we have

$$\lim_{n\to\infty}\|x_{n+1}-u_n\|=0. \tag{35}$$

Since

$$\begin{aligned}\|u_n-x_n\| &= \|u_n-x_{n+1}+x_{n+1}-x_n\|\\ &\leq \|u_n-x_{n+1}\|+\|x_{n+1}-x_n\|\end{aligned}$$

It follows from (32) and (35), that

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{36}$$

Since J is uniformly norm-to-norm continuous on bounded subsets of E, we also have

$$\lim_{n\to\infty}\|Ju_n-Jx_n\|=0. \tag{37}$$

Next, we will show that $x_n\to p\in F:=\cap_{j=1}^m EP(f_j)\cap(\cap_{i=1}^\infty F(S_i))\cap VI(A,C)$.

(i) We show that $x_n\to p\in\cap_{i=1}^\infty F(S_i)$. Since $x_{n+1}=\Pi_{C_{n+1}}x_0\in C_{n+1}\subset C_n$, it follow from (25), we have

$$\phi(x_{n+1},z_n)\leq\phi(x_{n+1},x_n)+\theta_n,$$

by (30) and (31), we get

$$\lim_{n\to\infty}\phi(x_{n+1},z_n)=0 \tag{38}$$

again form Lemma 2.4, that

$$\lim_{n\to\infty} \|x_{n+1} - z_n\| = 0. \tag{39}$$

Since

$$\|z_n - x_n\| \leq \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\|$$

from (32) and (39), we have

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{40}$$

By using the triangle inequality, we obtain

$$\|x_{n+1} - z_n\| \leq \|x_{n+1} - x_n\| + \|x_n - z_n\|. \tag{41}$$

By (32) and (40), we get

$$\lim_{n\to\infty} \|x_{n+1} - z_n\| = 0. \tag{42}$$

Since J is uniformly norm-to-norm continuous, we obtain

$$\lim_{n\to\infty} \|Jx_{n+1} - Jz_n\| = 0. \tag{43}$$

From (67), we note that

$$\begin{aligned}
\|Jx_{n+1} - Jz_n\| &= \|Jx_{n+1} - (\alpha_{n,0}Jx_n + \textstyle\sum_{i=1}^{\infty} \alpha_{n,i}JS_i^n v_n)\| \\
&= \|\alpha_{n,0}Jx_{n+1} - \alpha_{n,0}Jx_n + \textstyle\sum_{i=1}^{\infty} \alpha_{n,i}Jx_{n+1} - \sum_{i=1}^{\infty} \alpha_{n,i}JS_i^n v_n\| \\
&= \|\alpha_{n,0}(Jx_{n+1} - Jx_n) + \textstyle\sum_{i=1}^{\infty} \alpha_{n,i}(Jx_{n+1} - JS_i^n v_n)\| \\
&= \|\textstyle\sum_{i=1}^{\infty} \alpha_{n,i}(Jx_{n+1} - JS_i^n v_n) - \alpha_{n,0}(Jx_n - Jx_{n+1})\| \\
&\geq \textstyle\sum_{i=1}^{\infty} \alpha_{n,i}\|Jx_{n+1} - JS_i^n v_n\| - \alpha_{n,0}\|Jx_n - Jx_{n+1}\|,
\end{aligned}$$

and hence

$$\|Jx_{n+1} - JS_i^n v_n\| \leq \frac{1}{\sum_{i=1}^{\infty} \alpha_{n,i}}(\|Jx_{n+1} - Jz_n\| + \alpha_{n,0}\|Jx_n - Jx_{n+1}\|). \tag{44}$$

From (33), (43) and $\liminf_{n\to\infty} \sum_{i=1}^{\infty} \alpha_{n,i} > 0$, we get

$$\lim_{n\to\infty} \|Jx_{n+1} - JS_i^n v_n\| = 0. \tag{45}$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|x_{n+1} - S_i^n v_n\| = 0. \tag{46}$$

Using the triangle inequality, that

$$\begin{aligned}
\|x_n - S_i^n v_n\| &= \|x_n - x_{n+1} + x_{n+1} - S_i^n v_n\| \\
&\leq \|x_n - x_{n+1}\| + \|x_{n+1} - S_i^n v_n\|.
\end{aligned}$$

From (32) and (46), we have

$$\lim_{n\to\infty} \|x_n - S_i^n v_n\| = 0. \tag{47}$$

On the other hand, we observe that

$$\phi(q,x_n)-\phi(q,u_n)+\theta_n = \|x_n\|^2-\|u_n\|^2-2\langle q, Jx_n-Ju_n\rangle+\theta_n.$$

It follows from $\theta_n \longrightarrow 0,\ \|x_n-u_n\| \longrightarrow 0$ and $\|Jx_n-Ju_n\| \longrightarrow 0$, that

$$\phi(q,x_n)-\phi(q,u_n)+\theta_n \longrightarrow 0 \text{ as } n\longrightarrow\infty. \tag{48}$$

