Existence Results for Semilinear Fractional Functional Differential Equations with State-Dependent Delay*

Yong-Kui Chang[1], M.Mallika Arjunan[2], G. M. N'Guérékata[3] and V. Kavitha[4]

[1]*Department of Mathematics, Lanzhou Jiaotong University, Lanzhou, Gansu 7300070, P.R. China*

[2]*Department of Mathematics, Karunya University, Karunya Nagar, Coimbatore- 641 114, Tamil Nadu, India*

[3]*Department of Mathematics, Morgan State University, 1700 E. Cold Spring Lane, Baltimore, M.D. 21251, USA*

[4]*Department of Mathematics, Karunya University, Karunya Nagar, Coimbatore- 641 114, Tamil Nadu, India*

Email: lzchangyk@163.com, arjunphd07@yahoo.co.in,

Gaston.N'Guerekata@morgan.edu, kavi_velubagyam@yahoo.co.in

Abstract *According to theories for α-resolvent family $(S_\alpha(t))_{t\geq 0}$ and fixed point methods, this paper is mainly concerned with existence of mild solutions to a semilinear fractional functional differential equation with state-dependent delay in a complex Banach space $\mathbb{X}$. Some sufficient conditions are established without the compactness of $(S_\alpha(t))_{t\geq 0}$.*

Keywords *Fractional differential equations α-resolvent family State-dependent delay Fixed point*

1. Introduction

In this paper, we establish the existence of mild solutions to the following fractional func-

*This research is supported by NNSF of China (10901075), the Key Project of Chinese Ministry of Education (210226), the Scientific Research Fund of Gansu Provincial Education Department (0804-08) and "Qing Lan" Talent Engineering Funds (QL-05-16A) by Lanzhou Jiaotong University.

tional differential equation with state-dependent delay

$$D^{\alpha}x(t) = Ax(t) + f(t, x_{\rho(t,x_t)}), \quad t \in J = [0, b], \tag{1}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{2}$$

where $b > 0$, $0 < \alpha < 1$ and $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is the infinitesimal generator of an α-resolvent family $(S_{\alpha}(t))_{t\geq 0}$ defined on a complex Banach space $\mathbb{X}$. The function $x_s : (-\infty, 0] \to \mathbb{X}$, $x_s(\theta) = x(s + \theta)$, belongs to some phase space $\mathcal{B}$ that will be defined later (see Section 2), $f : J \times \mathcal{B} \to \mathbb{X}$, $\rho : J \times \mathcal{B} \to (-\infty, b]$ are appropriate functions and φ belongs to the phase space $\mathcal{B}$ with $\varphi(0) = 0$. The fractional derivative D^{α} is understood here in the Riemann-Liouville sense.

The theory of functional differential equations has emerged as an important branch in nonlinear analysis. It is worth mentioning that several important practical problems have lead to investigations of functional differential equations of various types (see the books of Hale et al. [11], Wu [31], and the references therein). On the other hand, functional differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason, the study of this type of equation has gained great attention in the last decades, we refer to [4, 5, 9, 14, 15, 22] and the references therein.

Differential equations of fractional order play a very important role in describing some real world problems. For example some problems in physics, mechanics and other fields can be described with the help of fractional differential equations, see [2, 7, 17, 24, 25] and references therein. The theory of differential equations of fractional order has recently received much attention and now constitutes a significant branch in differential equations. Lots of research papers and monographs have appeared devoted to fractional differential equations, for example see [1, 3, 6, 8, 18, 19, 20, 21, 26, 27, 28, 30, 33, 34] and the references therein.

Motivated by the above mentioned works, the purpose of this paper is to investigate the existence results of mild solutions to a semilinear fractional functional differential equation with state-dependent delay described in the general abstract form (1)-(2). The main technique is based upon the α-resolvent family $(S_{\alpha}(t))_{t\geq 0}$ combined with suitable fixed point theorems.

This paper is organized as follows. In Section 2, we introduce notations, definitions and some lemmas which are used in the sequel. In section 3, we prove the existence of mild solutions for the problem (1)-(2).

2. Preliminaries

From now on, we set $J = [0, b]$. We denote by $\mathbb{X}$ a complex Banach space with norm $\|\cdot\|, \mathbb{C}(J, \mathbb{X})$ the space of all $\mathbb{X}$-valued continuous functions on J, endowed with the topology of uniform convergence with norm

$$\|x\|_\infty := \sup_{t \in J} \|x(t)\|.$$

And $\mathcal{L}(\mathbb{X})$ the Banach space of all linear and bounded operators on $\mathbb{X}$. Moreover, $B_r(z_0, \mathbb{Z})$ denotes the closed ball with center at z_0 and radius $r > 0$ in $\mathbb{Z}$.

