Characterization of L_c^p-solutions for the Dilation Equations on $\mathbb{R}^2$

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Abstract *In this paper, the author discussed the existence of compactly supported L^p-solutions for the dilation equations on the plane. Furthermore, two examples are given to illustrate the general theory.*

Keywords *Dilation equation Compactly supported L^p-solutions Iteration function system*

1. Introduction

A α-scale dilation equation is a functional equation of the form

$$f(x) = \sum_{n=0}^{N} c_n f(\alpha x - \beta_n)$$

where $f : \mathbb{R} \to \mathbb{R}(\text{or } \mathbb{C}), \alpha > 1, \beta_0 < \beta_1 < \cdots < \beta_N$ are real constants, and c_n are real (complex) constants. The equation is called a lattice k-scale dilation equation if

$$f(x) = \sum_{n=0}^{N} c_n f(kx - n)$$

for an integer $k \geq 2$. A special case of the functional equation ($k = 3, N = 4$, and $c_n = 1, 2/3, 1/3, 1$) was first studied by de Rham[1] as an example of a continuous nowhere differentiable function. Recently this equation has attracted a lot of attention, especially for the lattice case with $k = 2$. In wavelet theory, the study of multiresolution and the search of various orthogonal, compactly supported wavelets has lead to the investigation of the existence, uniqueness, and smoothness of such continuous integrable solutions[2]. The equation also plays an important role in the “subdivision schemes” and “interpolation schemes” of constructing continuous spline curves, surfaces and fractal objects [3, 4] .

There are two major approaches to the equation: the Fourier method(the frequency domain approaches) and the iteration method(the time-domain approaches). Using Fourier transformation, Daubechies and Lagarias[3] proved that the equation has a nonzero integrable solution. By using the Fourier transform of f and the Paley-Wiener theorem, it was proved in [3] that f has compact support in $[0, \beta_N/(\alpha - 1)]$. The Fourier method, however, does not give sharp criteria for the existence of L^1- solutions in terms of the coefficients$\{c_n\}$. Some partial results are given in [5, 6].

The iteration method is restricted to the lattice case. It applies particularly well in the case of compactly supported solutions. The basic idea is to identify a given function f supported by

$[0, N]$ with the vector-valued function

$$\mathbf{f}(x) = [f(x), \cdots, f(x + (N-1))]^T, x \in [0,1],$$

and to use the right side of the dilation equation to construct two $N \times N$ matrices T_0 and T_1. A constant vector v is used as the initial condition, followed by iteration with the matrices T_0 and T_1. The limit, if the sequence converges, will be the solution of the dilation equation. Such an approach was used by Daubechies and Lagarias[4], and independently by Michelli and Prautzsch[7]. It was also used by Collela and Heil[8] and [9] and Ka-sing Lau and Jianrong Wang[10].

Similarly, on the plane the dilation equation is defined as the form

$$f(x) = \sum_{m=0}^{M}\sum_{n=0}^{N} c_{mn} f(Ax - Q_{mn}) \tag{1}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ or $\mathbb{C}$, A is an expand matrix, Q_{mn} are vectors, c_{mn} are real (or complex) constants. The equation is called a lattice dilation equation if

$$f(x) = \sum_{m=0}^{M}\sum_{n=0}^{N} c_{mn} f(Ax - \begin{pmatrix} m \\ n \end{pmatrix}) \tag{2}$$

where A is an integer expand matrix.

In this paper we will study the existence of the compactly supported L^p-solutionof the equation(2) on the plane with $c_{mn} \in \mathbb{R}$ and $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

As usually the basic assumption on the coefficients is $\sum\limits_{m=0}^{M}\sum\limits_{n=0}^{N} c_{mn} = |\det A| = 4$.

Let $d_1 = \binom{0}{0}, d_2 = \binom{0}{1}, d_3 = \binom{1}{0}, d_4 = \binom{1}{1}, \varphi_k(x) = A^{-1}(x + d_k), k = 1,2,3,4$, there exists an attractor $T = [0,1] \times [0,1]$ satisfing $T = \bigcup\limits_{k=1}^{4} \varphi_k(T) \triangleq \bigcup\limits_{k=1}^{4} T_k$.