From (19), (20) and (24), we compute

$$\begin{aligned}
\phi(q,u_n) &\le \phi(q,y_n)\\
&\le \beta_n\phi(q,x_n)+(1-\beta_n)\phi(q,z_n)\\
&\le \beta_n\phi(q,x_n)+(1-\beta_n)[\alpha_{n,0}\phi(q,x_n)+\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n\phi(q,v_n)\\
&\quad -\alpha_{n,0}\alpha_{n,j}g\|Jv_n-JS_j^nv_n\|]\\
&= \beta_n\phi(q,x_n)+(1-\beta_n)\alpha_{n,0}\phi(q,x_n)+(1-\beta_n)\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n\phi(q,v_n)\\
&\quad -(1-\beta_n)\alpha_{n,0}\alpha_{n,j}g\|Jv_n-JS_j^nv_n\|\\
&\le \beta_n\phi(q,x_n)+(1-\beta_n)\alpha_{n,0}\phi(q,x_n)+(1-\beta_n)\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n\phi(q,v_n)\\
&\le \beta_n\phi(q,x_n)+(1-\beta_n)\alpha_{n,0}\phi(q,x_n)+(1-\beta_n)\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n[\phi(q,x_n)-\\
&\quad 2\lambda_n(\alpha-\tfrac{2}{c^2}\lambda_n)\|Ax_n-Aq\|^2]\\
&\le \beta_n\phi(q,x_n)+(1-\beta_n)\alpha_{n,0}k_n\phi(q,x_n)+(1-\beta_n)\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n\phi(q,x_n)\\
&\quad -(1-\beta_n)\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n2\lambda_n(\alpha-\tfrac{2}{c^2}\lambda_n)\|Ax_n-Aq\|^2\\
&= \beta_nk_n\phi(q,x_n)+(1-\beta_n)k_n\phi(q,x_n)-(1-\beta_n)\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n2\lambda_n(\alpha-\\
&\quad \tfrac{2}{c^2}\lambda_n)\|Ax_n-Aq\|^2\\
&\le k_n\phi(q,x_n)-(1-\beta_n)\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n2\lambda_n(\alpha-\tfrac{2}{c^2}\lambda_n)\|Ax_n-Aq\|^2]\\
&\le \phi(q,x_n)+\sup_{q\in F}(k_n-1)\phi(q,x_n)-\\
&\quad (1-\beta_n)\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n2\lambda_n(\alpha-\tfrac{2}{c^2}\lambda_n)\|Ax_n-Aq\|^2\\
&\le \phi(q,x_n)+\theta_n-(1-\beta_n)\textstyle\sum_{i=1}^{\infty}\alpha_{n,i}k_n2\lambda_n(\alpha-\tfrac{2}{c^2}\lambda_n)\|Ax_n-Aq\|^2
\end{aligned}$$

and hence

$$\begin{aligned}
2a(\alpha-\tfrac{2b}{c^2})\|Ax_n-Aq\|^2 &\le 2\lambda_n(\alpha-\tfrac{2}{c^2}\lambda_n)\|Ax_n-Aq\|^2\\
&\le \tfrac{1}{(1-\beta_n)\sum_{i=1}^{\infty}\alpha_{n,i}k_n}(\phi(q,x_n)-\phi(q,u_n)+\theta_n).
\end{aligned} \tag{49}$$

From (48), $\{\lambda_n\}\subset[a,b]$ for some a,b with $0<a<b<c^2\alpha/2$, $\liminf_{n\longrightarrow\infty}(1-\beta_n)>0$ and $\liminf_{n\longrightarrow\infty}\alpha_{n,0}\alpha_{n,i}>0$, for $i\ge 0$ and $k_n\longrightarrow 1$ as $n\longrightarrow\infty$, we obtain that

$$\lim_{n\to\infty}\|Ax_n-Aq\|=0. \tag{50}$$

From Lemma 2.6, Lemma 2.7 and (23), we compute

$$\begin{aligned}
\phi(x_n,v_n) &= \phi(x_n,\Pi_CJ^{-1}(Jx_n-\lambda_nAx_n))\\
&\le \phi(x_n,J^{-1}(Jx_n-\lambda_nAx_n))\\
&= V(x_n,Jx_n-\lambda_nAx_n)\\
&\le V(x_n,(Jx_n-\lambda_nAx_n)+\lambda_nAx_n)-2\langle J^{-1}(Jx_n-\lambda_nAx_n)-x_n,\lambda_nAx_n\rangle\\
&= \phi(x_n,x_n)+2\langle J^{-1}(Jx_n-\lambda_nAx_n)-x_n,-\lambda_nAx_n\rangle\\
&= 2\langle J^{-1}(Jx_n-\lambda_nAx_n)-x_n,-\lambda_nAx_n\rangle\\
&\le \tfrac{4\lambda_n^2}{c^2}\|Ax_n-Aq\|^2\\
&\le \tfrac{4b^2}{c^2}\|Ax_n-Aq\|^2.
\end{aligned}$$