Definition 2.1 *[25] Assume that $f \in \mathbb{C}^m(\mathbb{R}_+, \mathbb{X})$. If $\alpha \in (m-1, m)$, where $m \in \mathbb{N}$, then the Riemann-Liouville fractional derivative of order $\alpha \in (m-1, m)$ is the expression*

$$D_t^\alpha f(t) = \frac{d^m}{dt^m} \int_0^t g_{m-\alpha}(t-s) f(s) ds,$$

where for $\beta > 0$

$$g_\beta(t) = \begin{cases} \frac{t^{\beta-1}}{\Gamma(\beta)} \ for\ t > 0, \\ 0 \quad for\ t \geq 0. \end{cases}$$

Definition 2.2 *[25] Let $\alpha > 0$ and $f : \mathbb{R}^+ \to \mathbb{R}$ be in $L^1(\mathbb{R}_+, \mathbb{X})$. Then the Riemann-Liouville integral is given by:*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds.$$

Recall that the Laplace transform of a function $f \in L^1(\mathbb{R}_+, \mathbb{X})$ is defined by:

$$\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t) dt, \quad Re(\lambda) > \omega,$$

if the integral is absolutely convergent for $Re(\lambda) > \omega$.

Definition 2.3 *[26] Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space $\mathbb{X}$ and $\alpha > 0$. Let $\rho(A)$ be the resolvent set of A. We call A the generator of an α-resolvent family if there exists $\omega \geq 0$ and a strongly continuous function $S_\alpha : \mathbb{R}_+ \to \mathcal{L}(\mathbb{X})$ satisfying $S_\alpha(0) = I$ such that $\{\lambda^\alpha : Re(\lambda) > \omega\} \subset \rho(A)$ and*

$$(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x dt, \quad Re(\lambda) > \omega, \ x \in \mathbb{X}.$$

In this case, $S_\alpha(t)$ is called the α-resolvent family generated by A.

For construction of solution by using α-resolvent family, we refer to [26] and the references therein. We also refer to [23, 29] for more information about resolvent or solution operator.

Remark 2.1 *[26] Note that if A is the generator of an α-resolvent family $(S_\alpha(t))_{t\geq 0}$ then the Laplace transform of $S_\alpha(t)$ is $\widehat{S_\alpha}(\lambda) = (\lambda^\alpha - A)^{-1}$.*

In this paper, we will employ the axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [12] and follow the terminology used in [16]. Thus, $(\mathcal{B}, \|\cdot\|_\mathcal{B})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ to $\mathbb{X}$, and satisfying the following axioms:

(A_1) If $x : (-\infty, b) \to \mathbb{X}$ with $b > 0$, is continuous on $[0, b]$ and $x_0 \in \mathcal{B}$, then for every $t \in [0, b)$ the following conditions hold:

(i) $x_t \in \mathcal{B}$;

(ii) There exists a positive constant H such that $||x(t)|| \leq H\|x_t\|_\mathcal{B}$;

(iii) There exist two functions $K(\cdot), M(\cdot) : \mathbb{R}_+ \to [1, +\infty)$ independent of $x(t)$ with K continuous and M locally bounded such that

$$\|x_t\|_\mathcal{B} \leq K(t) \sup\{||x(s)|| : 0 \leq s \leq t\} + M(t)\|x_0\|_\mathcal{B}.$$

Denote $K_b = \sup\{K(t) : t \in [0, b]\}$ and $M_b = \sup\{M(t) : t \in [0, b]\}$.

(A_2) For the function $x(\cdot)$ in (A_1), x_t is a $\mathcal{B}$-valued continuous function on $[0, b]$.

(A_3) The space $\mathcal{B}$ is complete.

An example of phase space $\mathcal{B}$ satisfying $(A_1)-(A_3)$ is the following space C_g^0 (see [31], pp.44), where $g:[-\infty,0]\to[0,\infty)$ is a given continuous nondecreasing function such that:

(i) $g(0)=1$ and $g(-\infty)=\infty$.

(ii) The function $G(t)=\sup\left\{\frac{g(t+s)}{g(t)}:-\infty<s\leq -t\right\}$ is locally bounded for $t\geq 0$.

Let

$$C_g^0=\{\phi:(-\infty,0]\to X;\phi \quad \text{is continuous and} \quad \lim_{s\to-\infty}\frac{|\phi(s)|}{g(s)}=0\}$$

Then C_g^0, together with the following norm:

$$\|\phi\|_{C_g^0}=\sup\frac{|\phi(s)|}{g(s)},$$

satisfies axioms $(A_1)-(A_3)$.