At the same time there exist vectors $\{e_{is} = \binom{i}{s}, 0 \le i \le M-1, 0 \le s \le N-1\}$ such that $\text{supp} f \subset \bigcup\limits_{i=0}^{M-1}\bigcup\limits_{s=0}^{N-1}(T + e_{is})$.

Let

$$P_{i0} = (c_{i,2s-t})_{0 \le s,\, t \le N-1},\ P_{i1} = (c_{i,2s-t+1})_{0 \le s,\, t \le N-1},\ i = 0, 1, \cdots, M.$$

For example,

$$P_{00} = (c_{0,2s-t})_{0 \le s,\, t \le N-1} = \begin{pmatrix} c_{0,0} & & & & & & & \\ c_{0,2} & c_{0,1} & c_{0,0} & & & & & \\ c_{0,4} & c_{0,3} & c_{0,2} & c_{0,1} & c_{0,0} & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & c_{0,N} & c_{0,N-1} \end{pmatrix}.$$

And let

$$M_1 = (P_{2i-j,0})_{0\le i,j\le M-1} = \begin{pmatrix} P_{0,0} \\ P_{2,0} & P_{1,0} & P_{0,0} \\ P_{4,0} & P_{3,0} & P_{2,0} & P_{1,0} & P_{0,0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & P_{M,0} & P_{M-1,0} \end{pmatrix},$$

$$M_2 = (P_{2i-j,1})_{0\le i,j\le M-1} = \begin{pmatrix} P_{0,1} \\ P_{2,1} & P_{1,1} & P_{0,1} \\ P_{4,1} & P_{3,1} & P_{2,1} & P_{1,1} & P_{0,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & P_{M,1} & P_{M-1,1} \end{pmatrix},$$

$$M_3 = (P_{2i-j+1,0})_{0\le i,j\le M-1} = \begin{pmatrix} P_{1,0} & P_{0,0} \\ P_{3,0} & P_{2,0} & P_{1,0} & P_{0,0} \\ P_{5,0} & P_{4,0} & P_{3,0} & P_{2,0} & P_{1,0} & P_{0,0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & P_{M,0} \end{pmatrix},$$

$$M_4 = (P_{2i-j+1,1})_{0\le i,j\le M-1} = \begin{pmatrix} P_{1,1} & P_{0,1} \\ P_{3,1} & P_{2,1} & P_{1,1} & P_{0,1} \\ P_{5,1} & P_{4,1} & P_{3,1} & P_{2,1} & P_{1,1} & P_{0,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & P_{M,1} \end{pmatrix}$$

We define a vector function: $F(x) = (f(x+e_{00}), f(x+e_{01}), f(x+e_{02}), \cdots, f(x+e_{0,N-1}), f(x+e_{10}), \cdots, f(x+e_{1,N-1}), \cdots, f(x+e_{M-1,N-1}))^T$ for $x \in T = [0,1] \times [0,1]$, then equation (2) will satisfy

$$F(x) = \begin{cases} M_1 F(\varphi_1^{-1}(x)) & x \in T_1 = [0,1/2) \times [0,1/2); \\ M_2 F(\varphi_2^{-1}(x)) & x \in T_2 = [0,1/2) \times [1/2,1); \\ M_3 F(\varphi_3^{-1}(x)) & x \in T_3 = [1/2,1) \times [0,1/2); \\ M_4 F(\varphi_4^{-1}(x)) & x \in T_4 = [1/2,1) \times [1/2,1); \\ 0 & x \in others. \end{cases} \tag{3}$$

Let v is 4-eigenvector of $(M_1 + M_2 + M_3 + M_4)$,we have $(M_1 + M_3 - 2I)v = -(M_2 + M_4 - 2I)v$.And let $\widetilde{v} = (M_1 + M_3 - 2I)v$, $H(\widetilde{v})$ be the subspace in $R^{M\times N}$ spanned by $\{M_\sigma \widetilde{v} : \sigma \in \Sigma^*\}$. Then the basic theorem is as follows.

