Existence and Nonexistence of Positive Solutions for a Kirchhoff-type Equation with Inhomogeneous Strong Allee Effect

Ruyun Ma and Guowei Dai

Department of Mathematics, Northwest Normal University, Lanzhou 730070, P.R. China

Email: mary@nwnu.edu.cn, daiguowei@uwnu.edu.cn

Abstract *In this paper, we deal with the nonlocal semilinear elliptic equation with inhomogeneous strong Allee effect*

$$\begin{cases} -M\left(\int_\Omega \frac{1}{2}|\nabla u|^2\,dx\right)\Delta u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the nonlocal coefficient $M\left(\int_\Omega \frac{1}{2}|\nabla u|^2\,dx\right)$ is a continuous function of $\int_\Omega \frac{1}{2}|\nabla u|^2\,\mathrm{d}x$. By means of variational approach, we prove that the equation has at least two positive solutions for large λ under suitable hypotheses about nonlinearity. We also prove some nonexistence results. In particular, we shall give a positive answer to the conjecture by Liu, Wang and Shi's of [1].

Keywords *Nonlocal differential equation Variational method Positive solutions Inhomogeneous strong Allee effect.*

1. Introduction

In this paper we study the following problem

$$\begin{cases} -M\left(\int_\Omega \frac{1}{2}|\nabla u|^2\,dx\right)\Delta u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ for $N \geq 1$, the nonlocal coefficient $M(t)$ is a continuous function of $t = \int_\Omega \frac{1}{2}|\nabla u|^2\,dx$. We shall give a positive answer to a conjecture by Liu, Wang and Shi's of [1].

The problem (1) is a generalization of a model introduced by Kirchhoff[2]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{2}$$

where ρ, ρ_0, h, E, L are constants, which extends the classical D'Alembert's wave equation, by considering the effect of the changing in the length of the string during the vibration. A distinguishing feature of equation (2) is that the equation contains a nonlocal coefficient $\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx$ which depends on the average $\frac{1}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx$ of the kinetic energy $\frac{1}{2}\left|\frac{\partial u}{\partial x}\right|^2$ on $[0, L]$, and hence the equation is no longer a pointwise identity. The equation

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2\,dx\right)\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

is related to the stationary analogue of the equation (2). Equation (3) received much attention only after Lions [3] proposed an abstract framework to the problem. Some important and interesting results can be found, for example, in [4-15].

In the context of population biology, the nonlinear function $f(x, u) \equiv ug(x, u)$ represents a density dependent growth if $g(x, u)$ is a function depending on the population density u. While traditionally $g(x, u)$ is assumed to be declining to reflect the crowding effect of the increasing population, Allee suggested that physiological and demographic precesses often possess an optimal density, with the response decreasing as either higher or lower densities. Such growth pattern is called an Allee effect. If the growth rate per capita is negative when u is small, we call it a strong Allee effect; if the growth rate per capita is small than the maximum but still positive for small u, we call it a weak Allee effect (for detail, see [16] or [17]).

Under the special case of equation (3) with $a = 1$, $b = 0$ and $f(x, u)$ satisfies inhomogeneous strong Allee effect growth pattern, Liu, Wang and Shi[16] prove that the equation

$$\begin{cases} -\Delta u = \lambda f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4}$$

has at least two positive solutions for large λ if $\int_0^{c(x)} f(x, s)\, ds > 0$ for x in an open subset of Ω, where $c(x) \in C^1(\overline{\Omega})$ such that $f(x, c(x)) = 0$(see the assumption of $(f2)$). They also prove some nonexistence results. In particular, they conjecture that the nonexistence holds if $\int_0^{c(x)} f(x, s)\, ds \leq 0$ for any $x \in \overline{\Omega}$ (see Remark 1.7 of [16]).

Motivated by above, we generalize existence and nonexistence results for the semilinear elliptic equation (4) to the case of nonlocal semilinear elliptic equation (1). More precisely, if $f(x, u)$ satisfies inhomogeneous strong Allee effect growth pattern and the nonlocal coefficient $M(t)$ satisfies some suitable conditions, we establish the existence of at least two positive solutions for the nonlocal problem (1) with λ large enough. We also prove some nonexistence results for the nonlocal problem (1). In particular, we shall give a positive answer to the conjecture by Liu, Wang and Shi. To the best of our knowledge, this is the first paper that discusses the nonlocal semilinear elliptic equation with inhomogeneous strong Allee effect via variational method.

We point out the nonlocal coefficient $M(t)$ raises some of the essential difficulties. For example, the way of proving the geometry condition of Mountain Pass Theorem in [16] can not be used here because the functional of (1) is not C^2 function under our assumptions. In order to overcome this difficulty, we divided Ω into B_1 and B_2 by comparing the value of $c(x)$ with $\underline{b}$, then use poincaré inequality to prove it(see Lemma 3.3).