Applying Lemma 2.4 and (50) that

$$\lim_{n\to\infty}\|x_n - v_n\| = 0 \tag{51}$$

and we also obtain

$$\lim_{n\to\infty}\|Jx_n - Jv_n\| = 0 \tag{52}$$

From S_i^n is continuous, for any $i \geq 1$

$$\lim_{n\to\infty}\|S_i^n x_n - S_i^n v_n\| = 0. \tag{53}$$

Again by the triangle inequality, we get

$$\|x_n - S_i^n x_n\| \leq \|x_n - S_i^n v_n\| + \|S_i^n v_n - S_i^n x_n\|.$$

From (47) and (53), we have

$$\lim_{n\to\infty}\|x_n - S_i^n x_n\| = 0, \quad \forall i \geq 1. \tag{54}$$

Since J is uniformly continuous on any bounded subset of E, we obtain

$$\lim_{n\to\infty}\|Jx_n - JS_i^n x_n\| = 0, \quad \forall i \geq 1. \tag{55}$$

Since $x_n \longrightarrow p$ and J is uniformly continuous, it yields $Jx_n \longrightarrow Jp$.
Hence, from (55), we get

$$JS_i^n x_n \longrightarrow Jp, \quad \forall i \geq 1. \tag{56}$$

Since $J^{-1} : E^* \longrightarrow E$ is norm-weake*-continuous, we have

$$S_i^n x_n \rightharpoonup p, \text{ for each } i \geq 1. \tag{57}$$

On the other hand, for each $i \geq 1$, we have

$$|\|S_i^n x_n\| - \|p\|| = |\|J(S_i^n x_n)\| - \|Jp\|| \leq ||J(S_i^n x_n) - Jp||.$$

By (56), we obtain $\|S_i^n x_n\| \longrightarrow \|p\|$ for each $i \geq 1$. Since E is uniformly convex Banach spaces then E has the Kadec-Klee property, we get

$$S_i^n x_n \longrightarrow p \quad \text{for each } i \geq 1.$$

By the assumption that $\forall i \geq 1$, S_i is uniformly L_i-Lipschitz continuous, hence we have.

$$\begin{aligned} \|S_i^{n+1}x_n - S_i^n x_n\| &\leq \|S_i^{n+1}x_n - S_i^{n+1}x_{n+1}\| + \|S_i^{n+1}x_{n+1} - x_{n+1}\| + \\ &\quad \|x_{n+1} - x_n\| + \|x_n - S_i^n x_n\| \\ &\leq (L_i + 1)\|x_{n+1} - x_n\| + \|S_i^{n+1}x_{n+1} - x_{n+1}\| + \|x_n - S_i^n x_n\|. \end{aligned} \tag{58}$$

By (32) and (54), it follows that $\|S_i^{n+1}x_n - S_i^n x_n\| \to 0$. From $S_i^n x_n \longrightarrow p$, we have $S_i^{n+1}x_n \to p$, that is $S_i S_i^n x_n \to p$. In view of closeness of S_i, we have $S_i p = p$, for all $i \geq 1$. This imply that $p \in \cap_{i=1}^{\infty} F(S_i)$.

(ii) We show that $x_n \to p \in \cap_{j=1}^m EP(f_j)$.
Applying (19) and (25), we get $\phi(p, y_n) \leq \phi(p, x_n) + \theta_n$. From Lemma 2.13 and $u_n = \Omega_n^m y_n$, when $\Omega_n^j = T_{r_{j,n}}^{Q_j} T_{r_{j-1,n}}^{Q_{j-1}} ... T_{r_{2,n}}^{Q_2} T_{r_{1,n}}^{Q_1}$, $j = 1, 2, 3, ..., m$, $\Omega_n^0 = I$, for $p \in F$, we observe that

$$\begin{aligned} \phi(p, u_n) &= \phi(p, \Omega_n^m y_n) \\ &\leq \phi(p, \Omega_n^{m-1} y_n) \\ &\vdots \\ &\leq \phi(p, \Omega_n^j y_n) \\ &\vdots \\ &\leq \phi(p, y_n) \\ &\leq \phi(p, x_n) + \theta_n \ \forall j = 1, 2, 3, ..., m. \end{aligned} \tag{59}$$