The next lemma is a consequence of the phase space axioms and is proved in [14].

Lemma 2.1 *([14]) Let $\varphi\in\mathcal{B}$ and $I=(\gamma,0]$ be such that $\varphi_t\in\mathcal{B}$ for every $t\in I$. Assume that there exists a locally bounded function $J^\varphi:I\to[0,\infty)$ such that $\|\varphi_t\|_{\mathcal{B}}\leq J^\varphi(t)\|\varphi\|_{\mathcal{B}}$ for every $t\in I$. If $x:(-\infty,b]\to\mathbb{R}$ is continuous on J and $x_0=\varphi$, then*

$$\|x_s\|_{\mathcal{B}}\leq(M_b+J^\varphi(\max\{\gamma,-|s|\})\|\varphi\|_{\mathcal{B}}+K_b\|x\|_{\max\{0,s\}},$$

for $s\in(\gamma,b]$, where we denoted $K_b=\sup_{t\in J}K(t)$ and $M_b=\sup_{t\in J}M(t)$, $\|x\|_{\max\{0,s\}}=\sup\{\|x(\theta)\|$, $\theta\in[0,\max\{0,s\}]\}$.

To conclude the current section, we recall the following well-known results.

Theorem 2.2 *[10, Theorem 6.5.4]. Let D be a closed convex subset of a Banach space $\mathbb{X}$ and assume that $0\in D$. Let $\Gamma:D\to D$ be a completely continuous map. Then, either the set $\{x\in D: x=\lambda\Gamma(x),\ 0<\lambda<1\}$ is unbounded or the map Γ has a fixed point in D.*

Lemma 2.3 ([13, 32]) *Suppose $\bar{b}\geq 0,\ \alpha>0$ and $a(t)$ is a nonnegative function locally integrable on $0\leq t<T$ (for some $T\leq+\infty$), and suppose $u(t)$ is nonnegative and locally integrable on $0\leq t<T$ with*

$$u(t)\leq a(t)+\bar{b}\int_0^t(t-s)^{\alpha-1}u(s)ds$$

on this interval; then

$$u(t) \leq a(t) + \int_0^t \Big[\sum_{n=1}^{\infty} \frac{(\bar{b}\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}a(s)\Big]ds.$$

3. Existence Results

In this section, we present and prove the existence results for the fractional differential problem (1)-(2). First, we present its mild solution.

Definition 3.1 *A function $x : (-\infty, b] \to \mathbb{X}$ is called a mild solution of (1)-(2) if $x_0 = \phi, x_{\rho(s,x_s)} \in \mathcal{B}$ for each $s \in J$ and*

$$x(t) = \int_0^t S_\alpha(t-s) f(s, x_{\rho(s,x_s)})ds, \quad \textit{for each} \quad t \in J. \tag{3}$$

We are now in a position to state and prove our existence result for the problem (1)-(2). For the study of this, we first list the following hypotheses:

(H1) There exist $M > 0$ and $\delta > 0$ such that $\|S_\alpha(t)\|_{\mathcal{L}(\mathbb{X})} \leq Me^{\delta t},\ t \in J.$

(H2) The function $f : J \to \mathcal{B} \to \mathbb{X}$ is completely continuous and there exists a continuous function $\mu : J \to (0, +\infty)$ such that

$$\|f(t,\psi)\| \leq \mu(t)\|\psi\|_{\mathcal{B}}, \quad (t,\psi) \in J \times \mathcal{B}.$$

(H3) The function $t \to \varphi_t$ is well defined and continuous from the set $\mathcal{R}(\rho^-) = \{\rho(s,\psi) : (s,\psi) \in J \times \mathcal{B},\ \rho(s,\psi) \leq 0\}$ into $\mathcal{B}$. Moreover, there exists a continuous and bounded function $J^\varphi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that $\|\varphi_t\|_{\mathcal{B}} \leq J^\varphi(t)\|\varphi\|_{\mathcal{B}}$ for every $t \in \mathcal{R}(\rho^-)$.

Remark 3.1 *For more details on the hypothesis (H3), we refer to [14].*

Theorem 3.1 *Assume that the hypotheses (H1)-(H3) hold, then the problem (1)-(2) has at least one mild solution on $(-\infty, b]$.*

Proof. Let $Y = \{u \in \mathbb{C}(J, \mathbb{X}) : u(0) = \varphi(0) = 0\}$ endowed with the uniform convergence topology and $N : Y \to Y$ be the operator defined by

$$Nx(t) = \int_0^t S_\alpha(t-s) f(s, \overline{x}_{\rho(s,\overline{x}_s)}) ds, \quad \text{for each} \quad t \in J. \tag{4}$$

where $\overline{x} : (-\infty, b] \to \mathbb{X}$ is such that $\overline{x}_0 = \varphi$ and $\overline{x} = x$ on J. From axiom (A_1) and our assumption on φ, we infer that $Nx(\cdot)$ is well defined and continuous.

