Dentability and Convexity*

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Abstract *In this paper, we introduced the notion of weak denting point of* $U(X)$ *(respectively, uniformly dentable) and described the characterization of reflexive very convex (respectively, uniformly convex) spaces by using the notion of weak denting point (respectively, uniformly dentable), and studied the properties of them.*

Keywords *Dentability Convexity Denting point Banach Space*

1. Introduction and Preliminaries

Throughout this paper, X will denote a real Banach space and X^* will denote its conjugate space . Set $S(X)=\{x: x\in X, \|x\|=1\}, U(X)=\{x: x\in X, \|x\|\le 1\}, S_x=\{f: f\in S(X^*), f(x)=1=\|x\|\}$. For a convex set $C\subset X$, $\operatorname{ext} C$ will denote the extreme point of C. For $f\in S(X^*)$ and $\delta>0$, set $F(f,\delta)$ will denote the slice $\{x: x\in U(X): f(x)\ge 1-\delta\}$. The weak topology of X is denoted by $\sigma(X,X^*)$ and the weak * topology of X^* is denoted by $\sigma(X^*,X)$. Let D be a subset of X. D is said to be dentable if for any $\varepsilon>0$ there is a $x_\varepsilon\in D$ such that $x_\varepsilon\notin\overline{co}(D\setminus B_\varepsilon(x_\varepsilon))$, where $B_\varepsilon(x_\varepsilon)=\{x: x\in X, \|x-x_\varepsilon\|<\varepsilon\}$. A point $x\in D$ is said to be denting point of D if for any $\varepsilon>0$, we have $x\notin\overline{co}(D\setminus B_\varepsilon(x))$.

M.A.Rieffel [1] first introduced the notion of dentability and proved that X has the Radon-Nikodym property whenever every bounded subset of X is dentable. H.B.Maynard[2] improved the result of M.A.Rieffel and proved that X has the Radon-Nikodym property if and only if X is dentable. It is known that there is a close connection between the extreme point and the denting point. For example, if M is a compact convex set in Banach space X, then x is extreme point of M if and only if x is denting point of M. In 1993, Congxin Wu and Yongjin Li[3] introduced the notion of strongly convex Banach spaces and proved that if X is reflexive Banach space, then X is dentable if and only if every point of $S(X)$ is denting point of $U(X)$. This is only a result about describing the straight relations between dentability and convexity.

* This research has been financially supported by the National Natural Science Foundation of China (No.11061022).

Definition 1 Let B be a bounded subset of X. A point $x_0 \in B$ is said to be weakly exposed point of B if there is a function $x^* \in S(X^*)$ such that $x^*(x_0) = \sup\{x^*(x): x \in B\}$ and for any sequence $\{x_n\} \subset B, x^*(x_n) \to x^*(x_0), (n \to \infty)$ imply that $x_n \xrightarrow{w} x_0, (n \to \infty)$.

Definition 2 A point $x_0 \in U(X)$ is said to be very extreme point of $U(X)$ if for $\forall \varepsilon > 0$ there exists a weak neighborhood V of 0 in weak topology $\sigma(X, X^*)$ such that for any $f \in S(X^*)$ does not exist $a, b \in U(X)+V$ satisfying that $f(a-b)=\varepsilon, \{ta+(1-t)b\} \subset U(X)+V$, $x_0 = \frac{1}{2}(a+b)$, where $\forall t \in [0,1]$.

Definition 3 Let D be a subset of X. D is said to be weak dentable if for any weak neig -hborhood V of 0 in weak topology $\sigma(X, X^*)$, there is a $x_V \in D$ such that $x_V \notin \overline{co}^w(D \backslash (x_V + V))$. A point $x \in D$ is said to be weak denting point of D if for any weak neighborhood V of 0 in weak topology $\sigma(X, X^*)$, we have $x \notin \overline{co}^w(D \backslash (x+V))$.

Definition 4 Let D^* be a subset of X^*. A point $x^* \in D^*$ is said to be weak* denting point of D^* if for any weak* neighborhood V of 0 in weak* topology $\sigma(X^*, X)$, we have $x^* \notin \overline{co}^{w^*}(D \backslash (x_* + V))$.

Definition5 $U(X)$ is said to be uniformly dentable. If for any $\varepsilon > 0, x \in S(X)$ there exists $\delta > 0$ such that $\inf\{x^*(x) - x^*(\overline{co}(U(X) \backslash B_\varepsilon(x))), \forall x^* \in S_x\} > \eta$.

Definition 6[4] A space X is said to be uniformly convex if and only if for any $\varepsilon > 0$ there is a $\delta > 0$ such that for $x, y \in S(X)$, if $\frac{1}{2}\|x+y\| > \delta$, then $\|x-y\| < \varepsilon$.

Definition 7[5] A space X is said to be very convex if and only if for any $x \in S(X)$, $\{x_n\} \subset S(X)$ and for some $x^* \in S_x$ there holds $x^*(x_n) \to 1, (n \to \infty)$, then $x_n \xrightarrow{w} x_0, (n \to \infty)$.

Definition 8[3] A space X is said to be strongly convex if and only if for any $x \in S(X)$,
$\{x_n\} \subset S(X)$ and for some $x^* \in S_x$ there holds $x^*(x_n) \to 1, (n \to \infty)$, then $x_n \to x_0, (n \to \infty)$.

Lemma 1[6] X is uniformly convex if and only if for any $\varepsilon > 0$, $f \in S(X^*)$, there is a $\delta > 0$ and some compact set C with $\dim C \le 1$ such that $F(f, \delta) \subset \{y \in X : d(y, C) < \varepsilon\}$.

Lemma 2 Let X^* be a strictly convex space. If $\{x_n^*\}_{n=1}^{\infty} \subset U(X^*), x_0 \in S(X), x^* \in S_{x_0}$, $x_n^*(x_0) \to x^*(x_0), (n \to \infty)$, then there exists a net $\{x_\alpha^*\} \subset \{x_n^*\}_{n=1}^{\infty}$ such that $x_\alpha^* \xrightarrow{w^*} x^*$.