Theorem 1.1. *For $1 \le p \le \infty$, the following are equivalent:*

(1) equation (2) has a nonzero compactly supportedL^p-solution;

(2) there exists a 4-eigenvector v of $(M_1 + M_2 + M_3 + M_4)$ satisfying

$$\lim_{l\to\infty} \frac{1}{4^l} \sum_{|\sigma|=l} \|M_\sigma \widetilde{v}\|^p = 0$$

(3) there exists a 4*-eigenvector* v *of* $(M_1+M_2+M_3+M_4)$ *such that there exists an integer* $l \geq 1$ *such that*

$$\frac{1}{4^l}\sum_{|\sigma|=l}\|M_\sigma u\|^p < 1 \text{ for all } u \in H(\widetilde{v}), \|u\| \leq 1$$

2. Preliminaries

Lemma 2.1. *If equation(2) exists compactly supported* L^p*-solution* f*, then* $\text{supp} f \subset [0,M]\times[0,N]$.

Proof Let $\text{supp} f \subset D$, take $x \in D$ with $f(x) \neq 0$ then $Ax - \binom{m}{n} \in D$, i.e. $x \in A^{-1}(D + \binom{m}{n})$. Let $E = \{0,1,2\cdots M\}\times\{0,1,2\cdots N\}$, then

$$\begin{aligned} D &\subset A^{-1}(D+E) = A^{-1}D + A^{-1}E \\ &\subset A^{-1}(A^{-1}D + A^{-1}E) + A^{-1}E = A^{-2}D + A^{-2}E + A^{-1}E \\ &\subset \cdots \\ &\subset A^{-t}D + A^{-t}E + A^{-(t-1)}E + \cdots + A^{-1}E \end{aligned}$$

let $t \to \infty$, then

$$D \subset \{\sum_{t=1}^{\infty} A^{-t}y : y \in E\} \subset [0,M]\times[0,N]$$

for the closed set E.

Proposition 2.2. *Let* f *be supported by* $[0,M]\times[0,N]$*, and let* F *be defined as above, then* f *is an* L_c^p-*solution of (2) if and only if* $F \in L^p$ *and* $F = MF$*, i.e.* F *satisfies equation(3).*

Proposition 2.3. *If* $\sum\limits_{m=0}^{M}\sum\limits_{n=0}^{N} c_{mn} = 4$*, then* 4 *is an eigenvalue of* $(M_1+M_2+M_3+M_4)$ *with left eigenvalue* $[1,1,\cdots,1]$.

Proof Obviously, the sum of each column is equal to 4 in the matrix $(M_1+M_2+M_3+M_4)$. □

It follows that the right 4-eigenvector of $(M_1+M_2+M_3+M_4)$ exists also; it will play a central role in the existence of the solution of equation(2). Let $f_\triangle$ be the average of f over $\triangle$, i.e., $f_\triangle = \frac{1}{L(\triangle)}\int_\triangle f$.

Proposition 2.4. *Let* f *be an compactly supported* L^p*-solutionof equation(2),* $v = [f_{T+e_{00}}, f_{T+e_{01}}\cdots f_{T+e_{M-1,N-1}}]^T$ *be the vector defined by the average of* f *on the* $M\times N$ *subintervals as indicated. Then* v *is* 4*-eigenvector of* $(M_1+M_2+M_3+M_4)$*.*

Proof According to Proposition2.2, $F = MF$, i.e.,

$$F(x) = \begin{cases} M_1F(\varphi_1^{-1}(x)) & x \in T_1 = [0,1/2)\times[0,1/2) \\ M_2F(\varphi_2^{-1}(x)) & x \in T_2 = [0,1/2)\times[1/2,1) \\ M_3F(\varphi_3^{-1}(x)) & x \in T_3 = [1/2,1)\times[0,1/2) \\ M_4F(\varphi_4^{-1}(x)) & x \in T_4 = [1/2,1)\times[1/2,1) \end{cases} \quad (4)$$