This paper is organized as follows. In Section 2, we present our main resuts and some necessary preliminary lemmas. In Sections 3, we use variational method and sub-supersolution method to prove the main results. In Section 4, we prove the conjecture of Liu, Wang and Shi's and give some examples which satisfy our hypotheses.

2. Main Resuts and Preliminaries

In this section, we give our main results and some necessary preliminary lemmas which will be used in the following proof. For simplicity we write $X = H_0^1(\Omega)$ with the norm $\|u\| =$

$\left(\int_0^1 |\nabla u|^2\, dx\right)^{\frac{1}{2}}$.

Hereafter, $f(x,t)$ and $M(t)$ are always supposed to verify the following assumptions:

$(f1)$ $f(x,u) \in C(\overline{\Omega} \times \mathbb{R}^+)$ and $f(x,\cdot) \in C^1(\mathbb{R}^+)$ for any $x \in \overline{\Omega}$;

$(f2)$ There exist $b(x) \in C(\overline{\Omega})$, $c(x) \in C^1(\overline{\Omega})$ such that $0 < b(x) < c(x)$ and $f(x,0) = f(x,b(x)) = f(x,c(x)) = 0$ for any $x \in \overline{\Omega}$;

$(f3)$ For almost all $x \in \overline{\Omega}$, $f(x,s) < 0$ for any $s \in (0,b(x)) \cup (c(x),+\infty)$ and $f(x,s) > 0$ for any $s \in (b(x),c(x))$.

Remark 2.1. Note that the weak maximum principle (Theorem 8.1 of [18]) and strong maximum principle (Theorem 8.1 of [18]) also hold for the nonlocal problem (1) because $M(t)$ satisfies the assumption (M).

(M) $\exists m_0 > 0$ such that

$$M(t) \geq m_0.$$

Definition 2.1. We say that $u \in X$ is a weak solution of (1), if

$$M\left(\int_\Omega \frac{1}{2}|\nabla u|^2\, dx\right)\int_\Omega \nabla u \nabla \varphi\, dx = \lambda \int_\Omega f(x,u)\varphi\, dx$$

for any $\varphi \in X$.

Define

$$\Phi(u) = \widehat{M}\left(\int_\Omega \frac{1}{2}|\nabla u|^2\, dx\right), \Psi(u) = \int_\Omega F(x,u)\, dx,$$

where $\widehat{M}(t) = \int_0^t M(s)\, ds, F(x,u) = \int_0^u f(x,t)\, dt$. We redefine $f(x,u)$, such that $f(x,u) \equiv 0$ when $u \in (-\infty,0) \cup (c(x),\infty)$, but it does not change the solution set of (1) by the weak maximum principle, since all the solution of (1) satisfies $0 \leq u(x) \leq c(x)$. Then the energy functional $I_\lambda(u) = \Phi(u) - \lambda\Psi(u) : X \to \mathbb{R}$ associated with problem (1) is well defined. Then it is easy to see that $I_\lambda \in C^1(X,\mathbb{R})$ is weakly lower semi-continuous and $u \in X$ is a weak solution of (1) if and only if u is a critical point of I_λ. From the regularity assumptions on $f(x,u)$, any critical point u of $I_\lambda(\cdot)$ is a classical solution of (1) (see [19, 20]), and from the strong maximum principle and the definition of modified $f(x,u)$ above, u is either zero or satisfies $0 < u(x) < c(x)$ for any $x \in \Omega$. Moreover, we have

$$\begin{aligned} I'_\lambda(u)v &= M\left(\int_\Omega \frac{1}{2}|\nabla u|^2\, dx\right)\int_\Omega \nabla u \nabla v\, dx - \lambda\int_\Omega f(x,u)v\, dx \\ &= \Phi'(u)v - \lambda\Psi'(u), \text{ for any } v \in X. \end{aligned}$$

From (M) and Lemma 4.1 of [21] we can easily see that Φ' is of (S_+) type, i.e. if $u_n \rightharpoonup u$ in X and $\overline{\lim\limits_{n\to+\infty}}\,(\Phi'(u_n) - \Phi'(u), u_n - u) \leq 0$, then $u_n \to u$ in X. It is clear that Ψ' is weak-strong continuous (or see Lemma 1.2 of [1]). So I'_λ is of (S_+) type.

Our main existence result is as follows:

Theorem 2.1. If $M(t)$ satisfies (M) and $f(x,u)$ satisfies $(f1)$–$(f3)$, and Ω_1 is an open subset of Ω such that

$$\int_0^{c(x)} f(x,s)\,ds > 0 \tag{5}$$

for $x \in \Omega_1$, then for λ large enough, (1) has at least two positive solutions, and (1) has no solution for small λ.