Let $\overline{\varphi} : (-\infty, b] \to \mathbb{X}$ be the extension of φ to $(-\infty, b]$ such that $\overline{\varphi}(0) = \varphi(0) = 0$ on J and $\overline{J}^{\varphi} = \sup\{J^{\varphi} : s \in \mathcal{R}(\rho^-)\}$.

We will prove that $N(\cdot)$ is completely continuous from $B_r(0, Y)$ to $B_r(0, Y)$.

Step 1: N is continuous on $B_r(0, Y)$.

Let $(x^n)_{n\in\mathbb{N}}$ be a sequence in $B_r(0, Y)$ and $x \in B_r(0, Y)$ such that $x^n \to x$ in Y. From the axiom (A_1), it is easy to see that $\overline{(x^n)}_s \to \overline{x}_s$ uniformly for $s \in (-\infty, b]$ as $n \to \infty$. By (H2) we have

$$\begin{aligned} &\|f(s, \overline{x^n}_{\rho(s,\overline{(x^n)}_s)}) - f(s, \overline{x}_{\rho(s,\overline{(x)}_s)})\| \\ &\quad \le \|f(s, \overline{x^n}_{\rho(s,\overline{(x^n)}_s)}) - f(s, \overline{x}_{\rho(s,\overline{(x^n)}_s)})\| + \|f(s, \overline{x}_{\rho(s,\overline{(x^n)}_s)}) - f(s, \overline{x}_{\rho(s,\overline{(x)}_s)})\|, \end{aligned}$$

which implies that $f(s, \overline{x^n}_{\rho(s,\overline{(x^n)}_s)}) \to f(s, \overline{x}_{\rho(s,\overline{(x)}_s)})$ as $n \to \infty$ for each $x \in J$. By axiom (A_1), Lemma 2.1 and dominated convergence theorem, we obtain

$$\begin{aligned} \|N(x^n) - N(x)\| &= \sup_{t\in J} \left\| \int_0^t S_\alpha(t-s) \Big[f(s, \overline{x^n}_{\rho(s,\overline{(x^n)}_s)}) - f(s, \overline{x}_{\rho(s,\overline{(x)}_s)}) \Big] ds \right\| \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Thus, $N(\cdot)$ is continuous.

Step 2: N maps bounded sets into bounded sets.

Let $\mu^* = \sup_{0\le\tau\le b} \mu(\tau)$. If $x \in B_r(0, Y)$, from Lemma 2.1, follows that

$$\|\overline{x}_{\rho(t,\overline{x}_t)}\|_{\mathcal{B}} \le r^* = (M_b + \overline{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + K_b r. \tag{5}$$

and so

$$\|N(x)(t)\| = \left\| \int_0^t S_\alpha(t-s) f(s, \overline{x}_{\rho(s,\overline{x}_s)}) ds \right\|$$

$$
\begin{aligned}
&\leq \int_0^t \|S_\alpha(t-s)\|_{\mathcal{L}(\mathbb{X})}\|f(s,\overline{x}_{\rho(s,\overline{x}_s)})\|ds \\
&\leq M\int_0^t e^{\delta(t-s)}\mu(s)\|\overline{x}_{\rho(s,\overline{x}_s)}\|_{\mathcal{B}}ds \\
&\leq M\mu^* r^*\int_0^t e^{\delta(t-s)}ds \\
&\leq M\mu^* r^*\frac{e^{\delta b}}{\delta}.
\end{aligned}
$$

This implies that

$$
\|N(x)\|_\infty \leq M\mu^* r^*\frac{e^{\delta b}}{\delta} = l.
$$

Step 3: N maps bounded sets into equicontinuous sets.

Let $t_1, t_2 \in J$ with $t_1 > t_2$ and $x \in B_r(0, Y)$. Then

$$
\begin{aligned}
\|N(x)(t_1) - N(x)(t_2)\| &= \bigg\|\int_{t_2}^{t_1} S_\alpha(t_1-s)f(s,\overline{x}_{\rho(s,\overline{x}_s)})ds \\
&\quad + \int_0^{t_2}\Big[S_\alpha(t_1-s) - S_\alpha(t_2-s)\Big]f(s,\overline{x}_{\rho(s,\overline{x}_s)})ds\bigg\| \\
&\leq I_1 + I_2,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \int_{t_2}^{t_1}\|S_\alpha(t_1-s)f(s,\overline{x}_{\rho(s,\overline{x}_s)})\|ds, \\
I_2 &= \int_0^{t_2}\Big\|\Big[S_\alpha(t_1-s) - S_\alpha(t_2-s)\Big]f(s,\overline{x}_{\rho(s,\overline{x}_s)})\Big\|\,ds.
\end{aligned}
$$

Here I_1 and I_2 tend to 0 as $t_1 \to t_2$ independently of $x \in B_r(0, Y)$.