when we integrate the expression over T_1, T_2, T_3 and T_4 separately, we have

$$[f_{[0,\frac{1}{2}]\times[0,\frac{1}{2}]}, \cdots, f_{[M-1,N-\frac{1}{2}]\times[M-1,N-\frac{1}{2}]}]^T = M_1 v$$

$$[f_{[0,\frac{1}{2}]\times[\frac{1}{2},1]}, \cdots, f_{[M-1,N-\frac{1}{2}]\times[M-\frac{1}{2},N]}]^T = M_2 v$$

$$[f_{[\frac{1}{2},1]\times[0,\frac{1}{2}]}, \cdots, f_{[M-\frac{1}{2},N]\times[M-1,N-\frac{1}{2}]}]^T = M_3 v$$

$$[f_{[\frac{1}{2},1]\times[\frac{1}{2},1]}, \cdots, f_{[M-\frac{1}{2},N]\times[M-\frac{1}{2},N]}]^T = M_4 v$$

On the other hand, note that on each interval $[i, i+1] \times [i, i+1]$ the average satisfies

$$f_{[i,i+\frac{1}{2}]\times[i,i+\frac{1}{2}]} + f_{[i,i+\frac{1}{2}]\times[i+\frac{1}{2},i+1]} + f_{[i+\frac{1}{2},i+1]\times[i,i+\frac{1}{2}]} + f_{[i+\frac{1}{2},i+1]\times[i+\frac{1}{2},i+1]} = 4f_{[i,i+1]\times[i,i+1]},$$

hence we conclude that $(M_1 + M_2 + M_3 + M_4)v = 4v$.

Let $\Sigma = \{1, 2, 3, 4\}$, $\Sigma^n = \{(i_1, i_2, \cdots, i_n) : i_j \in \Sigma\}$, $\Sigma^0 = \emptyset$, $\Sigma^* = \bigcup_{n=0}^{\infty} \Sigma^n$, $\Sigma^\infty = \{(i_1, i_2, \cdots) : i_j \in \Sigma\}$. For each $\sigma = (i_1, i_2, \cdots) \in \Sigma^\infty$, define $\sigma|_n = (i_1, i_2, \cdots, i_n)$. Let $\sigma = (i_1, i_2, \cdots, i_n) \in \Sigma^*, \tau = (j_1, j_2, \cdots, j_m) \in \Sigma^*$, define $(\sigma, \tau) := (i_1, i_2, \cdots, i_n, j_1, j_2, \cdots, j_m)$.$T_\sigma := \bigcup_{i=1}^{4} T_{(\sigma,i)}$, and $M_\sigma := M_{i_1} M_{i_2} \cdots M_{i_n}$. So for any $\sigma, \tau \in \Sigma^*$, we have $T_{(\sigma,\tau)} \subset T_\sigma$.

Lemma 2.5. *Let $F_0(x) = v$ for $x \in T$, and $F_{k+1} = MF_k$ for $k \geq 0$. Then $F_k = M_\sigma v$ for each $x \in T_\sigma$. Moreover, if f is an L_c^P-solution of equation(2) and v is the average vector of f defined in Proposition* 2.4*, then*

$$F_k = M_\sigma v = [f_{T_\sigma+e_{00}}, f_{T_\sigma+e_{01}} \cdots f_{T_\sigma+e_{M-1,N-1}}]^T,$$

where $(T_\sigma + j) = \{x + j : x \in T_\sigma\}$. Also, $F_k \to F$ in $L^p(T, R^{M\times N})$.

Proof We will use induction to show that $F_k(x) = M_\sigma v$ for $x \in T_\sigma$ with $|\sigma| = k$. Suppose that $F_k(x) = M_\sigma v$ for $x \in T_\sigma$. Let $x \in T_{(1,\sigma)} = \varphi_1(T_\sigma)$; then $\varphi_1^{-1}(x) \in T_\sigma$ and

$$F_{k+1}(x) = MF_k(x) = M_1 F_k(\varphi_1^{-1}(x)) = M_1 M_\sigma v = M_{(1,\sigma)} v.$$

Similarly, if $x \in T_{(i,\sigma)}$, then $F_{k+1}(x) = M_{(i,\sigma)} v, i = 2, 3, 4$.