In order to prove our main existence result we need the following lemma:

Lemma 2.1 (see [1].) Suppose that f satisfies $(f1)$–$(f3)$. If $u(x)$ is an integrable function in Ω, and there is a measurable subset Ω_0 of Ω with positive measure, such that

$$\int_0^{c(x)} f(x,s)\,ds > 0 \text{ in } \Omega_0 \quad \text{and} \quad \int_0^{c(x)} f(x,s)\,ds \le 0 \text{ in } \Omega \setminus \Omega_0,$$

then

$$\int_0^{u(x)} f(x,s)\,ds \le \int_0^{c(x)} f(x,s)\,ds \text{ in } \Omega_0 \quad \text{and} \quad \int_0^{u(x)} f(x,s)\,ds \le 0 \text{ in } \Omega \setminus \Omega_0,$$

Now we turn to the nonexistence of the positive solutions of (1) when (5) does not hold for any $x \in \overline{\Omega}$. We define $\overline{c} = \max_{x\in\overline{\Omega}} c(x)$, $\overline{f}(u) = \max_{x\in\overline{\Omega}} f(x,u)$. Our main nonexistence result is

Theorem 2.2. If $\int_0^{\overline{c}} \overline{f}(u)\,du \le 0$, then (1) has no positive solution for any $\lambda > 0$.

In order to prove our main nonexistence result, we recall a theorem in [22] for (1) with the special case of $M(t) \equiv 1$ and $f(x,u) \equiv f(u)$. In fact, the theorem also holds for the nonlocal problem (1) with $f(x,u) \equiv f(u)$. Because the proof is similar to the proof of[22], we omit it here (for detail, see the proof of Theorem 1 in [22]). Let us assume that $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function and let the following conditions hold: there exist $0 \le s_0 < s_1 < s_2$, such that

$$\begin{cases} f(s_i) = 0, & i = 1,2, \\ f(s_0) \le 0, & \\ f(s) < 0, & s_0 < s < s_1, \\ f(s) > 0, & s_1 < s < s_2 \end{cases} \tag{6}$$

and let

$$\int_{s_0}^{s_2} f(s)\,ds \le 0. \tag{7}$$

We have the following lemma.

Lemma 2.2. Assume that f satisfies (6) and (7). Let Ω be a bounded domain with smooth boundary. If (1) with $f(x,u) \equiv f(u)$ has a positive solution u, then u can not satisfy

$$\begin{cases} u_{\max} = \max_{x\in\Omega} u(x) \in (s_1, s_2), \\ u(x) > 0, \qquad x \in \Omega. \end{cases} \tag{8}$$

Remark 2.2. Note that our assumptions $(f1)$–$(f3)$ are weaker than $(f1)$–$(f4)$ of[1] even in the case of $M(t) \equiv 1$. In fact, from $(f1)$–$(f3)$, we can easily see that there exist a positive constant β such that $f(x,s) \le \beta s$ for any $s \ge 0$ and a.e. $x \in \Omega$, i.e., the condition $(f4)$ of [1]. We do not need the conditions of $b(x) \in C^{1,\alpha}(\Omega)(0 < \alpha < 1)$ and $f(\cdot,u) \in C^{1,\alpha}(\overline{\Omega})$ for any $u \ge 0$ because we do not need energy functional of (1) is a C^2 function in X in our proof.

Remark 2.3. The condition of $f(x,\cdot) \in C^1(\mathbb{R}^+)$ for any $x \in \overline{\Omega}$ can be relaxed to $f(x,\cdot)$ is locally lipschitz in $\mathbb{R}^+$ for any $x \in \overline{\Omega}$. In fact, Lemma 2.2 also holds when $f : \mathbb{R} \to \mathbb{R}$ is a locally lipschitz function because the symmetry results of [23] holds under this weaker condition.

3. Proof of Theorem 2.1 and 2.2

In this section we will prove Theorem 2.1 and 2.2.

Lemma 3.1. If $M(t)$ satisfies (M), and $f(x,u)$ satisfies $(f1)$–$(f3)$ and (5), then for λ large enough, $I_\lambda(\cdot)$ has a global minimum point u_1 such that $I_\lambda(u_1) < 0$.