In fact,

$$
\begin{aligned}
I_1 &= \int_{t_2}^{t_1}\|S_\alpha(t_1-s)f(s,\overline{x}_{\rho(s,\overline{x}_s)})\|ds \\
&\leq \int_{t_2}^{t_1}\|S_\alpha(t_1-s)\|_{\mathcal{L}(\mathbb{X})}\|f(s,\overline{x}_{\rho(s,\overline{x}_s)})\|ds \\
&\leq M\int_{t_2}^{t_1} e^{\delta(t_1-s)}\mu(s)\|\overline{x}_{\rho(s,\overline{x}_s)}\|_{\mathcal{B}}ds \\
&\leq M\mu^* r^*\int_{t_2}^{t_1} e^{\delta(t_1-s)}ds \\
&= M\mu^* r^*\Big[\frac{e^{\delta(t_1-t_2)}-1}{\delta}\Big].
\end{aligned}
$$

Hence $\lim_{t_1 \to t_2} I_1 = 0$. And

$$I_2 = \int_0^{t_2} \left\| \left[S_\alpha(t_1 - s) - S_\alpha(t_2 - s) \right] f(s, \overline{x}_{\rho(s,\overline{x}_s)}) \right\| ds$$
$$\to 0,$$

since f is compact and S_α is strongly continuous, $\left\| \left[S_\alpha(t_1 - s) - S_\alpha(t_2 - s) \right] f(s, \overline{x}_{\rho(s,\overline{x}_s)}) \right\|$ $\to 0$ as $t_1 \to t_2$ uniformly for $x \in B_r(0, Y)$. We conclude that $\lim_{t_1 \to t_2} I_2 = 0$.

Step 4: The operator N maps $B_r(0, Y)$ into a relatively compact set in $\mathbb{X}$.

Indeed from the strong continuity of $S_\alpha(\cdot)$ and (H2), the set $\{S_\alpha(t-s)f(s, x_{\rho(s,x_s)}), t, s \in [0, b],\ x \in B_r(0, Y)\}$ is relatively compact in $\mathbb{X}$. Moreover, for $x \in B_r(0, Y)$, using the mean value theorem for the Bochner integral, we obtain

$$Nx(t) \in t\overline{conv}\{S_\alpha(t-s)f(s, x_{\rho(s,x_s)}) : s \in [0, b], x \in B_r(0, Y)\}, \quad \text{for all} \quad t \in [0, b].$$

Consequently the set $\{Nx(t) : x \in B_r(\overline{\varphi}|_J, Y)\}$ is relatively compact in $\mathbb{X}$, for every $t \in [0, b]$.

Step 5: A priori bounds.

Set $\Lambda = \{x \in Y$ such that $x = \lambda N(x)$ for some $0 < \lambda < 1\}$. Let $x \in \Lambda$. Then for each $t \in [0, b]$, we have

$$\begin{aligned}
\|x(t)\| &\leq \lambda \int_0^t \|S_\alpha(t-s)\|_{\mathcal{L}(\mathbb{X})} \|f(s, \overline{x}_{\rho(s,\overline{x}_s)})\| ds \\
&\leq M \int_0^t e^{\delta(t-s)} \mu(s) \|\overline{x}_{\rho(s,\overline{x}_s)}\|_{\mathcal{B}} ds \\
&\leq M\mu^* \int_0^t e^{\delta(t-s)} \left[(M_b + \overline{J}^\varphi)\|\varphi\|_{\mathcal{B}} + K_b \|x\|_{\max\{0,s\}} \right] ds \\
&\leq M\mu^* \int_0^t e^{\delta(t-s)} \left[(M_b + \overline{J}^\varphi)\|\varphi\|_{\mathcal{B}} + K_b \|x\|_s \right] ds \\
&\leq M\mu^* (M_b + \overline{J}^\varphi)\|\varphi\|_{\mathcal{B}} \frac{e^{\delta b}}{\delta} + M\mu^* K_b \int_0^t e^{\delta(t-s)} (t-s)^{\alpha-1} (t-s)^{1-\alpha} \|x\|_s ds \\
&\leq \theta_1 + \theta_2 \int_0^t (t-s)^{\alpha-1} \|x\|_s ds,
\end{aligned}$$

where

$$\begin{aligned}
\theta_1 &= M\mu^* (M_b + \overline{J}^\varphi)\|\varphi\|_{\mathcal{B}} \frac{e^{\delta b}}{\delta} \\
\theta_2 &= M\mu^* K_b e^{\delta b} b^{1-\alpha}.
\end{aligned}$$