Follows from Lemma 2.13, that

$$\begin{aligned} \phi(u_n, \Omega_n^j y_n) &\leq \phi(p, \Omega_n^j y_n) - \phi(p, u_n) \\ &\leq \phi(p, x_n) - \phi(p, u_n) + \theta_n \\ &= \|p\|^2 - 2\langle p, Jx_n \rangle + \|x_n\|^2 - (\|p\|^2 - 2\langle p, Ju_n \rangle + \|u_n\|^2) + \theta_n \\ &= \|x_n\|^2 - \|u_n\|^2 - 2\langle p, Jx_n - Ju_n \rangle + \theta_n \\ &\leq \|x_n - u_n\|(\|x_n + u_n\|) + 2\|p\|\|Jx_n - Ju_n\| + \theta_n. \end{aligned} \tag{60}$$

From (36), (37), $\theta_n \to 0$ as $n \to \infty$ and Lemma 2.4, we get

$$\lim_{n\to\infty} \|u_n - \Omega_n^j y_n\| = 0 \, \forall j = 1, 2, 3, ..., m. \tag{61}$$

By using triangle inequality, we have

$$\|x_n - \Omega_n^j y_n\| \leq \|x_n - u_n\| + \|u_n - \Omega_n^j y_n\|.$$

From (36) and (61), we have

$$\lim_{n\to\infty} \|x_n - \Omega_n^j y_n\| = 0 \ \ \forall j = 1, 2, 3, ..., m. \tag{62}$$

Again by using triangle inequality, we have

$$\|\Omega_n^j y_n - \Omega_n^{j-1} y_n\| \leq \|\Omega_n^j y_n - x_n\| + \|x_n - \Omega_n^{j-1} y_n\|.$$

From (62),we also have

$$\lim_{n\to\infty} \|\Omega_n^j y_n - \Omega_n^{j-1} y_n\| = 0 \ \ \forall j = 1, 2, 3, ..., m. \tag{63}$$

Since J is uniformly norm-to-norm continuous, we obtain

$$\lim_{n\to\infty} \|J\Omega_n^j y_n - J\Omega_n^{j-1} y_n\| = 0 \ \ \forall j = 1, 2, 3, ..., m.$$

From $r_{j,n} > 0$ we have $\frac{\|J\Omega_n^j y_n - J\Omega_n^{j-1} y_n\|}{r_{j,n}} \to 0$ as $n \to \infty$ $\forall j = 1, 2, 3, ..., m$, and

$$f_j(\Omega_n^j y_n, y) + \frac{1}{r_{j,n}} \langle y - \Omega_n^j y_n, J\Omega_n^j y_n - J\Omega_n^{j-1} y_n \rangle \geq 0, \quad \forall y \in C.$$

By (A2), that

$$\begin{array}{rcl} \|y-\Omega_n^j y_n\| \frac{\|J\Omega_n^j y_n - J\Omega_n^{j-1} y_n\|}{r_n} & \geq & \frac{1}{r_{j,n}} \langle y-\Omega_n^j y_n, J\Omega_n^j y_n - J\Omega_n^{j-1} y_n \rangle \\ & \geq & -f_j(\Omega_n^j y_n, y) \\ & \geq & f_j(y, \Omega_n^j y_n), \quad \forall y \in C, \end{array}$$

and $\Omega_n^j y_n \to p$ we get $f(y,p) \leq 0$ for all $y \in C$. For $0 < t < 1$, define $y_t = ty + (1-t)p$. Then $y_t \in C$ which imply that $f_j(y_t, p) \leq 0$. From (A1), we obtain that

$$0 = f_j(y_t, y_t) \leq t f_j(y_t, y) + (1-t) f_j(y_t, p) \leq t f_j(y_t, y).$$

Thus $f_j(y_t, y) \geq 0$. From (A3), we have $f_j(p, y) \geq 0$ for all $y \in C$ and $j = 1, 2, 3, ..., m$. Hence $p \in EP(f_j)$ $\forall j = 1, 2, 3, ..., m$. This imply that $p \in \cap_{j=1}^m EP(f_j)$.

(iii) We show that $x_n \to p \in VI(A, C)$. Indeed, define $B \subset E \times E^*$ by

$$Bv = \begin{cases} Av + N_C(v), & v \in C; \\ \emptyset, & v \notin C. \end{cases} \tag{64}$$

By Lemma 2.8, B is maximal monotone and $B^{-1}0 = VI(A, C)$. Let $(v, w) \in G(B)$. Since $w \in Bv = Av + N_C(v)$, we get $w - Av \in N_C(v)$.
From $v_n \in C$, we have

$$\langle v - v_n, w - Av \rangle \geq 0. \tag{65}$$

On the other hand, since $v_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$. Then by Lemma 2.5, we have

$$\langle v - v_n, Jv_n - (Jx_n - \lambda_n Ax_n) \rangle \geq 0,$$

and thus

$$\langle v - v_n, \frac{Jx_n - Jv_n}{\lambda_n} - Ax_n \rangle \leq 0. \tag{66}$$