Proof If $\{x_n^*\}_{n=1}^{\infty} \subset U(X^*), x_0 \in S(X), x^* \in S_{x_0}, x_n^*(x_0) \to x^*(x_0), (n \to \infty)$, we may assume that $x_n^* \neq x_m^*$ for all $n \neq m$. $U(X^*)$ is weak* compact set, so there exists $x_0^* \in U(X^*)$ such that x_0^* is accumulation point of $\{x_n^*\}_{n=1}^{\infty}$ about weak* topology $\sigma(X^*, X)$. We construct a family of sets $\Delta = \{U_{x_0^*} : U_{x_0^*}$ is weak* neighborhood of x_0^* in weak* topology $\sigma(X^*, X)\}$ and

Define a order by inclusive relation, i.e., $U_{x_0^*} \subset U'_{x_0^*}$ if and only if $U_{x_0^*} \succ U'_{x_0^*}$, thus we obtain a ordered set Δ. We also construct another family of sets $\Delta = \{U_{x_0^*} \cap \{x_n^*\}_{n=1}^{\infty} : U_{x_0^*}$ is weak* neighborhood of x_0^* in weak* topology $\sigma(X^*, X)\}$, then by Zermelo axiom, there is a mapping f such that $f(U_{x_0^*} \cap \{x_n^*\}_{n=1}^{\infty}) \subset U_{x_0^*} \cap \{x_n^*\}_{n=1}^{\infty}$. Put $x_\alpha^* = f(U_{x_0^*} \cap \{x_n^*\}_{n=1}^{\infty})$, then $\{x_\alpha^*\}_{\alpha \in \Delta} \subset \{x_n^*\}_{n=1}^{\infty}$ is a net in X and $x_\alpha^* \xrightarrow{w^*} x^*$.

Denoting $r = |x_0^*(x_0)|$, then $r = 1$. Otherwise, $0 < r < 1$, we consider a neighborhood $\{z^* : |z^*(x_0) - x_0^*(x_0)| < \frac{1}{2}(1-r)\}$ of point x_0^*. Because $|x_n^*(x_0)| \to 1$, there is an integer N such that for all $n > N$ there holds inequality $|x_n^*(x_0)| > \frac{r+1}{2}$, hence $\left\{x_n^* : |x_n^*(x_0^*)| < \frac{r+1}{2}\right\}$ is finite set, furthermore, there exists a weak* neighborhood V in weak* topology $\sigma(X^*, X)$ such that $\left\{x_n^* : |x_n^*(x_0^*)| < \frac{r+1}{2}\right\} \cap V = \emptyset$ because of $\sigma(X^*, X)$ is Hausdorff topology. By $x_\alpha^* \xrightarrow{w^*} x_0^*$, we know that there is a α_0 such that for $\alpha \succ \alpha_0$ there holds $x_\alpha^* \in \{z^* : |z^*(x_0) - x_0^*(x_0)| < \frac{1}{2}(1-r)\} \cap V$. One hand, by $x_\alpha^* \in \{z^* : |z^*(x_0) - x_0^*(x_0)| < \frac{1}{2}(1-r)\}$, we have $|x_\alpha^*(x_0)| < \frac{r+1}{2}$. On the other hand, by $\{x_\alpha^*\} \subset \{x_n^*\}$, $x_\alpha^* \in \{z^* : |z^*(x_0) - x_0^*(x_0)| < \frac{1}{2}(1-r)\} \cap V$ and $\left\{x_n^* : |x_n^*(x_0^*)| < \frac{r+1}{2}\right\} \cap V = \emptyset$, we have $|x_\alpha^*(x_0)| > \frac{r+1}{2}$, which leads to a contradiction, this shows that $x_0^*(x_0) = \pm 1$. Noticing that $x^*(x_0) = 1$

and the hypothesis that X^* is strictly convex, we have $x_0^* = \pm x^*$, but $x_0^* = -x^*$ is impossible because of $x_\alpha^* \xrightarrow{w^*} x_0^*$, $x_n^*(x_0) \to x^*(x_0)$, $(n \to \infty)$ and $\{x_\alpha^*\}_{\alpha \in \Delta} \subset \{x_n^*\}_{n=1}^{\infty}$, which leads to $x_\alpha^* \xrightarrow{w^*} x_0^*$.

2. Main Results and Proof

Theorem 1 Let X be a reflexive Banach space. Then X is very convex if and only if every point of $S(X)$ is weak denting point of $U(X)$.

Proof We divide the proof into three parts. Firstly, we will prove that if every point of $S(X)$ is weak denting point of $U(X)$, then X is strictly convex.

Suppose that X is not strictly convex, then there exist three different point x_0, x_1, x_2 in $S(X)$ such that $x_0 = \frac{1}{2}x_1 + \frac{1}{2}x_2$. We choose a scalar $l > 0$ and define an linear functional f_0 on a subspace $X_0 = \{\alpha x_1 + \beta x_2 : \alpha, \beta \in R\}$ of X such that $f_0(\alpha x_1 + \beta x_2) = \alpha(l + \varepsilon) + \beta(l - \varepsilon)$

for some $\varepsilon>0$. Since X_0 is finite dimensional subspace of X, then there exists a real number $M>0$ such that $\| f_0 \| \le M$. By Hahn-Banach theorem, there exists $f \in S(X^*)$ such that $\| f \| \le M$ and $f(y) = f_0(y)$ whenever $y \in X_0$. Therefore we have

$$f(x_1) = f_0(x_1) = f_0(x_1 + 0x_2) = (l+\varepsilon) + 0(l-\varepsilon) = l+\varepsilon,$$

$$f(x_2) = f_0(x_2) = f_0(0x_1 + x_2) = 0(l+\varepsilon) + (l-\varepsilon) = l-\varepsilon,$$

$$f(x_0) = f\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right) = f_0\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right) = \frac{1}{2}(l+\varepsilon) + \frac{1}{2}(l-\varepsilon) = l.$$

We consider a weak neighborhood $x_0 + V = \left\{x : |f(x) - f(x_0)| < \frac{\varepsilon}{2}\right\}$ of x_0 in weak topology $\sigma(X, X^*)$, where V denotes the weak neighborhood of 0 in weak topology $\sigma(X, X^*)$, then $x_1, x_2 \notin x_0 + V$, hence $x_1, x_2 \in U(X) \setminus (x_0 + V)$, it follows that $x_0 \in \overline{co}^w(U(X) \setminus (x_0 + V))$. This contradicts that x_0 is weak denting point of $U(X)$.

Secondly, we will prove the necessity of theorem 1.