Moreover, $F = MF$ and $F(x) = M_\sigma F(\varphi_\sigma^{-1}(x))$ for $x \in T_\sigma$. Integrating this over the interval T_σ, we obtain $[f_{T_\sigma+e_{00}}, f_{T_\sigma+e_{01}}, \cdots, f_{T_\sigma+e_{M-1,N-1}}]^T = M_\sigma v$.

Lemma 2.6. *Let v be a 4-eigenvector of $(M_1 + M_2 + M_3 + M_4)$, and let F_k be defined as above;then for each k,*

$$\int_T F_k(x)dx = v. \tag{5}$$

Proof This follows from the following induction argument:

$$\begin{aligned}
\int_T F_{k+1}\,dx &= \int_{T_1} M_1 F_k(\varphi_1^{-1}(x))\,dx + \int_{T_2} M_2 F_k(\varphi_2^{-1}(x))\,dx \\
&\quad + \int_{T_3} M_3 F_k(\varphi_3^{-1}(x))\,dx + \int_{T_4} M_4 F_k(\varphi_4^{-1}(x))\,dx \\
&= \frac{1}{4}\Big(M_1 \int_T F_k(x)\,dx + M_2 \int_T F_k(x)\,dx + M_3 \int_T F_k(x)\,dx + M_4 \int_T F_k(x)\,dx\Big) \\
&= \frac{1}{4}\Big(M_1 + M_2 + M_3 + M_4\Big) \int_T F_k(x)\,dx \\
&= \frac{1}{4}(M_1 + M_2 + M_3 + M_4)v = v.
\end{aligned}$$

Theorem 2.7. *For $1 \le p \le \infty$, the following are equivalent:*

(1) equation (2) has a nonzero compactly supported L^p-solution;

(2) there exists a 4-eigenvector v of $(M_1 + M_2 + M_3 + M_4)$ satisfying

$$\lim_{l\to\infty} \frac{1}{4^l} \sum_{|\sigma|=l} \|M_\sigma \widetilde{v}\|^p = 0$$

(3) there exists a 4-eigenvector v of $(M_1 + M_2 + M_3 + M_4)$ such that there exists an integer $l \ge 1$ such that

$$\frac{1}{4^l} \sum_{|\sigma|=l} \|M_\sigma u\|^p < 1 \text{ for all } u \in H(\widetilde{v}), \|u\| \le 1 \tag{6}$$

Proof Let $F_0 = v$ and $F_{n+1} = MF_n$. By Lemma (2.5), for $x \in T_\sigma$ and $|\sigma| = n, F_n(x) = M_\sigma v$. Let $G_n = F_{n+1} - F_n$; then $F_{n+1} = F_0 + G_0 + \cdots + G_n$, where

$$G_n(x) = \begin{cases} M_{(\sigma,1)}v + M_{(\sigma,3)}v - 2M_\sigma v = M_\sigma \widetilde{v} & \text{if } x \in T_{(\sigma,1)} \cup T_{(\sigma,3)}, \\ M_{(\sigma,2)}v + M_{(\sigma,4)}v - 2M_\sigma v = -M_\sigma \widetilde{v} & \text{if } x \in T_{(\sigma,2)} \cup T_{(\sigma,4)}, \end{cases}$$

and

$$\|G_n\|^p = \frac{1}{4^n} \sum_{|\sigma|=n} \|M_\sigma \widetilde{v}\|^p.$$

Since (1) implies that $\|G_n\|$ converges to zero, (2) follows immediately.

To prove that (2) implies (3), we note that $H(\widetilde{v})$ is finite dimensional and has a finite basis of $M_\tau \widetilde{v}$'s. Let $u = M_\tau \widetilde{v}$ with $|\tau| = k$; then

$$\frac{1}{4^n} \sum_{|\sigma|=n} \|M_\sigma u\|^p = \frac{1}{4^n} \sum_{|\sigma|=n} \|M_\sigma M_\tau \widetilde{v}\|^p \le 4^k \frac{1}{4^{n+k}} \sum_{|\sigma|=n+k} \|M_\sigma \widetilde{v}\|^p \to 0$$

as $n \to \infty$, and the convergence is uniform for all $\|u\| \le 1$. Hence (6) follows by taking $l = n$ for n sufficiently large.