Proof. Since $\int_0^{c(x)} f(x,s)\,ds > 0$ in Ω_1, then there exists a measurable set $\Omega_0 \subset \Omega$ with positive measure, such that $\int_0^{c(x)} f(x,s)\,ds > 0$ in Ω_0, and $\int_0^{c(x)} f(x,s)\,ds \le 0$ in $\Omega \setminus \Omega_0$. From (M) and the definition of $\widehat{M}(t)$, we have $\widehat{M}(t) \ge m_0 t$. In view of Lemma 2.1, we have

$$\begin{aligned}
I_\lambda(u) &= \widehat{M}\left(\int_\Omega \frac{1}{2}|\nabla u|^2\,dx\right) - \lambda \int_\Omega F(x,u)\,dx \\
&\ge m_0 \int_\Omega \frac{1}{2}|\nabla u|^2\,dx - \lambda \int_\Omega \left(\int_0^{u(x)} f(x,s)\,ds\right) dx \\
&\ge m_0 \int_\Omega \frac{1}{2}|\nabla u|^2\,dx - \lambda \int_{\Omega_0} \left(\int_0^{u(x)} f(x,s)\,ds\right) dx - \lambda \int_{\Omega\setminus\Omega_0} \left(\int_0^{u(x)} f(x,s)\,ds\right) dx \\
&\ge m_0 \int_\Omega \frac{1}{2}|\nabla u|^2\,dx - \lambda \int_{\Omega_0} \left(\int_0^{c(x)} f(x,s)\,ds\right) dx \\
&\ge m_0 \int_\Omega \frac{1}{2}|\nabla u|^2\,dx - \lambda \int_{\Omega_0} A_1\,dx \\
&= \frac{m_0}{2}\|u\|^2 - \lambda|\Omega_0|A_1 \to +\infty, \quad \text{as} \quad \|u\| \to +\infty,
\end{aligned} \tag{9}$$

where $A_1 = \max_{x\in\overline{\Omega_0}} |F(x,c(x))|$. Since I_λ is weakly lower semi-continuous, I_λ has a minimum point u_1 in X.

Next we shall prove $I_\lambda(u_1) < 0$, thus u_1 is a positive solution of (1). In fact, we only need to verify that when λ is large there exists a $u_0 \in X$, such that $I_\lambda(u_0) < 0 = I_\lambda(0)$. We define $u_0(x) = 0$ in $\Omega \setminus \Omega_{1\varepsilon}$, and $u_0(x) = c(x)$ in Ω_1 and properly in $\Omega_{1\varepsilon} \setminus \Omega_1$ such that $u_0 \in X$, where $\Omega_{1\varepsilon} = \{x \in \Omega : \text{dist}(x,\Omega_1) \le \varepsilon\}$. Using the similar method with[1], we have

$$I_\lambda(u_0) \le \widehat{M}\left(\int_\Omega \frac{1}{2}|\nabla u_0|^2\,dx\right) - \lambda \int_{\Omega_1} F(x,c(x))\,dx - \lambda\left[-A_1\left(|\Omega_{1\varepsilon}| - |\Omega_1|\right)\right] dx. \tag{10}$$

Since $\int_0^{c(cx)} f(x,s)\,ds > 0$ when $x \in \Omega_1$ and $\int_0^{c(x)} f(x,s)\,ds$ is continuous, then there must exists an open subset Ω_2 with $\overline{\Omega}_2 \subset \Omega_1$ and $\delta > 0$, such that $|\Omega_2| > 0$ and $\int_0^{c(x)} f(x,s)\,ds \geq \delta$ for $x \in \Omega_2$. Choose ε small enough, such that $\delta|\Omega_2| + A(|\Omega_1| - |\Omega_{1\varepsilon}|) > 0$. Again using the similar method with [1], we have

$$I_\lambda(u_0) \leq \widehat{M}\left(\int_\Omega \frac{1}{2}|\nabla u_0|^2\,dx\right) - \lambda\left[\delta|\Omega_2| + A_1(|\Omega_1| - |\Omega_{1\varepsilon}|)\right].$$

Therefore when λ large enough, $I_\lambda(u_0) < 0$, and consequently when λ is large enough, (1) has a positive solution $u_1(x)$ satisfying $I_\lambda(u_1) = \inf_{u\in X} I_\lambda(u) < 0$.

Next we shall use Mountain Pass Theorem to prove that (1) has another positive solution u_2. First we prove $I_\lambda(u)$ satisfies Palais-Smale condition.

Definition 3.1. We say that I_λ satisfies (P.S.) condition in X, if any sequence $\{u_n\} \subset X$ such that $\{I_\lambda(u_n)\}$ is bounded and $I'_\lambda(u_n) \to 0$ as $n \to +\infty$, has a convergent subsequence, where (P.S.) means Palais-Smale.

Lemma 3.2. If $M(t)$ satisfies (M), f satisfies $(f1)$–$(f3)$ and (5), then I_λ satisfies (P.S.) condition.