By Hahn-Banach theorem, we know that for $\forall x \in S(X)$ there exists a $f \in S(X^*)$ such that $f(x)=1$. If $x_n \in U(X)$, $f(x_n) \to 1$, $(n \to \infty)$, then $\| x_n \| \to 1$, $(n \to \infty)$. Let $y_n = \frac{x_n}{\| x_n \|}$, then $f(y_n) \to 1$. Because X is very convex (which implies strictly convex), we $f(x) > f(U(X) \setminus \{x\})$ and $y_n \xrightarrow{w} x$, $(n \to \infty)$. On the other hand, $|f(x_n - x)| \le \|f\| \cdot \|x_n - y_n\| + \|f\| \cdot \|y_n - x\|$, hence $|f(x_n) - f(x)| \to 0$, $(n \to \infty)$. This shows that x is weakly exposed point of $U(X)$ and f is corresponding weakly exposing function. If $y \in U(X) \setminus (x+V)$ (where V is the weak neighborhood of 0 in weak topology $\sigma(X, X^*)$, then there exists scalar $m > 0$ such that $f(x) > f(y) + m$, hence $f(x) - m \ge \sup\{f(y) : y \in U(X) \setminus (x+V)\} = \sup\{f(y) : y \in \overline{co}^w(U(X) \setminus (x+V))\}$, This shows that $x \in \overline{co}^w(U(X) \setminus (x+V))$, hence x is weak denting point of $U(X)$.

Thirdly, we will prove the sufficiency of theorem 1.

Suppose that $\forall x \in S(X), \{x_n\}_{n=1}^{\infty} \subset S(X)$ and $f(x_n) \to 1, (n \to \infty)$ for some $f \in S_x$. By the reflexivity of X, we know that $U(X)$ is weak sequential compact, hence there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ such that $x_{n_k} \xrightarrow{w} x'$, $(k \to \infty)$. Because $U(X)$ is closed convex set, we have $U(X) = \overline{U(X)} = \overline{U(X)}^w$, hence $\|x'\| \le 1$. On the other hand, $\|x'\| \ge f(x) = 1$. This shows that $\|x'\| = 1$. By $f(x) = f(x') = 1$ and the fact that X is strictly convex, we have $x = x'$. Furthermore, we can deduce that $x_n \xrightarrow{w} x'$, $(n \to \infty)$, this shows that X is very convex. Assume the contrary,i.e., there exist weak neighborhood $x+V$ of x in weak topology $\sigma(X, X^*)$, and a subsequence $\{x_{n_i}\}_{i=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ such that $x_{n_i} \notin x+V$. By the assumption that every point of $S(X)$ is weak denting point of $U(X)$, we have $x \notin \overline{co}^w(U(X) \setminus (x+V))$. Hence there is a function $g \in X^* = \sigma(X, X^*)^*$ which separates x and $\overline{co}^w(U(X) \setminus (x+V))$, i.e., there is a scalar $r>0$ such that $g(x)-r>\sup g(\overline{co}^{w}(U(X)\setminus(x+V)))$. Evidently, $x_{n_i}\in\overline{co}^{w}(U(X)\setminus(x+V))$, thus $g(x)-g(x_{n_i})>r$. On the other hand, By the reflexivity of X, we know that there exist a subsequence $\{x_{n_{i_l}}\}_{l=1}^{\infty}\subset\{x_{n_i}\}_{i=1}^{\infty}$ such that $\{x_{n_{i_l}}\}_{l=1}^{\infty}$ converges weakly to x . Which contradicts that $g(x)-g(x_{n_i})>r$..

Theorem 2 X is uniformly convex if and only if $U(X)$ is uniformly dentable and reflexive.

Proof We divide the proof into three parts. Firstly, we will prove that X is uniformly convex if and only if for any $\varepsilon>0, x^*\in S(X^*)$, there exists $\delta>0, x_0\in S(X)$ such that $x^*(x_0)=1$, then $F(x^*,\delta)\subset\{x:\ x\in X,\ \|x-x_0\|<\varepsilon\}$.

Suppose that X is uniformly convex. By lemma 1 we know that for any $\varepsilon>0, x^*\in S(X^*)$, there exist a scalar $\delta>0$ and a compact set C with $\dim C\le 1$ such that $F(x^*,\delta)\subset\{x:\ x\in d(x,C)<\varepsilon\}$. Let $x\in F(x^*,\delta)$, then $x^*(x)\ge 1-\delta$. We select a $\delta>0, x_0\in S(X)$ such that $x^*(x_0)=1$ because of uniform convexity implies reflexivity, then $x^*(x)\ge x^*(x_0)-\delta$, it follows that $\left\|\dfrac{x+x_0}{2}\right\|>x^*(x_0)-\dfrac{\delta}{2}=1-\dfrac{\delta}{2}$. By the assumption that X is uniformly convex, we have $\|x-x_0\|<\varepsilon$. This shows that $F(x^*,\delta)\subset\{x:\ x\in X,\ \|x-x_0\|<\varepsilon\}$.

Conversely, suppose that for any $\varepsilon>0, x^*\in S(X^*)$, there exists $\delta>0, x_0\in S(X)$ such that $x^*(x_0)=1$, then $F(x^*,\delta)\subset\{x:\ x\in X,\ \|x-x_0\|<\varepsilon\}$. By lemma 1 we know that X is uniformly convex.

Secondly, we will prove the necessity of theorem 2.

Suppose that X is uniformly convex. Evidently, X is reflexive. Let $x\in S(X)$, $x^*\in S_x$.

By we have proved above, there exist $\varepsilon>0$ such that $F(x^*,\delta)\subset\{y:\ y\in X,\ \|y-x\|<\varepsilon\}$, i.e., if $x^*(x)-x^*(y)<\delta$, then $\|y-x\|<\varepsilon$. Hence, for $y\in U(X)\setminus B_\varepsilon(x)$, we have $\|y-x\|\ge\varepsilon$. (where $B_\varepsilon(x)=\{y:\ y\in X,\ \|y-x\|<\varepsilon\}$). We take $\eta=\dfrac{\delta}{2}$, then $x^*(x)-x^*(y)>\delta>\eta$. Therefore,

$\inf\{x^*(x)-x^*(\overline{co}(U(X)\setminus B_\varepsilon(x)))\}\ge\delta>\eta$. This shows that $U(X)$ is uniformly dentable.

Thirdly, we will prove the sufficiency of theorem 2.

By the reflexivity of X, we know that for $x^*\in S(X^*)$ there exists $x_0\in S(X)$ such that $x^*(x_0)=1$. Hence, there exists $\eta>0$ such that $\{x^*(x_0)-x^*(\overline{co}(U(X)\setminus B_\varepsilon(x_0)))\}>\eta$ because of $U(X)$ is uniformly dentable. We take $\eta=\delta$ and let $x\in F(x^*,\delta)$, then $x^*(x)\ge 1-\delta$, i.e., $x^*(x_0)-x^*(x)<\delta=\eta$. By the inequality $\{x^*(x_0)-x^*(\overline{co}(U(X)\setminus B_\varepsilon(x_0)))\}>\eta$, we have $x\in B_\varepsilon(x_0)=\{x\colon x\in X,\|x-x_0\|<\varepsilon\}$. This shows that $F(x^*,\delta)\subset\{y : y\in X,\|y-x_0\|<\varepsilon\}$. By we have proved above, we know that X is uniformly convex.