It follows from (65), (66) and A is monotone and $\frac{1}{\alpha}$-Lipschitz continuous, that

$$\begin{array}{rcl} \langle v - v_n, w \rangle & \geq & \langle v - v_n, Av \rangle \\ & \geq & \langle v - v_n, Av \rangle + \langle v - v_n, \frac{Jx_n - Jv_n}{\lambda_n} - Ax_n \rangle \\ & = & \langle v - v_n, Av - Ax_n \rangle + \langle v - zv_n, \frac{Jx_n - Jv_n}{\lambda_n} \rangle \\ & = & \langle v - v_n, Av - Av_n \rangle + \langle v - v_n, Av_n - Ax_n \rangle + \langle v - v_n, \frac{Jx_n - Jv_n}{\lambda_n} \rangle \\ & \geq & -\|v - v_n\| \frac{\|v_n - x_n\|}{\alpha} - \|v - v_n\| \frac{\|Jx_n - Jv_n\|}{a} \\ & \geq & -H\left(\frac{\|v_n - x_n\|}{\alpha} + \frac{\|Jx_n - Jv_n\|}{a}\right), \end{array}$$

where $H = \sup_{n \geq 1} \|v - v_n\|$. Take the limit as $n \to \infty$, (51) and (52), we obtain $\langle v - p, w \rangle \geq 0$. By the maximality of B we have $p \in B^{-1}0$, that is $p \in VI(A, C)$.

Finally, we show that $p = \Pi_F x_0$. From $x_n = \Pi_{C_n} x_0$, we have $\langle Jx_0 - Jx_n, x_n - z \rangle \geq 0$, $\forall z \in C_n$. Since $F \subset C_n$, we also have

$$\langle Jx_0 - Jx_n, x_n - y \rangle \geq 0, \quad \forall y \in F.$$

Taking limit $n \to \infty$, we obtain

$$\langle Jx_0 - Jp, p - y\rangle \geq 0, \quad \forall y \in F.$$

By Lemma 2.5, we can conclude that $p = \Pi_F x_0$ and $x_n \to p$ as $n \to \infty$. This completes the proof.

If $S_i = S$ for each $i \in \mathbb{N}$, then Theorem 3.1 is reduced to the following Corollary.

Corollary 3.2. *Let C be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. For each $j = 1, 2, ..., m$ let f_j be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)-(A4). Let A be an α-inverse-strongly monotone mapping of C into E^* satisfying $\|Ay\| \leq \|Ay - Au\|$, $\forall y \in C$ and $u \in VI(A, C) \neq \emptyset$. Let $S : C \to C$ be a closed L-Lipschitz continuous and quasi-ϕ-asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ such that $F := (F(S)) \cap (\cap_{j=1}^m EP(f_j)) \cap (VI(A, C))$ is a nonempty and bounded subset in C. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} v_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JS^n v_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n)Jz_n), \\ u_n = T^{f_m}_{r_{m,n}} T^{f_{m-1}}_{r_{m-1,n}} ... T^{f_2}_{r_{2,n}} T^{f_1}_{r_{1,n}} y_n, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1, \end{cases} \tag{67}$$

where J is the duality mapping on E, $\theta_n = \sup_{q\in F}(k_n - 1)\phi(q, x_n)$, $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0, 1]$, $\{r_{j,n}\} \subset [d, \infty)$ for some $d > 0$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E. If $\liminf_{n\longrightarrow\infty}(1 - \beta_n) > 0$ and $\liminf_{n\longrightarrow\infty} \alpha_n(1 - \alpha_n) > 0$, then $\{x_n\}$ converges strongly to $p \in F$, where $p = \Pi_F x_0$.

For a special case that $i = 1, 2$, we can obtain the following results on a pair of quasi-ϕ-asymptotically nonexpansive mappings immediately from Theorem 3.1.

Corollary 3.3. *Let C be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. For each $j = 1, 2, ..., m$ let f_j be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)-(A4). Let A be an α-inverse-strongly monotone mapping of C into E^* satisfying $\|Ay\| \leq \|Ay - Au\|$, $\forall y \in C$ and $u \in VI(A, C) \neq \emptyset$. Let $S, T : C \to C$ be two closed quasi-ϕ-asymptotically nonexpansive mappings and L_S, L_T-Lipschitz continuous, respectively with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ such that $F := F(S) \cap F(T) \cap (\cap_{j=1}^m EP(f_j)) \cap VI(A, C)$ is a nonempty and bounded subset in C. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} v_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = J^{-1}(\alpha_n Jx_n + \beta_n JS^n v_n + \gamma_n JT^n v_n), \\ y_n = J^{-1}(\delta_n Jx_n + (1 - \delta_n)Jz_n), \\ u_n = T^{f_m}_{r_{m,n}} T^{f_{m-1}}_{r_{m-1,n}} ... T^{f_2}_{r_{2,n}} T^{f_1}_{r_{1,n}} y_n, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases} \tag{68}$$

where J is the duality mapping on E, $\theta_n = \sup_{q\in F}(k_n - 1)\phi(q, x_n)$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $[0,1]$, $\{r_{j,n}\} \subset [d,\infty)$ for some $d > 0$ and $\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E. If $\alpha_n+\beta_n+\gamma_n = 1$ for all $n \geq 0$ and $\liminf_{n\longrightarrow\infty} \alpha_n\beta_n > 0$, $\liminf_{n\longrightarrow\infty} \alpha_n\gamma_n > 0$, $\liminf_{n\longrightarrow\infty} \beta_n\gamma_n > 0$ and $\liminf_{n\longrightarrow\infty} \delta_n(1-\delta_n) > 0$, then $\{x_n\}$ converges strongly to $p \in F$, where $p = \Pi_F x_0$.

