Existence, Uniqueness And Asymptotic Properties of Neutrastochastic Functional Differential Equations with Markovian Switching

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Abstract *This paper considers the existence and uniqueness of solution to neutral stochastic functional differential equation with Markovian switching with local Lipschitz condition but neither the linear growth condition. And we discuss the asymptotic properties of this solution including moment boundedness and moment average boundedness in time. A One-dimension nonlinear example is discussed to illustrate the theory.*

Keywords *Moment boundedness Lyapunov function Stochastic functional di fferential equations Markovian Switching Generalized Ito formula*

1. Introduction and Preliminaries

Many practical systems may experience abrupt changes in their structure and parameters caused by phenomena such as component failures or repairs, changing subsystem interconnections, and abrupt environmental disturbances. The hybrid systems driven by continuous-time Markov chains have recently been developed to cope with such situation, which have therefore received a great deal of attention, and have played a more and more important role in recent years. Stochastic functional differential equations with Markovian switching have been studied by many authors, and we here mention [8-12], in which they mainly discuss the asymptotic property of the solution, including the stability and moment boundedness and so on with the linear growth condition. Kolmanovskii [1] studied the neutral stochastic differential delay equations with Markovian switching, and discussed the existence and uniqueness of the solution of the equation and the moment asymptotic boundedness and moment exponential stability. Mao [2] discussed the almost surely asymptotic stability of NSDDE. In this paper, we will mainly consider neutral stochastic functional differential equations with Markovian switching and discuss the existence and uniqueness of a global solution without the linear growth condition, and asymptotic properties including moment boundedness and moment average boundedness in time of the this global solution.

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Consider the neutral stochastic functional differential equations with Markovian switching of the form:

$$d[x(t)-u(x_t, r(t))] = f(x(t), x_t, r(t))dt + g(x(t), x_t, r(t))dW(t), \qquad (1)$$

where $x_t(\theta) = x(t+\theta), \theta \in [-\tau, 0]$ which is regarded as in $C([-\tau,0]; R^n), r(t)(t \ge 0)$ is a right-continuous Markovian chain on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$. Moreover

$$f : R^n \times C([-\tau,0]; R^n) \times S \to R^n, \qquad g : R^n \times C([-\tau,0]; R^n) \times S \to R^{n\times m},$$

$$u : C([-\tau,0]; R^n) \times S \to R^{n\times m}.$$

Let $\{\Omega, F, \{F_t\}_{t\ge 0}, P\}$ be a complete probability space with a filtration $\{F_t\}_{t\ge 0}$ satisfying the usual conditions (i.e. it is right continuous and F_0 contains all P-null sets). Let $W(t)(t \ge 0)$ be an *m*-dimensional Brownian motion defined on this space. Let $\tau > 0$ and $C([-\tau,0]; R^n)$ denote the family of continuous functions φ from $[-\tau,0]$ to R^n with the norm $\|\varphi\| = \sup_{-\tau\le\theta\le 0} |\varphi(\theta)|$. If *A* is a vector or matrix, its transpose is denoted by A^T. Let $r(t) t \ge 0$, be a right-continuous Markovian chain on the probability space taking values in a finite state space $S = \{1,2,\cdots,N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\{r(t+\Delta) = j \mid r(t) = i\} = \begin{cases} \gamma_{ij} + o(\Delta), i \ne j \\ 1+\gamma_{ii}\Delta + o(\Delta), i = j \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \ge 0$ is the transition rate from *i* to *j* if $i \ne j$ while $\gamma_{ii} = -\sum_{j\ne i} \gamma_{ij}$. We assume that the Markovian chain $r(\cdot)$ is independent of the Brownian motion $W(\cdot)$.