Theorem 3 If X^* is strictly convex space, then weak* denting points of $U(X^*)$ are dense in $S(X^*)$.

Proof Firstly, we will prove that if for $x^*\in S(X^*)$ there exists $x_0\in S(X)$ such that $x^*(x_0)=1$, then x^* is weak* denting points of $U(X^*)$. For any weak* neighborhood V of 0 in weak* topology $\sigma(X^*,X)$, there exists a scalar $r>0$ such that for any $y^*\in U(X^*)\setminus(x^*+V)$ there holds inequality $x^*(x_0)>y^*(x_0)+r$. Assume the contrary, i.e., $x^*(x_0)=\sup x_0(U(X^*)\setminus(x^*+V))$, it is obvious that there exists $y_n^*\in U(X^*)\setminus(x^*+V)$ such that $y_n^*(x_0)\to x^*(x_0)=1,(n\to\infty)$. By Lemma 2, we know that there is a $\{x_\alpha^*\}_{\alpha\in\Delta}\subset\{y_n^*\}_{n=1}^{\infty}$ such that $x_\alpha^*\xrightarrow{w^*}x^*$, which contradicts to that $y_n^*\in U(X^*)\setminus(x^*+V)$. Thus

$$
\begin{aligned}
x^*(x_0)-r &\geq \sup\left\{y^*(x_0): y^*\in U(X^*)\setminus(x^*+V)\right\}\\
&= \sup\left\{y^*(x_0): y^*\in co(U(X^*)\setminus(x^*+V))\right\}\\
&\geq \sup\left\{y^*(x_0): y^*\in \overline{co}^{w^*}(U(X^*)\setminus(x^*+V))\right\},
\end{aligned}
$$

hence $x_0^*\notin\overline{co}^{w^*}(U(X^*)\setminus(x^*+V))$. This shows that x^* is weak* denting points of $U(X^*)$.

Secondly, we will prove that for any $x^*\in S(X^*)$ and $\forall\varepsilon>0$ there is a $y^*\in S(X^*)$ which attains its norm on $S(X)$ such that $\|y^*-x^*\|<\varepsilon$.

Let $x^*\in S(X^*)$. For $\forall\varepsilon>0$, take a $\eta\in(0,1)$ such that $0<\dfrac{25(1-\eta)}{4(5\eta-1)}<\varepsilon$. We consider the norm neighborhood $\left\{z^*:\|z^*-\eta x^*\|<\dfrac{1-\eta}{4}\right\}$ of ηx^* in norm topology $(X^*,\|\cdot\|)$, then $\dfrac{5\eta-1}{4}<\|z^*\|<\dfrac{3\eta+1}{4}$. Thus

$$
\begin{aligned}
\left\|\frac{z^*}{\|z^*\|}-x^*\right\| &\leq \left\|\frac{z^*}{\|z^*\|}-\frac{\eta x^*}{\|z^*\|}\right\|+\left\|\frac{\eta x^*}{\|z^*\|}-\frac{x^*}{\|z^*\|}\right\|+\left\|\frac{x^*}{\|z^*\|}-x^*\right\|\\
&\leq \frac{1-\eta}{4}\cdot\frac{4}{5\eta-1}+(\eta-1)\frac{4}{5\eta-1}+\left(\frac{4}{5\eta-1}-1\right)=\frac{25(1-\eta)}{4(5\eta-1)}<\varepsilon.
\end{aligned}
$$

By Bishp-Phelp theorem, we know that there exist $z_0^*\in X^*, z_0\in S(X)$ with $z_0^*(z_0)=\|z_0^*\|$ such that $z_0^*\in\left\{z^*:\|z^*-\eta x^*\|<\dfrac{1-\eta}{4}\right\}$. Taking $y^*=\dfrac{z_0^*}{\|z_0^*\|}$, then $y^*(z_0)=1$ and $\|y^*-x^*\|<\varepsilon$, this shows that weak* denting points of $U(X^*)$ are dense in $S(X^*)$.

Theorem4 Let D be a bounded closed convex set of X.Then D is weak dentable if and only if for any weak neighborhood V of 0 in weak topology $\sigma(X, X^*)$, there is a slice $S(x^*,\alpha,D)=\left\{y: y\in D, x^*(y)>\sup x^*(D)-\alpha\right\}$ and $x_0 \in S(x^*,\alpha,D)$ such that $S(x^*,\alpha,D)\subset x_0+V$. Where $\alpha>0$.

Proof Sufficiency. For any $f_1, f_2, \cdots, f_n \in X^*$ and $\varepsilon>0$, let $V=\bigcap_{i=1}^{n}\left\{x: |f_i(x)|<\varepsilon\right\}$. then, by the conditions given here, we know that there exist a slice $x_0 \in S(x^*,\alpha,D)$ and $x_0 \in S(x^*,\alpha,D)$ such that $S(x^*,\alpha,D)\subset x_0+V$, it follows that $x^*(x_0)>\sup x^*(D)-\alpha$.

Noticing that $D\setminus(V+x_0)\subset D\setminus S(x^*,\alpha,D).=\left\{y: y\in D,\ x^*(y)\le \sup x^*(D)-\alpha\right\}$ and $\left\{y: y\in D,\ x^*(y)\le \sup x^*(D)-\alpha\right\}$ is weak closed convex set of weak topology $\sigma(X,X^*)$,we know that $\overline{co}^w(D\setminus(x_0+V))\subset\left\{y: y\in D,\ x^*(y)\le \sup x^*(D)-\alpha\right\}$, it follows that $x_0\notin \overline{co}^w(D\setminus(x_0+V))$, this shows that D is weak dentable.

Necessity. Suppose that D is weak dentable. For any $f_1, f_2, \cdots, f_n \in X^*$ and $\varepsilon>0$, let $V=\bigcap_{i=1}^{n}\left\{x: |f_i(x)|<\frac{\varepsilon}{2}\right\}$. Then, there a is $x_0\in X$ such that $x_0\notin \overline{co}^w(D\setminus(x_0+V))$. Hence there is a function $x^*\in X^*$ which separates x_0 and $\overline{co}^w(D\setminus(x_0+V))$, i.e., there is a scalar $r>0$ such that $x^*(x_0)-r>\sup x^*(\overline{co}^w(D\setminus(x_0+V)))$.