Now assume (3) holds. Since $H(\widetilde{v})$ is finite dimensional, there is a constant $0 < c < 1$ such that for any $u \in H(\widetilde{v})$,

$$\frac{1}{4^l} \sum_{|\sigma|=l} \|M_\sigma u\|^p \leq c\|u\|^p.$$

For any $|\tau| = n$, let $u = M_\tau \widetilde{v} \in H(\widetilde{v})$; then

$$\frac{1}{4^l} \sum_{|\sigma|=l} \|M_\sigma M_\tau \widetilde{v}\|^p \leq c\|M_\tau \widetilde{v}\|^p.$$

Summing over all $|\tau| = n$, we have

$$\frac{1}{4^{l+n}} \sum_{|\sigma|=l+n} \|M_\sigma \widetilde{v}\|^p = \frac{1}{4^{l+n}} \sum_{|\sigma|=l} \sum_{|\tau|=n} \|M_\sigma M_\tau \widetilde{v}\|^p < \frac{c}{4^n} \sum_{|\tau|=n} \|M_\tau \widetilde{v}\|^p.$$

It follows from the expression of $\|G_n\|$ given above that

$$\|G_{n+l}\|^p \leq c\|G_n\|^p.$$

For each fixed n, $\{\|G_{n+kl}\|\}_{k=1}^{\infty}$ is dominated by a geometric series, hence $F_{n+1} = F_0 + G_0 + \cdots + G_n$ converges in L^p. The limit F is nonzero by Lemma (2.6), and so by Proposition (5), (1) follows.

Remark 2.8. we can also consider the equation

$$g(x) = \sum_{m=0}^{M} \sum_{n=0}^{N} d_{mn} g(Bx - \begin{pmatrix} m \\ n \end{pmatrix})$$

with $B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $\sum_{m=0}^{M} \sum_{n=0}^{N} d_{mn} = |\det B| = 2$, since iterating this equation again, we obtain the equation (2).

Corollary 2.9. *Under the same hypotheses of Theorem (2.7), assume that the solution f exists; then $v \notin H(\widetilde{v})$, and the dimension of $H(\widetilde{v})$ is at most $MN - 1$.*

Proof By Theorem (2.7)(2),

$$\frac{1}{4^n} \sum_{|\sigma|=n} \|M_\sigma u\|^p \to 0 \text{ for any } u \in H(\widetilde{v}).$$

It follows that if $v \in H(\widetilde{v})$, then

$$\|v\|^p = \frac{1}{4^{np}} \|(M_1 + M_2 + M_3 + M_4)^n v\|^p \leq \frac{1}{4^n} \sum_{|\sigma|=n} \|M_\sigma v\|^p \to 0$$

as $n \to \infty$. This contradicts $v \neq 0$.

3. Some Examples

Example 1: We consider a dilation equation :

$$f(x) = \sum_{m=0}^{1} \sum_{n=0}^{2} c_{mn} f(Ax - \binom{m}{n}) \tag{7}$$

where $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\sum\limits_{m=0}^{1} \sum\limits_{n=0}^{2} c_{mn} = 4$.

Theorem 3.1. *For $1 \leq p < \infty$, equation(7) has a (nonzero) L_c^p-solution if and only if either $c_{01} + c_{11} = 2$ and*

$$\frac{1}{4}(|c_{00} + c_{10}|^p + |2 - (c_{00} + c_{10})|^p) < 1$$

or $c_{00} + c_{10} = 2$ and $c_{02} + c_{12} = 2$.