For any $V(x,i) \in C^2(R^n \times S; R)$, define an operator LV from $R^n \times C([-\tau,0]; R^n) \times S$ to R by

$$LV(x,\varphi,i)=V_x(x-u(\varphi,i),i)f(x,\varphi,i)+\frac{1}{2}trace[g^T(x,\varphi,i)V_{xx}(x-u(\varphi,i),i)g(x,\varphi,i)]$$

$$+\sum_{j=1}^{N}\gamma_{ij}V(x-u(\varphi,i),\,j), \tag{2}$$

where

$$V_x(x,i)=\left[\frac{\partial V(x,i)}{\partial x_1},\cdots,\frac{\partial V(x,i)}{\partial x_n}\right],V_{xx}(x,i)=\left[\frac{\partial^2 V(x,i)}{\partial x_i\partial x_j}\right]_{n\times n}.$$

If x(t) is a solution to Eq.(1) and let $z(t)=x(t)-u(x_t,r(t))$ (as is the following), then by the generalized $It\hat{o}$ formula, we have

$$EV(z(t),r(t))-EV(z(0),r(0))=E\int_0^t LV(z(s),r(s))ds,$$

where $LV(z(t),r(t))=LV(x(t),x_t,r(t))$.

In this paper the following assumptions are imposed as standing hypothesis.

Assumption 1.1 Both f and g are locally Lipschitz continuous.

Assumption 1.2 For each $i\in S$, there is constant $\kappa_i\in(0,1)$ such that

$$|u(\varphi,i)-u(\psi,i)|\le\kappa_i\int_{-\tau}^{0}|\varphi(\theta)-\psi(\theta)|d\zeta(\theta), \tag{3}$$

where ζ is a probability measure and those $\varphi,\psi\in C([-\tau,0];R^n)$.

Assume moreover that u(0,i)=0, f(0,0,i)=0, g(0,0,i)=0.

In general, these assumptions will only guarantee a unique maximal local solution to Eq.(1) for any given initial data $\xi\in C([-\tau,0];R^n)$ and $r(0)=i_0\in S$. However, the additional conditions imposed in it, we will guarantee that this maximal local solution is in fact a unique global solution, which is denoted by $x(t,\xi,i_0)$, and this solution has properties

$$\limsup_{x\to\infty}E\left|x(t,\xi,i_0)\right|^p\le K_p,\quad \limsup_{x\to\infty}\frac{1}{t}\int_0^t E\left|x(t,\xi,i_0)\right|^{\alpha+p}ds\le K^*{}_{\alpha+p}, \tag{4}$$

where $\alpha\ge0$ and $p>0$ are proper parameters, K_p and $K^*{}_{\alpha+p}$ are positive constants independent of ξ and i_0.

For the convenience of reference, several elementary inequalities are given in the following which will be used frequently. For any $x, y \in R^n$,

$$x^\alpha y^\beta \le \frac{\alpha x^{\alpha+\beta} + \beta y^{\alpha+\beta}}{\alpha+\beta}, \alpha, \beta > 0. \tag{5}$$

$$(x+y)^p \le (1-\varepsilon)^{1-p} x^p + \varepsilon^{1-p} y^p, p \ge 1, 0 < \varepsilon < 1. \tag{6}$$

$$(x+y)^p \le x^p + y^p, 0 < p \le 1. \tag{7}$$

$$(x+y)^2 \le \frac{x^2}{\delta} + \frac{y^2}{1-\delta}, 0 < \delta < 1. \tag{8}$$

Before we state our main results, let us cite several useful lemmas.

Lemma 1.3 For any $h(x) \in C(R^n; R), \alpha, b > 0,$ when $|x| \to \infty,$ $h(x) = o(|x|^\alpha),$ then

$$\sup_{x \in R^n} [h(x) - b|x|^\alpha] < \infty.$$

In this paper, when we use the notation $o(|x|^\alpha)$, it is always under the condition $|x| \to \infty$. In addition, throughout this paper, *const* represents a positive constant, whose precise value or expression is not important. $I(x) \le const$ always implies that $I(x)(x \in R^n)$ is bounded above. Note that the notation $o(|x|^\alpha)$ includes the continuity. Hence Lemma 1.3 can be rewritten as

$$-b|x|^\alpha + o(|x|^\alpha) \le const.$$

In this paper, let $V(x,i) = (x^T Q_i x)^{p/2} (x \in R^n)$. $Q_i \in R^{n \times n} (i \in S)$ are positive definite matrices and $p > 0$. Clearly, we have

$$q_i^{p/2} |x|^p \le V(x,i) \le \|Q_i\|^{p/2} |x|^p, \tag{9}$$

here $q_i = \lambda_{\min}(Q_i)$. By (2), we have

$$LV(x,\varphi,i) = \frac{p}{2} (z^T Q_i z)^{p/2 - 1} [2 z^T Q_i f(x,\varphi,i) + g^T(x,\varphi,i) Q_i g(x,\varphi,i)]$$

$$+\frac{p(p-2)}{2}(z^TQ_iz)^{p/2-2}[z^TQ_ig(x,\varphi,i)]^2+\sum_j\gamma_{ij}(z^TQ_iz)^{p/2}, \quad (10)$$

where $z=x-u(\varphi,i)$.