Let $\alpha=\sup x^*(D)-x^*(x_0)+r$, then $x^*(x_0)=\sup x^*(D)+r-\alpha>\sup x^*(D)-\alpha$, this shows that $x_0\in S(x^*,\alpha,D)$. Furthermore, for any $y\in S(x^*,\alpha,D)$, we can deduce $x^*(y)>\sup x^*(D)-\alpha=\sup x^*(D)-\sup x^*(D)+x^*(x_0)-r=x^*(x_0)-r$, which leads to $y\in x_0+V$. Otherwise, $y\notin x_0+V$, furthermore $y\in D\setminus(x_0+V)\subset\overline{co}^w(D\setminus(x_0+V))$. By the inequality $x^*(x_0)-r>\sup x^*(\overline{co}^w(D\setminus(x_0+V)))$, we have $x^*(x_0)-r>x^*(y)$. A contradiction.

Theorem5 Let X be a reflexive Banach space. If every point of $S(X)$ is weak denting point of $U(X)$, then every point of $S(X)$ is weakly exposed point of $U(X)$.

Proof Suppose that $x_0\in S(X)$ is weak denting point of $U(X)$, by Hahn-Banach Theorem we know that there is a function $x_0^*\in S(X^*)$ such that $x_0^*(x_0)=1$. We will prove that if x_0^* attains its norm at another point $y_0\in S(X)$, then $x_0=y_0$. Otherwise, $x_0\ne y_0$, it is obvious that $1=\frac{1}{2}x_0^*\left(y_0+x_0\right)\le\left\|\frac{1}{2}\left(y_0+x_0\right)\right\|\le 1$, thus $\left\|\frac{1}{2}\left(y_0+x_0\right)\right\|=1$. By Hahn-Banach Theorem we know that there is a function $y_0^*\in S(X^*)$ such that $y_0^*\left(y_0\right)\ne y_0^*\left(x_0\right)$. We consider a weak neighborhood $V=\left\{x: \left|y_0^*(x-\frac{1}{2}(y_0+x_0))\right|<\frac{|y_0^*(y_0)-y_0^*(x_0)|}{4}\right\}$ of point $\frac{1}{2}(y_0+x_0)$, it is clear that $x_0, y_0 \notin V$, thus $x_0, y_0 \in U(X)\backslash V$, furthermore $\frac{1}{2}(y_0+x_0)\in\overline{co}^w(U(X)\backslash V)$, this shows that $\frac{1}{2}(y_0+x_0)$ is not weak denting point of $U(X)$, which contradicts to the hypothesis that every point of $S(X)$ is weak denting point of $U(X)$.

If $\{x_n\}_{n=1}^{\infty}\subset U(X)$, $x_0^*(x_n)\to x_0^*(x_0)=1$, $(n\to\infty)$, then, by the hypothesis that X is reflexive space we know that there exist $\{x_{n_k}\}_{k=1}^{\infty}\subset\{x_n\}_{n=1}^{\infty}$ and $x\in U(X)$ such that $x_{n_k}\xrightarrow{w}x,(k\to\infty)$. Hence $x_0^*(x_{n_k})\to x_0^*(x)=x_0^*(x_0)=1$, $(k\to\infty)$, it follows that $\|x\|\geq 1$, this shows that $x\in S(X)$. By the proof above, we have $x=x_0$, thus $x_{n_k}\xrightarrow{w}x_0,(k\to\infty)$ and we can deduce that $x_n\xrightarrow{w}x_0,(n\to\infty)$. If $\{x_n\}_{n=1}^{\infty}$ does not converges weakly to x_0, then there exist a weak neighborhood V_1 of 0 in weak topology $\sigma(X,X^*)$ and a subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $\{x_{n_i}\}_{i=1}^{\infty}\bigcap(x_0+V_1)=\varnothing$. Noticing that x_0 is weak denting point of $U(X)$, we know that $x_0\notin\overline{co}^w(U(X)\backslash(x_0+V_1))$. Hence there is a function $x^*\in X^*$ which separates $\overline{co}^w(U(X)\backslash(x_0+V_1))$ and x_0, i.e., there is a scalar $r>0$ such that $x^*(x_0)-r>\sup x^*(\overline{co}^w(U(X)\backslash(x_0+V_1)))$.

Noticing that $\{x_{n_i}\}_{i=1}^{\infty}\bigcap(x_0+V_1)=\varnothing$, we have $x_{n_i}\in U(X)\backslash(x_0+V_1)\subset\overline{co}^w(U(X)\backslash(x_0+V_1))$. By the inequality $x^*(x_0)-r>\sup x^*(\overline{co}^w(U(X)\backslash(x_0+V_1)))$, we have $x^*(x_0)-r>x^*(x_{n_i})$. On the other hand, by we have proved above, there exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_{n_i}\}_{i=1}^{\infty}$ such that $\{x_{n_j}\}_{j=1}^{\infty}$ converges weakly to x_0, which leads to $x^*(x_0)-r>x^*(x_0)$. A contradiction. Hence $x_0\in S(X)$ is weakly exposed point of $U(X)$.

Theorem6 Let X be a separable reflexive Banach space. If $x_0\in U(X)$ is extreme point of $U(X)$, then $x_0\in S(X)$ is weak denting point of $U(X)$.

Proof Suppose that $x_0\in U(X)$ is extreme point of $U(X)$. If $x_0\in S(X)$ is not weak denting point of $U(X)$, then there exists a weak neighborhood V of 0 in weak topology $\sigma(X,X^*)$ such that $x_0\in\overline{co}^w(U(X)\backslash(x_0+V))$. Noticing that $\overline{co}^w(U(X)\backslash(x_0+V))\subset U(X)$ and the assumption that X is reflexive Banach space, we know that $\overline{co}^w(U(X)\backslash(x_0+V))$ is weak compact set. By Krein-Milman theorem, we have $\overline{co}^w(ext\,\overline{co}^w(U(X)\backslash(x_0+V)))=\overline{co}^w(U(X)\backslash(x_0+V))$, hence $ext\,\overline{co}^w(U(X)\backslash(x_0+V))\subset\overline{U(X)\backslash(x_0+V)}^w$. In fact, by the assumption that X is separable Banach space, we know that weak topology $\sigma(X,X^*)$ is metrizable space, hence, for any $x\in ext\,\overline{co}^w(U(X)\backslash(x_0+V))$, there exists a sequence $\{t_nx_n+(1-t_n)y_n\}_{n=1}^{\infty}$ such that

$t_nx_n+(1-t_n)y_n\xrightarrow{w}x,(n\to\infty)$. Where $t_n\in[0,1]$, $x_n, y_n\in U(X)\backslash(x_0+V)$. By the reflexivity of X, we know that there exists a sequence $\{n_i\}\subseteq\{n\}$ such that $x_{n_i}\xrightarrow{w}x_0$, $y_{n_i}\xrightarrow{w}y_0$, $t_{n_i}\xrightarrow{w}t_0$, $(i\to\infty)$. It is obvious that $t_0\in[0,1]$, $x_0, y_0\in\overline{U(X)\backslash(x_0+V)}^w$ and $t_{n_i}x_{n_i}+(1-t_{n_i})y_{n_i}\xrightarrow{w}x$, thus $x=t_0x_0+(1-t_0)y_0$.