Proof. Suppose that $\{u_n\} \subset X$, $|I_\lambda(u_n)| \leq c_0$ and $I'_\lambda(u_n) \to 0$ as $n \to +\infty$. In view of (9), we have

$$c_0 \geq I_\lambda(u_n) \geq \frac{m_0}{2}\|u_n\|^2 - \lambda|\Omega_0|A.$$

Hence, $\{\|u_n\|\}$ is bounded. Without loss of generality, we assume that $u_n \rightharpoonup u$, then $I'(u_n)(u_n - u) \to 0$. Therefore, we have $u_n \to u$ by the (S_+) property of I'_λ.

Lemma 3.3. If $M(t)$ satisfies (M), f satisfies $(f1)$–$(f3)$, then there exist $\rho > 0$ and $\gamma > 0$ such that $I_\lambda(u) \geq \gamma$ for every $u \in X$ with $\|u\| = \rho$.

Proof. We define $\underline{b} = \min_{x\in\overline{\Omega}}$. For any $u(x) \in X$, we also define $B_1 = \{x \in \Omega : u(x) < \underline{b}\}$, $B_2 = \{x \in \Omega : u(x) \geq \underline{b}\}$. It is well known that the embedding of $X \hookrightarrow L^p(\Omega)$ is continuous when $2 < p \leq 2^*$, where 2^* is the critical exponent. From poincaré inequality, we have

$$\underline{b}|B_2|^{\frac{1}{p}} \leq \left(\int_{B_2} u^p\,dx\right)^{\frac{1}{p}} \leq c_1\left(\int_{B_2} |\nabla u|^2\,dx\right)^{\frac{1}{2}} \leq c_1\left(\int_\Omega |\nabla u|^2\,dx\right)^{\frac{1}{2}} = c_1\|u\|,$$

where c_1 is the embedding constant of $X \hookrightarrow L^p(\Omega)$. Thus, we have

$$\begin{aligned}
I_\lambda(u) &= \widehat{M}\left(\int_\Omega \frac{1}{2}|\nabla u|^2\,dx\right) - \lambda\int_\Omega F(x,u)\,dx \\
&\geq \frac{m_0}{2}\|u\|^2 - \lambda\int_{B_1} F(x,u)\,dx - \lambda\int_{B_2} F(x,u)\,dx \\
&\geq \frac{m_0}{2}\|u\|^2 - \lambda\int_{B_2} F(x,u)\,dx \\
&\geq \frac{m_0}{2}\|u\|^2 - \lambda A_2|B_2| \\
&\geq \frac{m_0}{2}\|u\|^2 - \lambda A_2\left(\frac{c_1}{\underline{b}}\right)^p\|u\|^p \\
&= \|u\|^2\left(\frac{m_0}{2} - \lambda A_2\left(\frac{c_1}{\underline{b}}\right)^p\|u\|^{p-2}\right),
\end{aligned}$$

where $A_2 = \max_{(x,s)\in\overline{B}_2\times[\underline{b},\overline{c}]}|F(x,s)|$. Therefore, there exist $\frac{m_0\underline{b}^p}{2\lambda A_2 c_1^p} > \rho > 0$ such that $I_\lambda(u) \geq \rho^2\left(\frac{m_0}{2} - \lambda A_2\left(\frac{c_1}{\underline{b}}\right)^p\rho^{p-2}\right) := \gamma > 0$ for every $\|u\| = \rho$ and fixed λ.

Proof of Theorem 2.1 concluded. First let us show that I_λ satisfies the conditions of Mountain Pass Theorem (see Theorem 2.10 of [24]). By Lemma 3.2, I_λ satisfies (P.S.) condition in X. By Lemma 3.3, for fixed $\lambda > 0$, there exist $\min\left\{\|u_0\|, \frac{m_0\underline{b}^p}{2\lambda A_2 c_1^p}\right\} > \rho > 0$, $\gamma > 0$ such that $I_\lambda(u) \geq \gamma > 0$ for every $\|u\| = \rho$, where u_0 comes from (10). On the other hand, since $I_\lambda(0) = 0$ and from the proof Lemma 3.1, there exists $u_0 \in X$ such that $I_\lambda(u_0) < 0$ and $\|u_0\| > \rho$. So from Mountain Pass Theorem, I_λ has another critical point u_2 such that

$$I_\lambda(u_2) \geq \gamma > 0 > I_\lambda(u_1).$$

Therefore, u_2 is another positive solution of (1).