Corollary 3.4. *Let C be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. For each $j = 1, 2, ..., m$ let f_j be a bifunction from $C\times C$ to $\mathbb{R}$ which satisfies conditions (A1)-(A4). Let A be an α-inverse-strongly monotone mapping of C into E^* satisfying $\|Ay\| \leq \|Ay - Au\|$, $\quad \forall y \in C$ and $u \in VI(A,C) \neq \emptyset$. Let $\{S_i\}_{i=1}^{\infty} : C \to C$ be an infinite family of closed quasi-ϕ- nonexpansive mappings such that $F := \cap_{i=1}^{\infty}F(S_i) \cap (\cap_{j=1}^{m}EP(f_j)) \cap VI(A,C) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1}x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} v_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = J^{-1}(\alpha_{n,0}Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i}JS_iv_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)Jz_n), \\ u_n = T_{r_{m,n}}^{f_m} T_{r_{m-1,n}}^{f_{m-1}} ... T_{r_{2,n}}^{f_2} T_{r_{1,n}}^{f_1} y_n, \\ C_{n+1} = \{z \in C_n : \phi(z,u_n) \leq \phi(z,x_n), \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \quad \forall n \geq 0, \end{cases} \tag{69}$$

where J is the duality mapping on E, $\{\alpha_{n,i}\}$ and $\{\beta_n\}$ is sequence in $[0,1]$, $\{r_{j,n}\} \subset [d,\infty)$ for some $d > 0$ and $\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E. If $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ for all $n \geq 0$, $\liminf_{n\longrightarrow\infty}(1-\beta_n) > 0$ and $\liminf_{n\longrightarrow\infty} \alpha_{n,0}\alpha_{n,i} > 0$ for all $i \geq 1$, then $\{x_n\}$ converges strongly to $p \in F$, where $p = \Pi_F x_0$.

Proof Since $\{S_i\}_{i=1}^{\infty} : C \longrightarrow C$ is an infinite family of closed quasi-ϕ-nonexpansive mappings, it is an infinite family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings with sequence $k_n = 1$. Hence the conditions appearing in Theorem 3.1 F is a bounded subset in C and for each $i \geq 1, S_i$ is uniformly L_i-Lipschitz continuous are of no use here. By virtue of the closeness of mapping S_i for each $i \geq 1$, it yields that $p \in F(S_i)$ for each $i \geq 1$, that is, $p \in \cap_{i=1}^{\infty}F(S_i)$. Therefore all conditions in Theorem 3.1 are satisfied. The conclusion of Corollary 3.4 is obtained from Theorem 3.1 immediately.

Corollary 3.5. [44, Theorem 3.2] *Let C be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying $(A1)$-$(A4)$. Let A be an α-inverse-strongly monotone mapping of C into E^* satisfying $\|Ay\| \leq \|Ay - Au\|$, $\quad \forall y \in C$ and $u \in VI(A,C) \neq \emptyset$. Let $\{S_i\}_{i=1}^{N} : C \to C$ be a finite family of closed quasi-ϕ- nonexpansive mappings such that $F := \cap_{i=1}^{N}F(S_i) \cap EP(f) \cap VI(A,C) \neq \emptyset$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1}x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} z_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ y_n = J^{-1}(\alpha_0 Jx_n + \sum_{i=1}^{N} \alpha_i JS_iz_n), \\ f(u_n,y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z,u_n) \leq \phi(z,x_n), \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \quad \forall n \geq 0, \end{cases} \tag{70}$$

where J is the duality mapping on E, $\{\alpha_{n,i}\}$ is sequence in $[0,1]$, $\{r_n\} \subset [d,\infty)$ for some $d > 0$ and $\{\lambda_n\} \subset [a,b]$ for some a,b with $0 < a < b < c^2\alpha/2$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E. If $\alpha_i \in (0,1)$ such that $\sum_{i=0}^{N} \alpha_i = 1$, then $\{x_n\}$ converges strongly to $p \in F$, where $p = \Pi_F x_0$.