Case (I): If $t_0=0$, then $x=y_0\in\overline{U(X)\setminus(x_0+V)}^w$;

Case (II): If $t_0=1$, then $x=x_0\in\overline{U(X)\setminus(x_0+V)}^w$;

Case (III): If $t_0\in(0,1)$, then, by $x_0,y_0\in\overline{U(X)\setminus(x_0+U)}^w\subset\overline{co}^w(U(X)\setminus(x_0+V))$ and $x\in ext\,\overline{co}^w(U(X)\setminus(x_0+V))$, we have $x=x_0=y_0\in\overline{U(X)\setminus(x_0+V)}^w$. Combining case (I),(II) and (III), we have $ext\,\overline{co}^w(U(X)\setminus(x_0+V))\subset\overline{U(X)\setminus(x_0+V)}^w$. Because $ext\,(U(X)\cap\overline{co}^w(U(X)\setminus(x_0+V)))\subset ext\,\overline{co}^w(U(X)\setminus(x_0+V))$, we have $x_0\in ext\,\overline{co}^w(U(X)\setminus(x_0+V))\subset\overline{U(X)\setminus(x_0+V)}^w$. This is a contradiction.

Theorem7 If $x_0\in S(X)$ is weak denting point of $U(X)$, then x_0 is very extreme point of $U(X)$.

Proof Suppose that $x_0\in S(X)$ is not very extreme point of $U(X)$, then there exist $\varepsilon_0>0$, $f_0\in S(X^*)$ such that for any neighborhood V of 0 in weak topology $\sigma(X,X^*)$, there exists $a,b\in U(X)+V$ satisfying that $x_0=\frac{1}{2}(a+b)$, $|f(a-b)|=\varepsilon_0$, $\{ta+(1-t)b\}\subset U(X)+V$, where $\forall t\in[0,1]$.

We consider a weak neighborhood $V=\left\{x:|f_0(x)|<\frac{\varepsilon_0}{6}\right\}$ of 0 in weak topology $\sigma(X,X^*)$

and a family of sets $\{V_\delta\}$ (where V_δ is balanced convex neighborhood of 0 in weak topology $\sigma(X,X^*)$, then weak topology $\sigma(X,X^*)$ has a neighborhood base $\{V_\delta\cap V\}$ of 0. For any $V_\delta\cap V$, there exist $a_\delta,b_\delta\in U(X)+(U_\delta\cap V)$ satisfying that $x_0=\frac{1}{2}(a_\delta+b_\delta)$, $|f(a_\delta-b_\delta)|=\varepsilon_0$, $\{ta_\delta+(1-t)b_\delta\}\subset U(X)+V_\delta\cap V$, where $\forall t\in[0,1]$.

Let $a_\delta=a'_\delta+a''_\delta$, $b_\delta=b'_\delta+b''_\delta$ (where $a'_\delta,b'_\delta\in U(X)$, $a''_\delta,b''_\delta\in V_\delta\cap V$), then

$$x_0=\frac{1}{2}(a'_\delta+b'_\delta)+\frac{1}{2}(a''_\delta+b''_\delta),\quad |f_0(a_\delta-x_0)|=|f_0(\frac{1}{2}(a_\delta-b_\delta))|=\frac{\varepsilon_0}{2},$$

$$|f_0(a'_\delta-x_0)|\geq|f_0(a_\delta-x_0)|-|f_0(a'_\delta-a_\delta)|\geq\frac{\varepsilon_0}{2}-\frac{\varepsilon_0}{6}=\frac{\varepsilon_0}{3}.$$

Noticing that $U(X)$ is convex set and $V_\delta\cap V$ is balanced convex set, we know that

$$\frac{1}{2}(a'_\delta+b'_\delta)\in U(X),\quad \frac{1}{2}(a''_\delta+b''_\delta)\in V_\delta\cap V,\quad -\frac{1}{2}(a''_\delta+b''_\delta)\in V_\delta\cap V.$$

Let $V_0=\left\{x:|f_0(x)|<\frac{\varepsilon_0}{3}\right\}$. It is obvious that $a'_\delta\notin x_0+V_0$. Similarly, $b'_\delta\notin x_0+V_0$. Hence

$a'_\delta, b'_\delta \in U(X)\setminus(x_0+V)$, furthermore, $\frac{1}{2}(a'_\delta+b'_\delta)\in co(U(X)\setminus(x_0+V_0))$.

On the other hand, $\frac{1}{2}(a'_\delta+b'_\delta)=x_0-\frac{1}{2}(a''_\delta+b''_\delta)$ and $-\frac{1}{2}(a''_\delta+b''_\delta)\in V_\delta\bigcap V$, so we have $\frac{1}{2}(a'_\delta+b'_\delta)\in x_0+V_\delta\bigcap V$. Because $\{V_\delta\bigcap V\}$ is neighborhood base of 0 in weak topology $\sigma(X,X^*)$, from the definition of weak closure about weak topology $\sigma(X,X^*)$, we know that $x_0\in\overline{co}^w(U(X)\setminus(x_0+V_0))$. Which contradicts that $x_0\in S(X)$ is weak denting point of $U(X)$.

Theorem8 Let X be a reflexive separable Banach space. $X\times X$ is a Banach space with the norm $\|(x,y)\|=\max\{\|x\|,\|y\|\}$. If x_1,x_2 are weak denting points of $U(X)$, then (x_1,x_2) is weak denting point of $U(X)\times U(X)=U(X\times X)$..