Finally we show that (1) has no positive solution when λ is small. We assume (1) has a positive solution u, let $(\Lambda_1, \varphi_1(x))$ be the principal eigen-pair of the problem

$$\begin{cases} -\Delta\phi = \Lambda\phi & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{11}$$

such that $\varphi_1(x) > 0$ in Ω. We rewrite (1) as the following form

$$\begin{cases} -\Delta u = \lambda\frac{f(x,u)}{M\left(\int_\Omega \frac{1}{2}|\nabla u|^2\,dx\right)} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{12}$$

Multiplying (11) by u, multiplying (12) by φ_1, subtracting and integrating in Ω, we obtain

$$0 = \int_\Omega \left[\Lambda_1 u\varphi_1 - \lambda\varphi_1\frac{f(x,u)}{M(t)}\right]dx = \int_\Omega \frac{u\varphi_1}{M(t)}\left[M(t)\Lambda_1 - \lambda\frac{f(x,u)}{u}\right]dx, \tag{13}$$

where $t = \int_\Omega \frac{1}{2}|\nabla u|^2\,dx$. If $\lambda < \frac{m_0\Lambda_1}{\beta}$, then by Remark 2.2, we have

$$M(t)\Lambda_1 - \lambda\frac{f(x,u)}{u} \geq m_0\Lambda_1 - \lambda\frac{f(x,u)}{u} > m_0\Lambda_1 - \lambda\beta > 0.$$

That is contrary to (13). So for small λ, (1) has no positive solution.

Proof of Theorem 2.2. The proof is similar to the proof of [1]. For the sake of completeness, we include it here. If there exists a positive solution (λ, u_*) for (1), then u_* is a subsolution of

$$\begin{cases} M\left(\int_\Omega \frac{1}{2}|\nabla u|^2\,dx\right)\Delta u + \lambda\overline{f}(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{14}$$

since $M\left(\int_\Omega \frac{1}{2}|\nabla u_*|^2\,dx\right)\Delta u_* + \lambda\overline{f}(u_*) \geq M\left(\int_\Omega \frac{1}{2}|\nabla u_*|^2\,dx\right)\Delta u_* + \lambda f(x, u_*)$. And $\overline{c}$ is supersolution of (13). So by the standard comparison arguments, (13) has a positive solution $\overline{u}$ such that $u_* \leq \overline{u} \leq \overline{c}$. But if we let $s_0 = 0$, $s_1 = \underline{b}$ and $s_2 = \overline{c}$, $\overline{f}$ satisfies (6) and (7), then by Lemma 2.2, (13) has no positive solution. This is a contradiction. So (1) has no positive solution if $\int_0^{\overline{c}} \overline{f}(u)\,du \leq 0$.

4. Proof of a Conjecture and Some Examples

In this section we will prove the conjecture of Liu, Wang and Shi's and give some typical consequences of Theorem 2.1 to Theorem 2.2.

In [1], Liu, Wang and Shi conjecture that the nonexistence holds with a weaker condition:

$$\int_0^{c(x)} f(x,s)\,ds \leq 0 \text{ for any } x \in \overline{\Omega}. \tag{15}$$

In fact, as we will see in the following Proposition, the condition (15) is more strong than $\int_0^{\overline{c}} \overline{f}(s)\,ds \leq 0$. Therefore, by Theorem 2.2, the conjecture is right.

Proposition 4.1. If $f(x, u)$ satisfies $(f1)$–$(f3)$ and $\int_0^{c(x)} f(x,s)\,ds \leq 0$ for any $x \in \overline{\Omega}$, we have $\int_0^{\overline{c}} \overline{f}(s)\,ds \leq 0$.

Proof. From $(f1)$–$(f3)$, we can easily see that $f(x, s) \leq 0$ when $s \in [c(x), \overline{c}]$. Thus, we have $\int_{c(x)}^{\overline{c}} f(x,s)\,ds \leq 0$. Then, for any $x \in \overline{\Omega}$, we have

$$0 \geq \int_0^{c(x)} f(x,s)\,ds = \int_0^{\overline{c}} f(x,s)\,ds - \int_{c(x)}^{\overline{c}} f(x,s)\,ds \geq \int_0^{\overline{c}} f(x,s)\,ds.$$

In particular, $\int_0^{\overline{c}} \overline{f}(s)\,ds \leq 0$.

Now, we give some examples which satisfy our hypotheses.

Example 4.1. Let $M(t) = a + bt$ with $t = \int_\Omega \frac{1}{2}|\nabla u|^2\,dx$, here a, b are two positive constants and $f(x,u) = u(u - b(x))(c(x) - u)$ with $b(x) \in C(\overline{\Omega})$, $c(x) \in C^1(\overline{\Omega})$ such that $0 < b(x) < c(x)$ for any $x \in \overline{\Omega}$. It is clear that $M(t)$ and $f(x, u)$ verify our assumptions (M) and $(f1)$–$(f3)$.