Lemma 1.4 Let I be the last term of (10), then we have

$$I\le M_p|z|^p, \quad (11)$$

Where $M_p=\max_i(\gamma_{ii}q_i^{\frac{p}{2}}+\sum_{j\ne i}\gamma_{ij}\|Q_j\|^{\frac{p}{2}})\ge 0.$

Proof Clearly, (11) is obtained directly. We only need to prove that $M_p\ge 0$. We may suppose $\|Q_1\|\le\|Q_2\|\le\cdots\le\|Q_N\|$. Noting that $\|Q_1\|\ge q_1$, then

$$M_p\ge\gamma_{11}q_1^{\frac{p}{2}}+\sum_{j>1}\gamma_{1j}\|Q_j\|^{\frac{p}{2}}\ge\gamma_{11}q_1^{\frac{p}{2}}+\sum_{j>1}\gamma_{1j}\|Q_1\|^{\frac{p}{2}}\ge q_1^{\frac{p}{2}}\sum_{j>1}\gamma_{1j}=0.$$

Lemma 1.5 Assume $p\ge 1$, let x(t) be a solution of Eq.(1) with $x_0=\xi$, we have

$$\limsup_{t\to\infty}E|x(t)|^p\le(1-\breve{\kappa})^{-p}\limsup_{t\to\infty}E|z(t)|^p, \quad (12)$$

Where $z(t)=x(t)-u(x_t,r(t)),\breve{\kappa}=\max_{1\le i\le n}\{\kappa_i\}$.

Proof By (3) and (6), we have

$$E|x(t)|^p\le(1-\breve{\kappa})^{1-p}+\breve{\kappa}\int_{-\tau}^0E|x(t+\theta)|^p d\zeta(\theta)$$
$$\le(1-\breve{\kappa})^{1-p}\sup_{0\le s\le t}E|z(s)|^p+\breve{\kappa}\sup_{-\tau\le s\le t}E|x(s)|^p.$$

This implies $\sup_{-\tau\le s\le t}E|x(s)|^p\le\|\xi\|_0^{\ p}+(1-\breve{\kappa})^{1-p}\sup_{0\le s\le t}E|z(s)|^p+\breve{\kappa}\sup_{-\tau\le s\le t}E|x(s)|^p$. So, we can get $\sup_{-\tau\le s\le t}E|x(s)|^p<\infty$. Then $\limsup_{t\to\infty}E|x(t)|^p\le(1-\breve{\kappa})^{-p}\limsup_{t\to\infty}E|z(t)|^p$.

2. A Basic Lemma

The following lemma plays a key role in this paper.

Lemma 2.1 Under Assumptions 1.1 and 1.2, let $p \geq 1,$ if there exist constants $\alpha \geq 0, a_i, \alpha_j, \varepsilon, K_{i0}, K_{ij} > 0 (i \in S, 1 \leq j \leq m),$ positive definite matrices Q_i and probability measures μ_j, such that

$$LV(x, \varphi, i) + \varepsilon V(x - u(\varphi, i), i) \leq \\ -a_i |x|^{\alpha+p} + K_{i0} + \sum_j K_{ij} (\int_{-\tau}^{0} |\varphi(\theta)|^{\alpha_j} \, d\mu_j(\theta) - e^{\varepsilon\tau} |x|^{\alpha_j}), \tag{13}$$

then for any initial data $\xi \in C([-\tau, 0], R^n)$ and $r(0) = i_0 \in S$ there exists a unique global solution $x(t, \xi, i_0)$ to Eq.(1) and this solution satisfies (4).

Proof For any given initial data $\xi \in C([-\tau, 0]; R^n)$ and $i_0 \in S$, write $x(t, \xi, i_0) = x(t)$, we will divide the whole proof into three steps.

