Time-Optimal Control of Infinite Variables Parabolic Systems with Time Lags Given in Integral Form

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Abstract *In this paper, the time-optimal control problem for second order parabolic system and also for $(n \times n)$– parabolic systems with infinite number of variables involving constant time lags appearing in integral form in both the state equation and in the boundary condition is presented. Some specific properties of the optimal control are discussed.*

Keywords *Time-optimal control* $(n \times n)$ *Parabolic systems* *Operator with an infinite number of variables* *Time lag*

1. Introduction

Distributed parameters systems with delays can be used to describe many phenomena in the real world. As is well known, heat conduction, properties of elastic-plastic material, fluid dynamics, diffusion-reaction processes, the transmission of the signals at a certain distance by using electric long lines, etc., all lie within this area. The object that we are studying (temperature, displacement, concentration, velocity, etc.) is usually referred to as the state.

The time-optimal control problems of distributed second order parabolic systems with finite number of variables involving time lags appearing in the boundary condition have been widely discussed in many papers and monographs. A fundamental study of such problems is given by (Wang, 1975) and was next developed by (Knowles, 1978) and (Wong, 1987). It was also intensively investigated by (Kowalewski, 1988; 1990a; 1990b; 1993; 1998; 1999; 2009), (Kowalewski and Duda, 1992), (Kowalewski and Krakowiak, 1994; 2000; 2006; 2008), (Kotarski, 1997), (Kotarski & El-Saify and Bahaa, 2002b), (Kotarski and Bahaa, 2007) and (El-Saify, 2005; 2006) in which linear quadratic problem for parabolic systems with time delays given in the different form (constant time delays, time-varying delays, time delays given in the integral form, etc.) were presented.

The necessary and sufficient conditions of optimality for systems consists of only one equation and for $(n \times n)$ systems governed by different types of partial differential equations defined on spaces of functions of infinitely many variables and also for infinite order systems are discussed for example in (Gali, I. M. & El-Saify, H. A. 1982; 1983), (El-Saify & Bahaa, 2001; 2003), (El-Saify, H. A., Serag, H. M, & Bahaa, G. M. 2000), (El-Saify, 2005; 2006), (Kowalewski, 2009) and (Kowalewski and Krakowiak, 2008) in which the argument of (Lions, 1971 and Lions & Magenes, 1972) were used.

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Making use of the Dubovitskii-Milyutin Theorem in (Kotarski, El-Saify & Bahaa, 2002a,b), (Bahaa, 2003; 2005a,b; 2008) and (Bahaa and Kotarski, 2008), the necessary and sufficient conditions of optimality for similar systems governed by second order operator with an infinite number of variables and also for infinite order systems were investigated. The interest in the study of this class of operators is stimulated by problems in quantum field theory.

In particular, the papers of (Kowalewski & Krakowiak, 2006, 2008), the time-optimal boundary control problem for a second order distributed parabolic systems with finite number of variables in which constant time lags appear in integral form in both the state equation and the boundary condition is presented. Some particular properties of optimal control are discussed.

In this paper we recall the problem in a more general formulation. We consider the time-optimal distributed and boundary control problem for second order parabolic system and also for $(n \times n)$ –second order parabolic systems with infinite number of variables involving constant time lags appearing in integral form in both the state equation and in the boundary condition simultaneously. Such an infinite variables parabolic systems can be treated as a generalization of the mathematical model for a plasma control process. The quadratic performance functional defined over a fixed time horizon are taken and some constraints are imposed on the boundary control. Following a line of the Lions scheme (Lions, 1971) and (Lions & Magenes, 1972), necessary and sufficient optimality conditions for the Neumann and Dirichlet problem applied to the above systems were derived. The optimal control is characterized by the adjoint equations.

This paper is organized as follows. In section 2, we introduce Sobolev spaces with infinite number of variables. In section 3, we formulate the mixed Neumann problem for infinite variables parabolic systems involving time lags. In section 4, the time-distributed control problem for this case is formulated, then we give the necessary and sufficient conditions for the time control to be an optimal. In section 5, we concluded and generalized our results.