In view of Lemma 2.3, we have for all $t \in J$,

$$\begin{aligned}\|x(t)\| &\le \theta_1\Big[1 + \int_0^t \sum_{n=1}^{\infty} \frac{(\theta_2\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}\Big]ds \\ &\le \theta_1\Big[1 + \sum_{n=1}^{\infty} \frac{(\theta_2\Gamma(\alpha))^n b^{n\alpha}}{n\alpha\Gamma(n\alpha)}\Big] \\ &= \theta_1\Big[1 + \sum_{n=1}^{\infty} \frac{[\theta_2\Gamma(\alpha)b^{\alpha}]^n}{\Gamma(n\alpha+1)}\Big] \\ &\le \theta_1\Lambda_\alpha[\theta_2\Gamma(\alpha)b^\alpha],\end{aligned}$$

where $\Lambda_\alpha[\theta_2\Gamma(\alpha)b^\alpha] = \sum_{n=0}^{\infty} \frac{[\theta_2\Gamma(\alpha)b^{\alpha}]^n}{\Gamma(n\alpha+1)}$ is the Mittag-Leffer function. This implies that

$$\|x\|_\infty \le \theta_1\Lambda_\alpha[\theta_2\Gamma(\alpha)b^\alpha].$$

Hence combining Step 1–Step 5 and using the Theorem 2.2, we obtain that N has a fixed point which is a mild solution of (1)-(2) on $(-\infty, b]$.

Next, we give an existence result when the nonlinearity f has a sublinear growth with its state variable. Let us list the following condition:

(H2*) The function $f : J \to \mathcal{B} \to \mathbb{X}$ is completely continuous such that there exist a continuous function $\mu : J \to (0, +\infty)$ and a continuous nondecreasing function $W : [0, +\infty) \to (0, +\infty)$ satisfying

$$\|f(t, \psi)\| \le \mu(t)W(\|\psi\|_{\mathcal{B}}), \quad (t, \psi) \in J \times \mathcal{B}, \quad \liminf_{\xi \to +\infty} \frac{W(\xi)}{\xi} = \gamma < +\infty.$$

Theorem 3.2 *Assume that the hypotheses (H1), (H2*) and (H3) are satisfied. Then the problem (1)-(2) admits at least one mild solution on* $(-\infty, b]$ *provided that*

$$\frac{Me^{\delta b}\mu^* K_b}{\delta}\gamma < 1, \textit{where } \mu^* = \sup_{0\le\tau\le b} \mu(\tau). \tag{6}$$

Proof. Let N be the operator defined by (4). We shall complete the proof by Schauder's fixed point theorem. Let r^* be defined as (5).

We claim that there exists a positive number r such that $NB_r(0, Y) \subseteq B_r(0, Y)$. If it is not true, then for each $r > 0$, there exists $x^r(\cdot) \in B_r(0, Y)$, but $Nx^r \notin B_r(0, Y)$, that is,

$\|N(x^r)(t)\| > r$ for some $t(r) \in J$, where $t(r)$ denotes t depending on r. However, on the other hand, we have from (H1), (H2*) that

$$\begin{aligned} r &< \|N(x^r)(t)\| \\ &= \left\| \int_0^t S_\alpha (t-s) f\left(s, \overline{x}_{\rho\left(s, \overline{x^r}_s\right)}\right) ds \right\| \\ &\leq M \int_0^t e^{\delta(t-s)} \mu(s) W(r^*) ds \\ &\leq M \frac{e^{\delta b}}{\delta} \mu^* W(r^*). \end{aligned}$$

Dividing both sides by r and taking the lower limit, we get

$$\frac{M e^{\delta b} \mu^* K_b}{\delta} \gamma \geq 1,$$

where contradicts (6). Hence for some positive r, $NB_r(0, Y) \subseteq B_r(0, Y)$.

Just the same as the proof in Theorem 3.1, we can show that N is continuous on $B_r(0, Y)$ and N maps $B_r(0, Y)$ into a relatively compact set in $\mathbb{X}$. Next we prove that the family $\{Nx : x \in B_r(0, Y)\}$ is an equicontinuous family of functions.