Step 1 Let us first show the existence of the global solution x(t). Under Assumption 1.1 and 1.2, Eq.(1) admits a unique maximal local solution $x(t)(-\tau \leq t < \sigma)$, where σ is the explosion time. Let $z(t) = x(t) - u(x_t, r(t))$, define the stopping time

$$\sigma_k = \inf \left\{ 0 \leq t < \sigma : V(z(t), r(t)) > k \right\}, (k \in N).$$

Since ξ is bounded, when k is large enough, $V(z(\theta), r(\theta)) \leq k$ for $-\tau \leq \theta < \sigma$, thus, $\sigma_k \geq 0$. If $\sigma < \infty$, when $t \to \sigma$, z(t) may explode. Hence,

$$\left\{ -\tau \leq t < \sigma : V(z(t), r(t)) > k \right\} \neq \varnothing, (k \in N)$$

shows that $\sigma_k \leq \sigma$. Thus, we may assume $0 \leq \sigma_k \leq \sigma (k \in N)$. Obviously, σ_k is increasing and $\sigma_k \to \sigma_\infty \leq \sigma (k \to \infty) a.s.$. If we could show $\sigma_\infty = \infty, a.s.$, then $\sigma = \infty a.s.$. Thus it need only, for any $t > 0$, $P(\sigma_k \leq t) \to 0$ as $k \to \infty$.

Fix $t > 0$. Now we prove that $P(\sigma_k \leq t) \to 0$ as $k \to \infty$. First note that if $\sigma_k < \infty$, then by the continuity of x(t) and the right continuity of r(t), $V(z(\sigma_k), r(\sigma_k)) = k$. Hence, by (13), we have

$$kP(\sigma_k \le t) = V(z(\sigma_k), r(\sigma_k))P(\sigma_k \le t) \le EV(z(t\wedge\sigma_k), r(t\wedge\sigma_k))$$

$$\le EV(z(0), i_0) + \int_0^{t\wedge\sigma_k} LV(z(s), r(s))ds$$

$$\le EV(z(0), i_0) + E\int_0^{t\wedge\sigma_k}\left\{K_{r0} + \sum_j K_{rj}[\int_{-\tau}^0 |x(s+\theta)|^{\alpha_j}\, d\mu_j(\theta) - |x(s)|^{\alpha_j}]\right\}ds$$

$$\le EV(z(0), i_0) + \breve{K}_0 t + \sum_j \breve{K}_j E\left\{\int_{-\tau}^0 d\mu_j(\theta)\int_{-\tau}^{t\wedge\sigma_k} |x(s)|^{\alpha_j}\, ds - \int_0^{t\wedge\sigma_k} |x(s)|^{\alpha_j}\, ds\right\}$$

$$\le V(\xi(0) - u(\xi(0), i_0), i_0) + \breve{K}_0 t + \sum_j \breve{K}_j \int_{-\tau}^0 |\xi(\theta)|^{\alpha_j}\, d\theta =: K_t,$$

where the index r represents r(t), $\breve{K}_j = \max_i K_{ij} (0 \le j \le m)$, and K_t is a positive constant independent of k. So we can get $P(\sigma_k \le t) \le k^{-1}K_t \to 0 (k \to \infty).$ That shows that x(t) is a global solution to Eq.(1).

Step 2 Let us now show inequality (4). By (13), we obtain that

$$e^{\varepsilon\tau}EV(z(t), r(t)) = EV(z(0), r(0)) + E\int_0^t L[e^{\varepsilon s}V(z(s), r(s))]ds$$

$$\le EV(z(0), r(0)) + E\int_0^t e^{\varepsilon s}\left\{K_{r0} + \sum_j K_{rj}[\int_{-\tau}^0 |x(s+\theta)|^{\alpha_j}\, d\mu_j(\theta) - e^{\varepsilon\tau}|x(s)|^{\alpha_j}]\right\}ds$$

$$\le V(\xi(0) - u(\xi(0), i_0), i_0) + \varepsilon^{-1}\breve{K}_0(e^{\varepsilon t} - 1) + \sum_j \breve{K}_j \int_{-\tau}^0 e^{\varepsilon(\theta+\tau)}|\xi(\theta)|^{\alpha_j}\, d\theta =: c_1 + Ke^{\varepsilon t},$$

where c_1 is a positive constant independent of t and $K = \varepsilon^{-1}\breve{K}_0$ is a positive constant independent of ξ and i_0. Hence, we have $\limsup_{t\to\infty} EV(z(t), r(t)) \le K.$ Then the required assertion (4) follows from (9) and (12).