Proof Note that

$$M_1 = \begin{pmatrix} c_{00} & 0 \\ c_{02} & c_{01} \end{pmatrix}, M_2 = \begin{pmatrix} c_{01} & c_{00} \\ 0 & c_{02} \end{pmatrix}, M_3 = \begin{pmatrix} c_{10} & 0 \\ c_{12} & c_{11} \end{pmatrix}, M_4 = \begin{pmatrix} c_{11} & c_{10} \\ 0 & c_{12} \end{pmatrix},$$

and $M_1 + M_2 + M_3 + M_4 = \begin{pmatrix} c_{00} + c_{10} + c_{01} + c_{11} & c_{00} + c_{10} \\ c_{02} + c_{12} & c_{01} + c_{11} + c_{02} + c_{12} \end{pmatrix}$.

If $(c_{00} + c_{10}, c_{02} + c_{12}) = (0, 0)$, then $M_1 + M_2 + M_3 + M_4 = 4I$. Any nonzero vector $v = [x, y]^T$ will be a 4-eigenvector. It is a direct calculation that $v \in H(\widetilde{v})$ and, by Corollary(2.9), no nonzero L_c^p-solution exists.

We assume that $(c_{00} + c_{10}, c_{02} + c_{12}) \neq (0, 0)$, then 4-eigenvector of $M_1 + M_2 + M_3 + M_4$ is $v = [c_{00} + c_{10}, c_{02} + c_{12}]^T$, so that

$$\widetilde{v} = (M_1 + M_3 - 2I)v = \begin{pmatrix} (c_{00} + c_{10})(c_{00} + c_{10} - 2) \\ (c_{02} + c_{12})(2 - (c_{02} + c_{12})) \end{pmatrix}. \tag{8}$$

For a L_c^p-solution exists, $H(\widetilde{v})$ can only be $\{0\}$ or one-dimensional(Corollary(2.9)).

In the first case, $\widetilde{v} = 0$, condition(6) is automatically satisfied. The only possible cases are

$$(c_{00} + c_{10}, c_{02} + c_{12}) = \{(2, 2), (0, 2), (2, 0)\}.$$

In the second case, $\widetilde{v} \neq 0$. Since $H(\widetilde{v})$ is invariant under $M_1 + M_3$ and $M_2 + M_4$, $(M_1 + M_3)\widetilde{v} = c\widetilde{v}$ for some c. Expression (8) yields the following cases:

(a) $c_{00} + c_{10} = 0$ or $c_{02} + c_{12} = 0$. In this case $v \in H(\widetilde{v})$ and Corollary(2.9) implies that (7) has no L_c^p-solution.

(b) $c_{00} + c_{10} = 2$ or $c_{02} + c_{12} = 2$. In this case a direct calculation shows that $(M_1 + M_3)\widetilde{v}, (M_2 + M_4)\widetilde{v}$ are independent. Since $H(\widetilde{v})$ is 2-dimensional and by Corollary(2.9) no L_c^p-solution exists.

(c) $c_{00}+c_{10} \neq 0, 2$ and $c_{02}+c_{12} \neq 0, 2$. Let $a = c_{00}+c_{10}, b = c_{02}+c_{12}$. By equating (8) and

$$(M_1 + M_3)\widetilde{v} = \begin{pmatrix} a^2(a-2) \\ b[(2-b)(4-a-b)+(a-2)a] \end{pmatrix} \tag{9}$$

with $(M_1 + M_3)\widetilde{v} = c\widetilde{v}$, we have $c = a$, so that by (8) and (9),

$$b[(2-b)(4-a-b)+(a-2)a] = ab(2-b);$$

that is

$$(a+b-4)(a+b-2) = 0.$$

Hence, either (i) or (ii) below holds.

(i) $a+b=4$. In this case $v = [a, 4-a]^T$ and $\widetilde{v} = (a-2)v$. Once again $v \in H(\widetilde{v})$ and no L_c^p-solution exists.

(ii) $a+b=2$. In this case a direct calculation shows that $(M_1+M_3)\widetilde{v} = a\widetilde{v}, (M_2+M_4)\widetilde{v} = b\widetilde{v}$. By Theorem(6), equation(7) has an L_c^p-solution if and only if there exists an integer $l \geq 1$ such that

$$\frac{1}{4^l}(|a|^p + |b|^p)^l \|\widetilde{v}\|^p = \frac{1}{4^l}\sum_{|\sigma|=l} \|M_\sigma \widetilde{v}\|^p < \|\widetilde{v}\|^p.$$

This is equivalent to

$$\frac{1}{4}(|a|^p + |2-a|^p) < 1,$$

i.e.,

$$\frac{1}{4}(|c_{00}+c_{10}|^p + |2-(c_{00}+c_{10})|^p) < 1.$$

The theorem follows by summarizing all the cases.