Example 4.2. We consider a special case of Example 4.1:

$$\begin{cases} \Delta u + \lambda u(u - b(x))(c(x) - u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{16}$$

where $b(x) \in C(\overline{\Omega})$, $c(x) \in C^1(\overline{\Omega})$ such that $0 < b(x) < c(x)$ for any $x \in \overline{\Omega}$. We have known that $f(x,u)$ satisfies $(f1)$–$(f3)$ from Example 4.1. Moreover, we have

$$\begin{aligned}\int_0^{c(x)} f(x,s)\,ds &= \int_0^{c(x)} s(s-b(x))(c(x)-s)\,ds \\ &= \frac{1}{12}[c(x)]^3(c(x)-2b(x)).\end{aligned}$$

Then by Theorem 2.1, if there exists an open subset $\Omega_1 \subset \Omega$, such that $c(x) > 2b(x)$ in Ω_1, then (16) has at least two positive solutions for large λ.

If $c(x) \equiv 1$ for all $x \in \overline{\Omega}$, we obtain

$$\begin{aligned}\int_0^1 \overline{f}(s)\,ds &= \int_0^1 \max_{x\in\overline{\Omega}} s(s-b(x))(1-s)\,ds \\ &= \int_0^1 \max_{x\in\overline{\Omega}}[s^2-s^3+b(x)(s^2-s)]\,ds \\ &= \frac{1}{12}-\frac{\underline{b}}{6},\end{aligned}$$

since $s^2-s \le 0$ for $s \in [0,1]$. Then by Theorem 2.2, if $\underline{b} = \min_{x\in\overline{\Omega}} b(x) \ge \frac{1}{2}$, then (16) has no positive solution for any $\lambda > 0$.

Example 4.3. Let $M(t) \equiv 1$ and $f(x,s) = s(s-1)(c(x)-s)$ with $\frac{3}{2} \le c(x)$ for any $x \in \overline{\Omega}$. We can easily obtain

$$\begin{aligned}\int_0^{\overline{c}} \overline{f}(s)\,ds &= \int_0^{\overline{c}} \max_{x\in\overline{\Omega}} s(s-1)(c(x)-s)\,ds \\ &= \int_0^{\overline{c}} (c(x)s^2-s^3+s^2-c(x)s)\,ds \\ &= \frac{\overline{c}^3}{3}-\frac{\overline{c}^4}{4}+\int_0^{\overline{c}} \max_{x\in\overline{\Omega}} c(x)(s^2-s)\,ds \\ &= \frac{\overline{c}^3}{3}-\frac{\overline{c}^4}{4}+\max_{x\in\overline{\Omega}} c(x)\left(\frac{\overline{c}^3}{3}-\frac{\overline{c}^2}{2}\right)\,ds \\ &= \frac{\overline{c}^3}{3}-\frac{\overline{c}^4}{4}+\overline{c}\left(\frac{\overline{c}^3}{3}-\frac{\overline{c}^2}{2}\right)\,ds \\ &= \frac{\overline{c}^3}{12}[\overline{c}-2].\end{aligned}$$

So $\int_0^{\overline{c}} \overline{f}(s)\,ds \le 0$ if and only if $\overline{c} \le 2$.

On the other hand,

$$\begin{aligned}\int_0^{c(x)} f(x,s)\,ds &= \int_0^{\overline{c}} s(s-1)(c(x)-s)\,ds - \int_{c(x)}^{\overline{c}} s(s-1)(c(x)-s)\,ds \\ &\ge \int_0^{\overline{c}} s(s-1)(c(x)-s)\,ds \\ &= -\frac{\overline{c}^4}{4}+\frac{1+c(x)}{3}\overline{c}^3-\frac{c(x)}{2}\overline{c}^2.\end{aligned}$$

If $\int_0^{c(x)} f(x,s)\,ds \le 0$ for any $x \in \overline{\Omega}$, we have

$$\begin{aligned} 0 &\ge -\frac{\overline{c}^4}{4} + \frac{1+c(x)}{3}\overline{c}^3 - \frac{c(x)}{2}\overline{c}^2 \\ &\Rightarrow 4(1+c(x))\overline{c} - 6c(x) \le 3\overline{c}^2. \end{aligned}$$

In particular, we have

$$4(1+\overline{c})\overline{c} - 6\overline{c} \le 3\overline{c}^2 \Rightarrow \overline{c} \le 2.$$

However, it is clear that

$$\int_0^{\overline{c}} \overline{f}(s)\,ds \le 0 \not\Rightarrow \int_0^{c(x)} f(x,s)\,ds \le 0 \text{ for any } x \in \overline{\Omega}.$$

Therefore, the condition "$\int_0^{c(x)} f(x,s)\,ds \le 0$ for any $x \in \overline{\Omega}$" is more strong than the condition "$\int_0^{\overline{c}} \overline{f}(s)\,ds \le 0$" in this example, which verifies Proposition 4.1 by a concrete example.