Step 3 Finally, using (13), we obtain that

$$\hat{a}\int_0^t E|x(s)|^{\alpha+P}\, ds$$

$$\le E\int_0^t\left\{-LV(z(s), r(s)) + K_{r0} + \sum_j K_{rj}[\int_{-\tau}^0 |x(s+\theta)|^{\alpha_j}\, d\mu_j(\theta) - |x(s)|^{\alpha_j}]\right\}ds$$

$$\le V(\xi(0) - u(\xi(0), i_0), i_0) + \breve{K}_0 t + \sum_j \breve{K}_j \int_{-\tau}^0 |\xi(\theta)|^{\alpha_j}\, d\theta =: c_2 + \breve{K}_0 t,$$

where $\hat{a} = \min_i a_i$ and c_2 is a positive constant independent of t. The assertion (4) follows directly. The proof is therefore complete.

Denote the left side of (13) by ϕ and establish the inequality

$$\phi \le \sum_j K_{ij} \left(\int_{-\tau}^{0} |\varphi(\theta)|^{\alpha_j} \, d\mu_j(\theta) - e^{\varepsilon\tau} |x|^{\alpha_j} \right) + I, \tag{14}$$

where

$$I = -a_i |x|^{\alpha+p} + o(|x|^{\alpha+p}). \tag{15}$$

By Lemma 2.1, we have $-\frac{a_i}{2}|x|^{\alpha+p} + o(|x|^{\alpha+p}) \le const.$ This together with (15) yields $I \le -\frac{a_i}{2}|x|^{\alpha+p} + const.$ Substituting this into (14) shows that the condition (13) are required. To get (14) and (15), some conditions imposing on the coefficients f and g. These conditions are considered in the next section.

3. Main Results

Recall ϕ to denote the left hand of (13). If p>2, by (9) (10) and (11)

$$\begin{aligned}\phi \le\ & p(z^T Q_i z)^{p/2-1} x^T Q_i f(x,\varphi,i) + p(z^T Q_i z)^{p/2-1} |u(\varphi,i)| \|Q_i\| |f(x,\varphi,i)| \\ & + \frac{p(p-1)}{2} \|Q_i\|^{p/2} |z|^{p-2} |g(x,\varphi,i)|^2 + [M_p + \varepsilon \|Q_i\|^{p/2}] |z|^p =: I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{16}$$

We firstly list the following conditions that we will need:

(H1) There exist $\alpha, a_i, \sigma_i \ge 0,$ positive-definite matrices Q_i and a probability measure μ, such that

$$x^T Q_i f(x,\varphi,i) \le -a_i |x|^{\alpha+2} + \sigma_i \int_{-\tau}^{0} |\varphi(\theta)|^{\alpha+2} \, d\mu(\theta) + o(|x|^{\alpha+2}).$$

(H2) There exist $\alpha > 0, r_i, \overline{r}_i \ge 0$ and a probability measure μ, such that

$$|f(x,\varphi,i)| \le r_i |x|^{\alpha+1} + \overline{r}_i \int_{-\tau}^{0} |\varphi(\theta)|^{\alpha+1} \, d\mu(\theta) + o(|x|^{\alpha+1}).$$

(H3) There exist $\beta > 0, \lambda_i, \overline{\lambda}_i \ge 0,$ positive-definite matrices Q_i and a probability measure ν, such that

$$|g(x,\varphi,i)| \le \lambda_i |x|^{\beta+1} + \overline{\lambda}_i \int_{-\tau}^{0} |\varphi(\theta)|^{\beta+1} \, d\nu(\theta) + o(|x|^{\beta+1}).$$

We can now state our main result in this paper.

Theorem 4.1 Under Assumptions 1.1 and 1.2, if the conditions (H1)-(H3) hold, $\alpha \ge 2\beta, 2 < p \le 3$ and

$$a_i > \frac{1+\kappa_i^{p-2}}{1-\kappa_i^{p-2}}[\sigma_i + \|Q_i\| \frac{(r_i+\bar{r}_i)(p\kappa_i^{p-2}+p-2)}{(p-1)(1+\kappa_i^{p-2})}$$

$$+\|Q_i\| \frac{(p-1)(\lambda_i+\bar{\lambda}_i)^2}{2}(1-\operatorname{sgn}(\alpha-2\beta))], \tag{17}$$

then for any initial data $\xi \in C([-\tau,0],R^n)$ and $r(0)=i_0 \in S$ there exists a unique global solution $x(t,\xi,i_0)$ to Eq.(1) and this solution satisfies (4).