Proof Suppose that x_1,x_2 are weak denting points of $U(X)$. By theorem 7 we know that x_1,x_2 are very extreme points of $U(X)$. It is obvious that x_1,x_2 are extreme points of $U(X)$. We will prove that (x_1,x_2) is extreme point of $U(X)\times U(X)$. If $(y_1,y_2),(z_1,z_2)\in U(X)\times U(X)$

and $(x_1,x_2)=\frac{1}{2}(y_1,y_2)+\frac{1}{2}(z_1,z_2)=(\frac{1}{2}y_1+\frac{1}{2}z_1,\frac{1}{2}y_2+\frac{1}{2}z_2)$, then $x_1=\frac{1}{2}(y_1+z_1)$, $x_2=\frac{1}{2}(y_2+z_2)$. Hence $x_1=y_1=z_1$, $x_2=y_2=z_2$. This shows that $(x_1,x_2)=(y_1,y_2)=(z_1,z_2)$. Therefore (x_1,x_2) is extreme point of $U(X)\times U(X)$. By Theorem 6, (x_1,x_2) is weak denting point of $U(X)\times U(X)=U(X\times X)$.

Theorem9 Let X be a very convex space and $X\times X$ is a Banach space with the norm $\|(x,y)\|=\max\{\|x\|,\|y\|\}$. If $x_1,y_1\in S(X)$, then (x_1,y_1) is weak denting point of $U(X\times X)$. If $\|x_1\|=1,\|y_1\|<1$, then (x_1,y_1) is not weak denting point of $U(X\times X)$.

Proof Firstly, we will prove that $\sigma(X\times X,(X\times X)^*)=\sigma(X,X^*)\times\sigma(X,X^*)$.

For $\forall\varepsilon>0$, $f\in(X\times X)^*$, we consider a eighborhood $\{(x,y):|f(x,y)|<\varepsilon\}$ of $(0,0)$ in weak topology $\sigma(X\times X,(X\times X)^*)$ and define $f_1(x)=f(x,0)$, $f_2(x)=f(0,x)$ for any $x\in X$, then $\|f_i(x)\|\le\|f\|\cdot\|x\|$ and f_i $(i=1,2)$ are linear functions. This shows that $f_i\in X^*,(i=1,2)$. We construct a set $G=\left\{x:|f_1(x)|<\frac{\varepsilon}{2}\right\}\times\left\{x:|f_2(x)|<\frac{\varepsilon}{2}\right\}$, then G is open set in $\sigma(X,X^*)\times\sigma(X,X^*)$.

For any $(x,y)\in G$, we have $|f(x,y)|\le|f(x,0)|+|f(0,y)|\le|f_1(x)|+|f_2(y)|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$, hence $G\subset\{(x,y):|f(x,y)|<\varepsilon\}$. This shows that topology $\sigma(X,X^*)\times\sigma(X,X^*)$ is smaller than topology $\sigma(X\times X,(X\times X)^*)$.

For $\forall\varepsilon>0$, $f_1,f_2\in X^*$, we also consider the above open set G in topology and define

$f(x,y)=f_1(x)$, $g(x,y)=f_2(x)$ for $(x,y)\in X\times X$, then $\|f(x,y)\|\le\|f_1\|.\|x\|\le\|f_1\|\|(x,y)\|$, $\|g(x,y)\|\le\|f_2\|.\|x\|\le\|f_2\|\|(x,y)\|$ and $f(x,y)$, $g(x,y)$ are linear functions. This shows that $f,g\in(X\times X)^*$, we consider a neighborhood $\left\{(x,y):|f(x,y)|<\frac{\varepsilon}{2}\right\}\cap\left\{(x,y):|g(x,y)|<\frac{\varepsilon}{2}\right\}$ of $(0,0)$ in weak topology $\sigma(X\times X,(X\times X)^*)$, then $\left\{(x,y):|f(x,y)|<\frac{\varepsilon}{2}\right\}\cap\left\{(x,y):|g(x,y)|<\frac{\varepsilon}{2}\right\}\subset G.$ This shows that topology $\sigma(X\times X,(X\times X)^*)$ is smaller than topology $\sigma(X,X^*)\times\sigma(X,X^*)$. This completes the proof that $\sigma(X\times X,(X\times X)^*)=\sigma(X,X^*)\times\sigma(X,X^*)$.

Secondly, we will prove that if $x_1,y_1\in S(X)$, then (x_1,y_1) is weak denting point of $U(X\times X)$.

If $x_1,y_1\in S(X)$, then, by Hahn Banach Theorem we know that there exists $f_1\in S(X^*)$ such that $f_1(x_1)=1$. We choose $x_n\in U(X)$ with $\|x_n\|=1$, such that $f_1(x_n)\to f_1(x_1)=1,(n\to\infty)$.

By the assumption that X is very convex space, we have $x_n\xrightarrow{w}x_1,(n\to\infty)$. It follows that for any weak neighborhood V of 0 in weak topology $\sigma(X,X^*)$, there exists $r_1>0$ such that for any $x\in U(X)\backslash(x_1+V)$ there holds $f_1(x_1)=1>f_1(x)+r_1$. Otherwise, there exists $x_n\in U(X)\backslash(x_1+V)$ such that $f_1(x_n)\to 1,(n\to\infty)$, which leads to $x_n\xrightarrow{w}x_1,(n\to\infty)$. This contradicts that $x_n\in U(X)\backslash(x_1+V)$ .Similarly, there exists $f_2\in S(X^*)$ such that $f_2(y_1)=1$ and for any weak neighborhood V of 0 in weak topology $\sigma(X,X^*)$, there exists $r_2>0$ such that for any $y\in U(X)\backslash(y_1+V)$ there holds $f_2(y_1)=1>f_2(y)+r_2$. For any $(x,y)\in X\times X$, let $f(x,y)=f_1(x)+f_2(y)$, then f is linear function on $X\times X$, and

$$|f(x,y)|\le|f_1(x)|+|f_2(y)|\le\|f_1\|.\|x\|+\|f_2\|.\|y\|\le 2\sqrt{\|f_1\|^2+\|f_2\|^2}\sqrt{\|x\|^2+\|y\|^2}$$

$$\le 4\sqrt{\|f_1\|^2+\|f_2\|^2}\max\{\|x\|,\|y\|\}=4\sqrt{\|f_1\|^2+\|f_2\|^2}\|(x,y)\|.$$

This shows that $f\in(X\times X)^*$.