Remark 4.1. In [25], Dancer and Yan proved when $c(x) \equiv 1$ and $\{x \in \Omega : b(x) < 1/2\}$ is of positive measure, then (16) may have many positive solutions of local minimum type. The results of Example 4.2 shows that the condition $\int_0^1 \overline{f}(s)\,ds \le 0$ is optimal for the nonexistence of positive solution of (16). However, we do not know whether $\int_0^{\overline{c}} \overline{f}(s)\,ds \le 0$ is optimal for the nonexistence of positive solution of (1).

References

[1] G.Liu, ,Y. Wang, & J. Shi, Existence and nonexistence of positive solutions of semilinear elliptic equation with inhomogeneous strong Allee effect, Appl. Math. Mech. -Engl. Ed. 30(11), 1461-1468 (2009)

[2] G.Kirchhoff, Mechanik, Teubner, Leipzig. (1883).

[3] J. L. Lions, On some questions in boundary value problems of mathematical physics, in: Proceedings of International Symposium on Continuum Mechanics and Partial Differential Equations. Rio de Janeiro, 1977, in: de la Penha, Medeiros (Eds.), Math. Stud., vol. 30, North-Holland, 1978, pp. 284-346.

[4] A. Arosio, & S.Pannizi, On the well-posedness of the Kirchhoff string. Trans. Amer. Math. Soc. 348 (1996) 305-330.

[5] Cavalcante, Cavalcante M. M., , V.N., & J.A. Soriano, Global existence and uniform decay rates for the Kirchhoff-Carrier equation with nonlinear dissipation. Adv. Differential Equations 6 (2001) 701-730.

[6] F. J. S. A. a Corrê, S.D.B.Menezes, &J. Ferreira, On a class of problems involving a nonlocal operator. Appl. Math. Comput. 147 (2004) 475-489.

[7] F. J. S. A. Corrêa, & G.M. Figueiredo, On a elliptic equation of p-kirchhoff type via variational methods. Bull. Austral. Math. Soc. Vol. 74 (2006) 263-277.

[8] P.D'Ancona, , & S. Spagnolo, Global solvability for the degenerate Kirchhoff equation with real analytic data. Invent. Math. 108 (1992) 247-262.

[9] M.Chipot,, & B. Lovat, Some remarks on nonlocal elliptic and parabolic problems. Nonlinear Anal. 30 (1997) 4619-4627.

[10] M.Dreher, The Kirchhoff equation for the p-Laplacian. Rend. Semin. Mat. Univ. Politec. Torino 64 (2006) 217-238.

[11] M.Dreher, The ware equation for the p-Laplacian. Hokkaido Math. J. 36 (2007) 21-52.

[12] G.ai, & R. Hao, Existence of solutions for a $p(x)$-Kirchhoff-type equation. J. Math. Anal. Appl. 359 (2009) 275-284.

[13] Dai, G., & Liu, D. Infinitely many positive solutions for a $p(x)$-Kirchhoff-type equation. J. Math. Anal. Appl. 359 (2009) 704-710.

[14] Fan, X. L. On nonlocal $p(x)$-Laplacian Dirichlet problems. Nonlinear Anal. 72 (2010) 3314-3323.

[15] He, X., & Zou, W. Infinitely many positive solutions for Kirchhoff-type problems. Nonlinear Anal. 70 (2009) 1407-1414.

[16] Allee, W. C. The social life of animals. W.W Norton, New York(1938).

[17] Cantrell, R. S., & Cosner, C. Spatial ecology via reaction-diffusion equation. Wiley series in mathematical and computational biology, John Wiley SonsLtd.

[18] Gilbarg, D., & Trudinger, S. Elliptic partial differential equations of second order. Springer-Verlag, Berlin.

[19] Lieberman, G. M. Boundary regularity for solutions of degenerate elliptic equations. Nonlinear Anal. TMA 12 (1988) 1203-1219.

[20] Lieberman, G. M. The natural generalization of the natural conditions of Ladyzenskaja and Ural'tzeva for elliptic equations. Comm. Partial Differential Equations 16 (1991) 311-361.

[21] G.Dai, Nonsmooth version of Fountain theorem and its application to a Dirichlet-type differential inclusion problem. Nonlinear Analysis 72 (2010) 1454-1461

[22] E. N.Dancer, & K. Schmitt, On positive solution of semilinear elliptic equations. Proc. Amer. Math.Soc. 101 (1987), no.3, 445-452.

[23] B. Gidas,, W. Ni, W. & Nirenberg, L. Symmetry and related properties via the maximum principle. Comm. Math. Phys. 68 (1979), 209-243.

[24] M. Willem, Minimax Theorems. Birkhäuser, Boston.

[25] E. N.Dancer,, & S. Yan, Construction of various types of solutions for an elliptic problem. Calculus Variations and Partial Differential Equations 20(1)(2004) , 93-118.