Proof Let $x(t)=x(t,\xi,i_0)$ and ε be sufficiently small. Now we estimate $I_1 - I_4$ respectively. First, by the condition (H1), the inequalities (5) and (7), we can have

$$I_1 \le -a_i p \|Q_i\|^{p/2-1} |x|^{\alpha+2} [|x|^{p-2} - \kappa_i^{p-2} \int_{-\tau}^0 |\varphi|^{p-2} d\zeta]$$

$$+p\|Q_i\|^{p/2-1}[|x|^{p-2} + \kappa_i^{p-2}\int_{-\tau}^0 |\varphi|^{p-2} d\zeta][\sigma_i \int_{-\tau}^0 |\varphi|^{\alpha+2} d\mu + o(|x|^{\alpha+2})]$$

$$\le p\|Q_i\|^{p/2-1}[-a_i|x|^{\alpha+p} + a_i\kappa_i^{p-2}\int_{-\tau}^0 \frac{(p-2)|\varphi|^{\alpha+p}+(\alpha+2)|x|^{\alpha+p}}{\alpha+p} d\zeta$$

$$+\sigma_i \int_{-\tau}^0 \frac{(p-2)|x|^{\alpha+p}+(\alpha+2)|\varphi|^{\alpha+p}}{\alpha+p} d\mu + \int_{-\tau}^0 o(|\varphi|^{\alpha+p}) d\zeta + o(|x|^{\alpha+p})$$

$$+\kappa_i^{p-2}\sigma_i \int_{-\tau}^0\int_{-\tau}^0 \frac{(p-2)|\varphi(s)|^{\alpha+p}+(\alpha+2)|\varphi(\theta)|^{\alpha+p}}{\alpha+p} d\zeta(s) d\mu(\theta)]. \tag{18}$$

Next, by the condition (H2), the inequalities (5) and (7), we obtain

$$I_2 \le \frac{p}{p-1}\|Q_i\|^{p/2}[(p-2)|x|^{p-1} + p\kappa_i^{p-1}\int_{-\tau}^0 |\varphi|^{p-1} d\zeta]$$

$$[r_i|x|^{\alpha+1} + \bar{r}_i \int_{-\tau}^0 |\varphi(\theta)|^{\alpha+1} d\mu(\theta) + o(|x|^{\alpha+1})]$$

$$\le \frac{p}{p-1}\|Q_i\|^{p/2}[(p-2)r_i|x|^{\alpha+p} + (p-2)\bar{r}_i \int_{-\tau}^0 \frac{(p-1)|x|^{\alpha+p}+(\alpha+1)|\varphi|^{\alpha+p}}{\alpha+p} d\mu$$

$$+p\kappa_i^{\,p-1}r_i\int_{-\tau}^{0}\frac{(p-1)|\varphi|^{\alpha+p}+(\alpha+1)|x|^{\alpha+p}}{\alpha+p}d\zeta+\int_{-\tau}^{0}o(|\varphi|^{\alpha+p})d\zeta+o(|x|^{\alpha+p})$$

$$+p\kappa_i^{\,p-1}\overline{r}_i\int_{-\tau}^{0}\int_{-\tau}^{0}\frac{(p-1)|\varphi(s)|^{\alpha+p}+(\alpha+1)|\varphi(\theta)|^{\alpha+p}}{\alpha+p}d\zeta(s)d\mu(\theta)]. \qquad (19)$$