Let $t_1, t_2 \in J$ with $t_1 > t_2$ and $x \in B_r(0, Y)$. Then

$$\begin{aligned} \|N(x)(t_1) - N(x)(t_2)\| = &\left\| \int_{t_2}^{t_1} S_\alpha(t_1 - s) f(s, \overline{x}_{\rho(s, \overline{x}_s)}) ds \right. \\ &+ \left. \int_0^{t_2} \Big[S_\alpha(t_1 - s) - S_\alpha(t_2 - s) \Big] f(s, \overline{x}_{\rho(s, \overline{x}_s)}) ds \right\| \\ &\leq I_1 + I_2. \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_{t_2}^{t_1} \|S_\alpha(t_1 - s) f(s, \overline{x}_{\rho(s, \overline{x}_s)})\| ds, \\ I_2 &= \int_0^{t_2} \left\| \Big[S_\alpha(t_1 - s) - S_\alpha(t_2 - s) \Big] f(s, \overline{x}_{\rho(s, \overline{x}_s)}) \right\| ds. \end{aligned}$$

Here I_1 and I_2 tend to 0 as $t_1 \to t_2$ independently of $x \in B_r(0, Y)$.

Indeed,

$$I_1 = \int_{t_2}^{t_1} \|S_\alpha(t_1 - s) f(s, \overline{x}_{\rho(s, \overline{x}_s)})\| ds$$

$$
\begin{aligned}
&\leq \int_{t_2}^{t_1} \|S_\alpha(t_1-s)\|_{\mathcal{L}(\mathbb{X})}\|f(s,\overline{x}_{\rho(s,\overline{x}_s)})\|ds\\
&\leq M\int_{t_2}^{t_1} e^{\delta(t_1-s)}\mu(s)W(\|\overline{x}_{\rho(s,\overline{x}_s)}\|_{\mathcal{B}})ds\\
&\leq M\mu^* W(r^*)\int_{t_2}^{t_1} e^{\delta(t_1-s)}ds\\
&= M\mu^* W(r^*)\left[\frac{e^{\delta(t_1-t_2)}-1}{\delta}\right].
\end{aligned}
$$

Hence $\lim\limits_{t_1\to t_2} I_1 = 0$. And

$$
\begin{aligned}
I_2 &= \int_0^{t_2} \left\|\Big[S_\alpha(t_1-s)-S_\alpha(t_2-s)\Big]f(s,\overline{x}_{\rho(s,\overline{x}_s)})\right\| ds\\
&\to 0,
\end{aligned}
$$

since f is compact and S_α is strongly continuous, $\left\|\Big[S_\alpha(t_1-s)-S_\alpha(t_2-s)\Big]f(s,\overline{x}_{\rho(s,\overline{x}_s)})\right\|$ $\to 0$ as $t_1\to t_2$ uniformly for $x\in B_r(0,Y)$. We deduce that $\lim\limits_{t_1\to t_2} I_2 = 0$.

Thus, by the Arzela-Ascoli theorem N is a completely continuous operator. In view of Schauder's fixed point theorem, we deduce that N has a fixed point which is a mild solution of (1)-(2) on $(-\infty,b]$. This finishes the proof.

According to Theorem 3.2, we can easily obtain the following consequence for the sub-linear growth case.

(H2**) The function $f: J\to\mathcal{B}\to\mathbb{X}$ is completely continuous such that there exist a continuous function $\mu: J\to(0,+\infty)$ and a contant $\vartheta\in(0,1)$ satisfying

$$
\|f(t,\psi)\| \leq \mu(t)\left[1+(\|\psi\|_{\mathcal{B}})^{\vartheta}\right],\quad (t,\psi)\in J\times\mathcal{B}.
$$

Corollary 3.1 *Suppose that (H1),(H2**) and (H3)hold. Then the problem (1)-(2) has at least one mild solution on* $(-\infty,b]$.

References

[1] R.P. Agarwal, Y. Zhou and Y. He, Existence of fractional neutral functional differential equations, *Comp. Math. Appl.*, **59**(2010), 1095-1100.

[2] R.P. Agarwal, M. Belmekki and M. Benchohra, A survey on semilinear differential equations and inclusions involving Riemann-Liouville fractional derivative, *Advances in Difference Equations*, **2009**(2009), Article ID 981728, 47 pages.

[3] B. Ahmad, J. J. Nieto, Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions, *Comp. Math. Appl*, **58**(2009), 1838-1843.

[4] A. Anguraj, M. Mallika Arjunan, Hernández, E, Existence results for an impulsive neutral functional differential equation with state-dependent delay, *Appl. Anal.,* **86**(2007), 861–972.

[5] W. Aiello, H.I.Freedman, J. Wu, Analysis of a model representing stage-structured population growth with state-dependent time delay, *SIAM J. Appl. Math.,* **52**(1992), 855–869.