We consider any weak neighborhood W of $(0,0)$ in weak topology $\sigma(X\times X,(X\times X)^*)$ and notice that $x_1,y_1\in S(X)$, then, by $\sigma(X\times X,(X\times X)^*)=\sigma(X,X^*)\times\sigma(X,X^*)$, we know that there exists a weak neighborhood V of 0 in weak topology $\sigma(X,X^*)$ such that $(x_1+V)\times(y_1+V)\subset(x_1,y_1)+W$. For $(x,y)\in(U(X)\times U(X))\backslash((x_1+V)\times(y_1+V))$, we have $x\in U(X)\backslash(x_1+V)$ or $x\in U(X)\backslash(y_1+V)$. Without loss of generality, we may assume that $x\in U(X)\setminus\left(x_1+V\right)$, then $f(x_1,y_1)-f(x,y)=(f_1(x_1)-f_1(x))+(f_2(y_1)-f_2(y))>k_1$. Thus

$$f(x_1,y_1)-k_1\ge\sup\left\{ f(x,y):(x,y)\in U(X)\times U(X)\setminus(x_1+V)\times(y_1+V)\right\}$$

$$\ge\sup\left\{ f(x,y):(x,y)\in U(X)\times U(X)\setminus((x_1,y_1)+W)\right\}$$

$$=\sup\left\{ f(x,y):(x,y)\in \overline{co}(U(X)\times U(X)\setminus((x_1,y_1)+W))\right\}$$

$$=\sup\left\{ f(x,y):(x,y)\in \overline{co}^{w}(U(X)\times U(X)\setminus((x_1,y_1)+W))\right\}.$$

Hence $(x_1,y_1)\notin\overline{co}^{w}(U(X)\times U(X)\setminus((x_1,y_1)+W))$. Obviously,
$U(X\times X)=U(X)\times U(X)$,
which leads to $(x_1,y_1)\notin\overline{co}^{w}(U(X\times X)\setminus((x_1,y_1)+W))$, this shows that (x_1,y_1) is weak denting point of $U(X\times X)$.

Thirdly, we will prove that if $\|x_1\|=1, \|y_1\|<1$, then (x_1,y_1) is not weak denting point of $U(X\times X)$.

Case (i). If $x_1\in S(X), \|y_1\|=a, a\in(0,1)$, then, there exist $y'=\left(1+\frac{2-2a}{3a}\right)\cdot y_1, y''=\left(1-\frac{2-2a}{3a}\right)\cdot y_1$ such that $\|y'\|\le\|y'-y_1\|+\|y_1\|=\frac{2}{3}(1-a)+a<1$, $\|y''\|\le\|y''-y_1\|+\|y_1\|=\frac{2}{3}(1-a)+a<1$. By Hahn-Banach Theorem, there exists $f\in X^*$ such that $f(y_1)=1, f(y')=1+\frac{2-2a}{3a}, f(y'')=1-\frac{2-2a}{3a}$, hence $f(y'-y_1)=\frac{2-2a}{3a}, f(y''-y_1)=-\frac{2-2a}{3a}$ and $\varepsilon_0=\frac{2-2a}{6a}$. We consider a weak neighborhood $\{x:|f(x-y_1)|<\varepsilon_0\}$ of y_1 in weak topology $\sigma(X,X^*)$, then $\{x:|f(x-y_1)|<\varepsilon_0\}$ can be written by $\{x:|f(x-y_1)|<\varepsilon_0\}=y_1+V'$, where V' is a weak neighborhood of 0 in weak topology $\sigma(X,X^*)$. Evidently, $y',y''\notin y_1+V'$, it follows that $(x_1,y'),(x_1,y'')\in(U(X)\times U(X)\setminus(x_1+V')(y_1+V'))$. By $\sigma(X\times X,(X\times X)^*)=\sigma(X,X^*)\times\sigma(X,X^*)$, we know that there exists a weak neighborhood V of 0 in weak topology $\sigma(X,X^*)$ such that $(x_1,y_1)+V\subset(x_1+V')\times(y_1+V')$. Noticing that $(x_1,y_1)=\frac{1}{2}(x_1,y')+\frac{1}{2}(x_1,y'')$ and the equality $U(X\times X)=U(X)\times U(X)$, we have $(x_1,y_1)\in(U(X)\times U(X)\setminus(x_1+V')\times(y_1+V'))\subset co(U(X\times X)\setminus((x_1,y_1)+V))$, furthermore, $(x_1,y_1)\in\overline{co}^{w}(U(X\times X)\setminus((x_1,y_1)+V))$. Hence, (x_1,y_1) is not weak denting point o $U(X\times X)$.

Case (ii). If $y_1=0, x_1\in S(X)$, let $z\in U(X), \|z\|=\frac{1}{2}$. By Hahn-Banach Theorem, there exists $f\in X^*$ such that $f(z)=1$. We consider a neighborhood $V'=\left\{x:\left|f(x)\right|<\frac{1}{2}\right\}$ of 0 in weak topology $\sigma(X, X^*)$, then $z, -z \notin V'$. Which leads to (x_1, z) , $(x_1, -z) \in (U(X) \times U(X) \setminus (x_1 + V') \times (0 + V'))$.

Similar to case (i), there exists a V such that $(x_1, 0) \subset (x_1 + V') \times (0 + V')$. Noticing that $(x_1, 0) = \frac{1}{2}(x_1, z) + \frac{1}{2}(x_1, -z)$, we have $(x_1, 0) \in \overline{co}^w(U(X \times X) \setminus (x_1 + V') \times (0 + V')) \subset \overline{co}^w(U(X \times X) \setminus ((x_1, 0) + V))$.

This shows that $(x_1, 0)$ is not weak denting point of $U(X \times X)$. .

Theorem10 Let X be a strongly convex space. $X \times X$ is a Banach space with the norm $\|(x, y)\| = \max\{\|x\|, \|y\|\}$. If $x_1, y_1 \in S(X)$, then (x_1, y_1) is denting point of $U(X \times X)$. If $\|x_1\| = 1, \|y_1\| < 1$, then (x_1, y_1) is not denting point of $U(X \times X)$.

Proof Case (i). $x_1, y_1 \in S(X)$. Using a similar method to that used in the proof of theorem 9, We can prove that (x_1, y_1) is denting point of $U(X \times X)$.

Case (ii). $\|x_1\| = 1, \|y_1\| < 1$. By the assumption that X is strongly convex space, we know that X is very convex space. By theorem 9, we know that (x_1, y_1) is not weak denting point of $U(X \times X)$, so there exists a weak neighborhood V of $(0, 0)$ in weak topology $\sigma(X, X^*)$, such that $(x_1, y_1) \in \overline{co}^w(U(X \times X) \setminus (x_1, y_1) + V)$. It is obvious that there exists a scalar $\varepsilon > 0$ such that $B_\varepsilon(x_1, y_1) \subset (x_1, y_1) + V$. Hence $(x_1, y_1) \in \overline{co}^w(U(X \times X) \setminus B_\varepsilon(x_1, y_1)) = \overline{co}(U(X \times X) \setminus B_\varepsilon(x_1, y_1))$.

This shows that (x_1, y_1) is not denting point of $U(X \times X)$.

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