Then by the condition (H3) and the inequalities (5), (7) and (8), we can get

$$I_3\le\frac{p(p-1)}{2\nu}\|Q_i\|^{p/2}[|x|^{p-2}+\kappa_i^{\,p-2}\int_{-\tau}^{0}|\varphi|^{p-2}d\zeta]$$

$$[\frac{\lambda_i^2|x|^{2\beta+2}}{\delta_i}+\frac{\overline{\lambda}_i^2\int_{-\tau}^{0}|\varphi|^{2\beta+2}d\nu}{1-\delta_i}+o(|x|^{2\beta+2})]$$

$$\le\frac{p(p-1)}{2\nu}\|Q_i\|^{p/2}[\frac{\lambda_i^2}{\delta_i}|x|^{2\beta+p}+\frac{\lambda_i^2\kappa_i^{\,p-2}}{\delta_i}\int_{-\tau}^{0}\frac{(p-2)|\varphi|^{2\beta+p}+(2\beta+2)|x|^{2\beta+p}}{2\beta+p}d\zeta$$

$$+\frac{\overline{\lambda}_i^2}{1-\delta_i}\int_{-\tau}^{0}\frac{(2\beta+2)|\varphi|^{2\beta+p}+(p-2)|x|^{2\beta+p}}{2\beta+p}d\nu+\int_{-\tau}^{0}o(|\varphi|^{2\beta+p})d\zeta+o(|x|^{2\beta+p})$$

$$+\frac{\overline{\lambda}_i^2\kappa_i^{\,p-2}}{1-\delta_i}\int_{-\tau}^{0}\int_{-\tau}^{0}\frac{(p-2)|\varphi(s)|^{2\beta+p}+(2\beta+2)|\varphi(\theta)|^{2\beta+p}}{2\beta+p}d\zeta(s)d\nu(\theta)], \qquad (20)$$

where $\delta_i,\nu\in(0,1)$ are constants.

It is easy to see that

$$I_4\le[M_p+\varepsilon\|Q_i\|^{p/2}][(1-\kappa_i)^{1-p}|x|^p+\kappa_i\int_{-\tau}^{0}|\varphi|^p d\zeta]. \qquad (21)$$

Then substituting (18)-(21) into (16), we can get ϕ whose form is similar to (14), where

$$I=p\|Q_i\|^{p/2-1}\{-a_i+a_i\kappa_i^{\,p-2}\frac{e^{\varepsilon\tau}(p-2)+(\alpha+2)}{\alpha+p}+\sigma_i\frac{(p-2)+e^{\varepsilon\tau}(\alpha+2)}{\alpha+p}$$

$$+\kappa_i^{\,p-2}\sigma_i e^{\varepsilon\tau}+\frac{\|Q_i\|^{p/2}}{p-1}[(p-2)\overline{r}_i\frac{(p-1)+e^{\varepsilon\tau}(\alpha+1)}{\alpha+p}+p\kappa_i^{\,p-1}r_i\frac{e^{\varepsilon\tau}(p-1)+(\alpha+1)}{\alpha+p}$$

$$+(p-2)r_i+p\kappa_i^{\,p-1}\overline{r}_i e^{\varepsilon\tau}]\}|x|^{\alpha+p}+\frac{p(p-1)}{2\nu}\|Q_i\|^{p/2}[\frac{\lambda_i^2}{\delta_i}+\frac{\overline{\lambda}_i^2\kappa_i^{\,p-2}}{1-\delta_i}e^{\varepsilon\tau}+\frac{\lambda_i^2\kappa_i^{\,p-2}}{\delta_i}$$

$$\frac{e^{\varepsilon\tau}(p-2)+(2\beta+2)}{2\beta+p}+\frac{\overline{\lambda}_i^2}{1-\delta_i}\frac{e^{\varepsilon\tau}(2\beta+2)+(p-2)}{2\beta+p}]|x|^{2\beta+p}+o(|x|^{\alpha+p})+o(|x|^{2\beta+p}).$$

If $\alpha>2\beta$, then we have $I=-p\|Q_i\|^{p/2-1}\overline{a}_i|x|^{\alpha+p}+o(|x|^{\alpha+p})$, where

$$\overline{a}_i=a_i[1-\kappa_i^{\,p-2}\frac{e^{\varepsilon\tau}(p-2)+(\alpha+2)}{\alpha+p}]-\{\sigma_i\frac{(p-2)+e^{\varepsilon\tau}(\alpha+2)}{\alpha+p}+\kappa_i^{\,p-2}\sigma_i e^{\varepsilon\tau}$$

$$+\frac{\|Q_i\|}{p-1}[(p-2)\overline{r}_i\frac{(p-1)+e^{\varepsilon\tau}(\alpha+1)}{\alpha+p}+p\kappa_i^{\,p-1}r_i\frac{e^{\varepsilon\tau}(p-1)+(\alpha+1)}{\alpha+p}$$

$$+(p-2)r_i+p\kappa_i^{\,p-1}\overline{r}_i e^{\varepsilon\tau}]\}=:\overline{a}_i(\varepsilon).$$

By (17), we have $\overline{a}_i(0)>0$. Since ε is sufficiently small, we get $\overline{a}_i>0$. Therefore, the form of I is similar to (15).