Corollary 3.6. *Let C be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)$-$(A4)$. Let $\{S_i\}_{i=1}^{\infty} : C \to C$ be an infinite family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1,\infty)$, $k_n \to 1$ and uniformly L_i-Lipschitz continuous such that $F := \cap_{i=1}^{\infty} F(S_i) \cap EP(f)$ is a nonempty and bounded subset in C. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} y_n = J^{-1}(\alpha_{n,0} Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i} JS_i^n x_n), \\ f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases} \tag{71}$$

where J is the duality mapping on E, $\theta_n = \sup_{q\in F}(k_n - 1)\phi(q, x_n)$, $\{\alpha_{n,i}\}$ is sequence in $[0,1]$, $\{r_n\} \subset [a,\infty)$ for some $a > 0$. If $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ for all $n \geq 0$ and $\liminf_{n\longrightarrow\infty} \alpha_{n,0}\alpha_{n,i} > 0$ for all $i \geq 1$, then $\{x_n\}$ converges strongly to $p \in F$, where $p = \Pi_F x_0$.

4. Deduced to Hilbert Spaces

If $E = H$, a Hilbert space, then E is 2-uniformly convex (we can choose $c = 1$) and uniformly smooth real Banach space and closed relatively quasi-nonexpansive map reduces to closed quasi-nonexpansive map. Moreover, $J = I$, identity operator on H and $\Pi_C = P_C$, projection mapping from H into C. Thus, the following corollaries hold.

Theorem 4.1. *Let C be a nonempty closed and convex subset of a Hilbert space H. For each $j = 1, 2, ..., m$ let f_j be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)-(A4), $B_j : C \longrightarrow E^*$ be a continuous and monotone mapping and $\varphi_j : C \to \mathbb{R}$ be a lower semicontinuous and convex function. Let A be an α-inverse-strongly monotone mapping of C into H satisfying $\|Ay\| \leq \|Ay - Au\|$, $\forall y \in C$ and $u \in VI(A,C) \neq \emptyset$. Let $\{S_i\}_{i=1}^{\infty} : C \to C$ be an infinite family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1,\infty)$, $k_n \to 1$ and uniformly L_i-Lipschitz continuous such that $F := \cap_{i=1}^{\infty} F(S_i) \cap (\cap_{j=1}^{m} GMEP(f_j, B_j, \varphi_j)) \cap VI(A,C)$ is a nonempty and bounded subset in C. For an initial point $x_0 \in H$ with $x_1 = P_{C_1} x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} z_n = P_C(x_n - \lambda_n Ax_n), \\ y_n = \alpha_{n,0} x_n + \sum_{i=1}^{\infty} \alpha_{n,i} S_i^n z_n, \\ u_n = T_{r_{m,n}}^{f_m} T_{r_{m-1,n}}^{f_{m-1}} ... T_{r_{2,n}}^{f_2} T_{r_{1,n}}^{f_1} y_n, \\ C_{n+1} = \{z \in C_n : \|z - u_n\| \leq \|z - x_n\| + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases} \tag{72}$$

where $\theta_n = \sup_{q\in F}(k_n - 1)\|q - x_n\|$, $\{\alpha_{n,i}\}$ is sequence in $[0,1]$, $\{r_{j,n}\} \subset [a,\infty)$ for some $a > 0$ and $\{\lambda_n\} \subset [a,b]$ for some a,b with $0 < a < b < \alpha/2$. If $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ for all $n \geq 0$

and $\liminf_{n\longrightarrow\infty}\alpha_{n,0}\alpha_{n,i} > 0$ *for all* $i \geq 1$*, then* $\{x_n\}$ *converges strongly to* $p \in F$*, where* $p = \Pi_F x_0$.

Remark 4.2. Theorem 4.1 improve and extend the Corollary 3.7 in Zegeye [44] in the aspect for the mappings, we extend the mappings from a finite family of closed relatively quasi-nonexpansive mappings to more general an infinite family of closed and uniformly quasi-ϕ-asymptotically nonexpansive mappings.

5. Applications

5.1 Zero Points of an Inverse-strongly Monotone Operator

Next, we consider the problem of finding a zero point of an inverse-strongly monotone operator of E into E^*. Assume that A satisfies the conditions:

(C1) A is α-inverse-strongly monotone,

(C2) $A^{-1}0 = \{u \in E : Au = 0\} \neq \emptyset$.