It follows directly from the theorem that if $a+b=2$ and if

(i) $a \in (-1, 3)$, then an $L_c^1 - solution$ exists;

(ii) $a \in (0, 2)$, then an $L_c^2 - solution$ exists;

(iii) $a = 1$, then an $L_c^p - solution$ exists for all $1 \leqslant p < \infty$.

Example 2: Considering a dilation equation as follows:

$$g(x) = \sum_{m=0}^{1}\sum_{n=0}^{1} d_{mn} g(Bx - \begin{pmatrix} m \\ n \end{pmatrix}) \tag{10}$$

where $B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $\sum\limits_{m=0}^{1}\sum\limits_{n=0}^{1} d_{mn} = 2.$. Iterating the equation once, we obtain the supp$g \subset [-1, 2] \times [0, 3]$.

Let $f(x) = g(x - \binom{1}{0})$, $A = B^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, and let

$$(c_{mn})_{0 \leq m, n \leq 3} = \begin{pmatrix} 0 & d_{00}d_{10} & d_{01}d_{10} & 0 \\ d_{00}^2 & d_{00}d_{01} + d_{10}^2 & d_{00}d_{11} + d_{10}d_{11} & d_{01}d_{11} \\ d_{00}d_{10} & d_{00}d_{11} + d_{00}d_{01} & d_{01}^2 + d_{10}d_{11} & d_{11}^2 \\ 0 & d_{01}d_{10} & d_{01}d_{11} & 0 \end{pmatrix}$$

then (10) is rewritten as the form

$$f(x) = \sum_{m=0}^{3}\sum_{n=0}^{3} c_{mn} f(Ax - \binom{m}{n}) \tag{11}$$

with supp$f \subset [0,3] \times [0,3]$, so the discussion of the equation (11) is similar with the Example 1.

References

[1] Rham, G.de.. Sur un example de fonction continue sans d$\acute{e}$riv$\acute{e}$e. Enseign. Math., **3**(1957) 71-72.

[2] Heil,C. Mathods of solving dilation equations, Proc.1991 Nato Adv.Sci.Ins.on Prob. and Stoch. Methods in Anal.with Appl.,J. Byrnes,ed.,Kluwer Academic Publishers,Dordrecht, 1993.

[3] Daubechies,I. & Lagarias J. Two-scale difference equation I.Existence and global regulatity of solutions. Siam J. Math.Anal., **22**(1991) 1388-1410.

[4] Daubechies,I. & Lagarias J. Two-scale difference equation II. Local regulatity, infinite products of matrices,and fractals, Siam J. Math.Anal., **23**(1992) 1031-1079.

[5] Lawton, W. Necessary and sufficient conditions for constructing orthonormal wavelet bases, J.Math.Phys., **32**(1991) 57-64.

[6] Mallalt, S. Multiresolution approximation and wavelet orthonormal bases for $L^2(\mathbb{R})$, Trans. Amer. Math. Soc., **315**(1989) 69-87.

[7] Michelli, C.A. & Prautzsch,H. Uniform refinement of curves, Linear Algebra Appl., **114/115**(1989) 841-870.

[8] Colella, D. & Heil,C. The characterization of continuous,four-coefficient scaling functions and wavelets, IEEE.Trans.Inform.Theory, **30**(1992) 876-881.

[9] Colella, D. & Heil,C. Characterizations of scaling functions, I.Continuous solutions,J.Math.Anal.Appl., **15**(1994) 496-518.

[10] Lau K.L. & Wang,J.R. *Characterization of* L^p-solutions for the two-scale dilation equations, SIAM J.Math.Anal., **26**(1995),no.4 1018-1046.