If $\alpha=2\beta$, then we can get $I=-p\|Q_i\|^{p/2-1}\tilde{a}_i|x|^{\alpha+p}+o(|x|^{\alpha+p})$, where

$$\tilde{a}_i=\overline{a}_i-\frac{p-1}{2v}\|Q_i\|[\frac{\lambda_i^2}{\delta_i}+\frac{\overline{\lambda}_i^2\kappa_i^{\,p-2}}{1-\delta_i}e^{\varepsilon\tau}+\frac{\lambda_i^2\kappa_i^{\,p-2}}{\delta_i}\frac{e^{\varepsilon\tau}(p-2)+(\alpha+2)}{\alpha+p}$$

$$+\frac{\overline{\lambda}_i^2}{1-\delta_i}\frac{e^{\varepsilon\tau}(\alpha+2)+(p-2)}{\alpha+p}]=:\tilde{a}_i(\varepsilon,v).$$

Choosing that $\delta_i=\lambda_i/(\lambda_i+\overline{\lambda}_i)$ and by (17), we get $\tilde{a}_i(\varepsilon,v)>0$. Then we also have $\tilde{a}_i>0$, and the form of I is similar to (15).

Thus, by Lemma 2.1, we can get that for any initial data ξ and i_0, there exists a unique global solution $x(t,\xi,i_0)$ to Eq.(1) and this solution satisfies (4).

Theorem 4.2 Under Assumptions 1.1 and 1.2, if the conditions (H1)-(H3) hold, $\alpha\ge 2\beta, 2<p\le 3$ and

$$a_i>\left(\frac{1+\kappa_i}{1-\kappa_i}\right)^{p-2}[\sigma_i+\|Q_i\|\frac{(r_i+\overline{r}_i)(\kappa_i^{\,p-1}+p\kappa_i^2-\kappa_i^2+p-2)}{(p-1)(1+\kappa_i)}$$

$$+\|Q_i\|\frac{(p-1)(\lambda_i+\overline{\lambda}_i)^2}{2}(1-\text{sgn}(\alpha-2\beta))], \tag{22}$$

then for any initial data $\xi \in C([-\tau,0],R^n)$ and $r(0)=i_0 \in S$ there exists a unique global solution $x(t,\xi,i_0)$ to Eq.(1) and this solution satisfies (4).

The proof is mostly the same as the one we provided previously, only when we estimate the $I_1 - I_3$, we use the inequality (6) not (7).

4. One-dimension Nonlinear Example

Let us discuss a one-dimension nonlinear neutral stochastic functional differential with Markovian switching to illustrate our theory.

$$d[x(t)-0.1x(t-1)] = [-b_r x^4(t) + cx^2(t) + d + \int_{-1}^{0} x^4(t+\theta)d\theta]dt + \lambda x^2(t)dW(t).$$

Let $p=3, Q_i \equiv E, \alpha=3, \beta=1, \kappa_i=0.1, b_i>0, \lambda>0$, then

$$x^T Q_i f(x,\varphi,i) = -b_i x^5 + cx^3 + dx + \int_{-1}^{0} x\varphi^4 d\theta \le -(b_i - \frac{1}{5})|x|^5 + \frac{4}{5}\int_{-1}^{0}|\varphi|^5 d\theta + o(|x|^5),$$

$$|f(x,\varphi,i)| \le b_i|x|^4 + \int_{-1}^{0}|\varphi|^4 d\theta + o(|x|^4), \qquad |g(x,\varphi,i)| \le \lambda|x|^2.$$

Now, $a_i = b_i - \frac{1}{5}, \sigma_i = \frac{4}{5}, r_i = b_i, \overline{r_i} = 1, \lambda_i = \lambda_i, \overline{\lambda}_i = 0.$ By Theorem 4.1, when $b_i > \frac{166}{25}$, we can conclude that there exists a unique global solution to Eq.(1), and the solution has properties (4) .

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