Theorem 5.1. *Let C be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. For each $j = 1, 2, ..., m$ let f_j be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)-(A4), $B_j : C \longrightarrow E^*$ be a continuous and monotone mapping and $\varphi_j : C \to \mathbb{R}$ be a lower semicontinuous and convex function. Let A be an operator of E into E^* satisfying (C1) and (C2). Let $\{S_i\}_{i=1}^{\infty} : C \to C$ be an infinite family of closed uniformly L_i-Lipschitz continuous and uniformly quasi-ϕ-asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ such that*

$$F := \cap_{i=1}^{\infty} F(S_i) \cap (\cap_{j=1}^{m} GMEP(f_j, B_j, \varphi_j)) \cap A^{-1}0$$

is a nonempty and bounded subset in C. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, we define the sequence $\{x_n\}$ as follows:

$$\begin{cases} z_n = J^{-1}(\alpha_{n,0} J x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J S_i^n v_n), \\ y_n = J^{-1}(\beta_n J x_n + (1-\beta_n) J z_n), \\ u_n = T_{r_{m,n}}^{f_m} T_{r_{m-1,n}}^{f_{m-1}} ... T_{r_{2,n}}^{f_2} T_{r_{1,n}}^{f_1} y_n, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases} \tag{73}$$

where J is the duality mapping on E, $\theta_n = \sup_{q\in F}(k_n - 1)\phi(q, x_n)$, for each $i \geq 0$, $\{\alpha_{n,i}\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$, $\{r_{j,n}\} \subset [d, \infty)$ for some $d > 0$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E. If $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ for all $n \geq 0$, $\liminf_{n\longrightarrow\infty}(1 - \beta_n) > 0$ and $\liminf_{n\longrightarrow\infty} \alpha_{n,0}\alpha_{n,i} > 0$ for all $i \geq 1$, then $\{x_n\}$ converges strongly to $p \in F$, where $p = \Pi_F x_0$.

Proof. Setting $C = E$ in Corollary 3.4, we also get $\Pi_E = I$. We also have $VI(A, C) = VI(A, E) = \{x \in E : Ax = 0\} \neq \emptyset$ and then the condition $\|Ay\| \leq \|Ay - Au\|$ holds for all $y \in E$ and $u \in A^{-1}0$. So, we obtain the result.

5.2 Complementarity Problems

Let K be a nonempty, closed convex cone in E. We define the *polar* K^* of K as follows:

$$K^* = \{y^* \in E^* : \langle x, y^* \rangle \geq 0, \forall x \in K\}. \tag{74}$$

If $A : K \longrightarrow E^*$ is an operator, then an element $u \in K$ is called a solution of the *complementarity problem* ([37]) if

$$Au \in K^*, \;\; and \;\; \langle u, Au \rangle = 0. \tag{75}$$

The set of solutions of the complementarity problem is denoted by $CP(A, K)$.

Theorem 5.2. *Let K be a nonempty closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. For each $j = 1, 2, ..., m$ let f_j be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)-(A4), $B_j : C \longrightarrow E^*$ be a continuous and monotone mapping and $\varphi_j : C \to \mathbb{R}$ be a lower semicontinuous and convex function. Let A be an α-inverse-strongly monotone mapping of K into E^* satisfying $\|Ay\| \leq \|Ay - Au\|, \quad \forall y \in K$ and $u \in CP(A, K) \neq \emptyset$. Let $\{S_i\}_{i=1}^{\infty} : K \to K$ be an infinite family of closed uniformly L_i-Lipschitz continuous and uniformly quasi-ϕ-asymptotically nonexpansive mappings with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ such that $F := \cap_{i=1}^{\infty} F(S_i) \cap (\cap_{j=1}^{m} GMEP(f_j, B_j, \varphi_j)) \cap CP(A, K)$ is a nonempty and bounded subset in K. For an initial point $x_0 \in E$ with $x_1 = \Pi_{K_1} x_0$ and $K_1 = K$, we define the sequence $\{x_n\}$ as follows:*

$$\begin{cases} v_n = \Pi_K J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = J^{-1}(\alpha_{n,0} Jx_n + \sum_{i=1}^{\infty} \alpha_{n,i} JS_i^n v_n), \\ y_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n) Jz_n), \\ u_n = T_{r_{m,n}}^{f_m} T_{r_{m-1,n}}^{f_{m-1}} ... T_{r_{2,n}}^{f_2} T_{r_{1,n}}^{f_1} y_n, \\ K_{n+1} = \{z \in K_n : \phi(z, u_n) \leq \phi(z, x_n) + \theta_n\}, \\ x_{n+1} = \Pi_{K_{n+1}} x_0, \quad \forall n \geq 0, \end{cases} \tag{76}$$

where J is the duality mapping on E, $\theta_n = \sup_{q \in F}(k_n - 1)\phi(q, x_n)$, for each $i \geq 0$, $\{\alpha_{n,i}\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$, $\{r_{j,n}\} \subset [d, \infty)$ for some $d > 0$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of E. If $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ for all $n \geq 0$, $\liminf_{n \longrightarrow \infty}(1 - \beta_n) > 0$ and $\liminf_{n \longrightarrow \infty} \alpha_{n,0}\alpha_{n,i} > 0$ for all $i \geq 1$, then $\{x_n\}$ converges strongly to $p \in F$, where $p = \Pi_F x_0$.

Proof. As in the proof of Takahashi in [37, Lemma 7.11], we get that $VI(A, K) = CP(A, K)$. So, we obtain the result.

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