[6] Z. Bai, H. Lü, Positive solutions for boundary value problem of nonlinear fractional differential equations, *J. Math. Anal. Appl.*, **311**(2005), 495-505.

[7] B. Bonila, M. Rivero, L. Rodriquez-Germa and J.J. Trujilio, Fractional differential equations as alternative models to nonlinear differential equaitons, *Appl. Math. Comput.,* **187**(2007), 79-88.

[8] Y. K. Chang, J. J. Nieto, Some new existence results for fractional differential inclusions with boundary conditions, *Math. Comput. Modelling*, **49**(2009), 605-609.

[9] C. Cuevas, G. M. N'Guérékata, M. Rabelo, Mild solutions for impulsive neutral functional differential equations with state-dependent delay, *Semigroup Forum*, **80**(2010), 375–390.

[10] A. Granas, J. Dugundji, *Fixed Point Theory.* Springer-Verlag, New York, 2003.

[11] J.K. Hale and S. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, New York, 1993.

[12] J. Hale and J. Kato, Phase space for retarded equations with infinite dealy, *Funkcial. Ekvac.,* **21**(1978), 11-41.

[13] D. Henry, *Geometric Theory of Semilinear Parabolic Equations* Lecture Notes in Mathematics, **Vol. 8**, Springer-Verlag, New York, 1981.

[14] E. Hernández, A. Prokopczyk, Luiz. Ladeira, A note on partial functional differential equations with state-dependent delay, *Nonlinear Anal. RWA,* **7** (2006), 510–519.

[15] E. Hernández, Mark A. Mckibben., On state- dependent delay partial neutral functional-differential equations, *Appl. math. Comput.,* **186**(2007), 294-301.

[16] Y. Hino, S. Murukami and T. Naito, *Functional Differential Equations with Unbounded Delay*, Lecture Notes in Mathematics, 1473, Springer-Verlag, Berlin, 1991.

[17] G. Jumarie, An approach via fractional analysis to non-linearity induced by coarse-graining in space, *Nonlinear Anal. RWA*, **11** (2010), 535-546.

[18] A. A. Kilbas, Hari M. Srivastava, and Juan J. Trujillo, *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[19] V. Lakshmikantham and J. V. Devi, Theory of fractional differential equations in a Banach space, *Eur. J. Pure Appl. Math.,* **1**(2008), 38-45.

[20] V. Lakshmikantham and A. S. Vatsala, Basic theory of fractional differential equations, *Nonlinear Anal.,* **69**(2008), 2677-2682.

[21] V. Lakshmikantham and A. S. Vatsala, General uniqueness and monotone iteration technique in fractional differential equations, *Appl. Math. Lett.,* **21**(2008), 828-834.

[22] W. S. Li, Y. K. Chang, J. J. Nieto, Solvability of impulsive neutral evolution differential inclusions with state-dependent delay *Math. Comput. Modelling,* **49**(2009), 605–609.

[23] C. Lizama, Regularized solutions for abstract volterra equations, *J. Math. Anal. Appl.,* **243**(2000), 278-292.

[24] Y. F. Luchko, M. Rivero, J. J. Trujillo and M. P. Velasco, Fractional models, nonlocality and complex systems, *Comp. Math. Appl.*, **59**(2010), 1048-1056.

[25] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, Inc., New York, 1993.

[26] G.M. Mophou and G. M. N'Guérékata, Existence of mild solutions of some semilinear neutral fractional functional evolution equations with infinite delay, *Appl. Math. Comput.,* **216**(2010), 61-69.

[27] G.M. N'Guérékata, A Cauchy problem for some fractional abstract differential equations with non local conditions, *Nonlinear Anal.*, **70**(2009), 1873-1876.

[28] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.

[29] J. Pruss, *Evolutionary Integral Equations and Applications*, Monographs Mathematics Volume **87**, Birkhuser-Verlag, 1993.

[30] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[31] J. Wu, *Theory and Applications of Partial Functional Differential Equations*, Applied Mathematical Sciences 119, Springer-Verlag, New York, 1996.

[32] H. Ye, J. Gao and Y. Ding, A generalized Gronwall inequality and its application to a fractional differential equations, *J. Math. Anal. Appl.,* **338**(2007), 1075-1081.

[33] S. Zhang, Monotone iterative method for initial value problem involving Riemann-Liouville fractional derivatives , *Nonlinear Anal.*, **71** (2009), 2087-2093.

[34] Y. Zhong, J. Feng and J. Li, Existence and uniqueness for fractional neutral differential equations with infinite delay, *Nonlinear Anal.*, **71** (2009), 3249-3256.