2. Sobolev Spaces with Infinite Number of Variables

This section covers the basic notations, definitions and properties, which are necessary to present this work (Berezanskii, 1975), (Gali & El-Saify 1982; 1983), (El-Saify & Serag & Bahaa, 2000) and (El-Saify & Bahaa, 2001).

Let $(p_k(t))_{k=1}^{\infty}$ be a sequence of weights, fixed in all that follows, such that;

$$0 < p_k(t) \in C^{\infty}(\mathbb{R}^1), \int_{\mathbb{R}^1} p_k(t)dt = 1,$$

with respect to it we introduce on the region $\mathbb{R}^{\infty} = \mathbb{R}^1 \times \mathbb{R}^1 \times \ldots$, the measure $d\rho(x)$ by setting,

$$d\rho(x) = p_1(x_1)dx_1 \otimes p_2(x_2)dx_2 \otimes \ldots, (\mathbb{R}^{\infty} \ni x = (x_k)_{k=1}^{\infty}, x_k \in \mathbb{R}^1).$$

On $\mathbb{R}^{\infty}$ we construct the space $L^2(\mathbb{R}^{\infty}, d\rho(x))$ with respect to this measure i.e., $L^2(\mathbb{R}^{\infty}, d\rho(x))$ is the space of quadratic integrable functions on $\mathbb{R}^{\infty}$. We shall often set $L^2(\mathbb{R}^{\infty}, d\rho(x)) = L^2(\mathbb{R}^{\infty})$.

It is classical result that $L^2(\mathbb{R}^\infty)$ is a Hilbert space for the scalar product

$$(\phi,\psi)_{L^2(\mathbb{R}^\infty)} = \int_{\mathbb{R}^\infty} \phi(x)\psi(x)d\rho(x).$$

We next consider a Sobolev space in the case of an unbounded region. For functions which are $\ell = 1,2,\dots$ times continuously differentiable up to the boundary Γ of $\mathbb{R}^\infty$ (Γ is meant to be the boundary of the support of the measure $d\rho(x)$) and which vanish in a neighborhood of ∞, we introduce the scalar product

$$(\phi,\psi)_{W^\ell(\mathbb{R}^\infty)} = \sum_{|\alpha|\le\ell}(D^\alpha\phi, D^\alpha\psi)_{L^2(\mathbb{R}^\infty)},$$

where D^α is defined by

$$D^\alpha = \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1}(\partial x_2)^{\alpha_2}\dots}, \quad |\alpha| = \sum_{i=1}^{\infty}\alpha_i,$$

and the differentiation is taken in the sense of generalized functions on $\mathbb{R}^\infty$, and after the completion, we obtain the Sobolev space $W^\ell(\mathbb{R}^\infty)$. So in short, Sobolev space $W^1(\mathbb{R}^\infty)$ is defined by :

$$W^1(\mathbb{R}^\infty) = \{\phi|\phi, D\phi \in L^2(\mathbb{R}^\infty)\}.$$

As in the case of a bounded region, the space $W^1(\mathbb{R}^\infty)$ form the space with positive norm $||.||_{W^1(\mathbb{R}^\infty)}$. We can construct the space $W^{-1}(\mathbb{R}^\infty) = (W^1(\mathbb{R}^\infty))^*$ with negative norm $||.||_{W^{-1}(\mathbb{R}^\infty)}$ with respect to the space $W^0(\mathbb{R}^\infty) = L^2(\mathbb{R}^\infty)$ with zero norm $||.||_{L^2(\mathbb{R}^\infty)}$, then we have the following equipped,

$$W^1(\mathbb{R}^\infty) \subseteq L^2(\mathbb{R}^\infty) \subseteq W^{-1}(\mathbb{R}^\infty),$$

$$||\phi||_{W^1(\mathbb{R}^\infty)} \ge ||\phi||_{L^2(\mathbb{R}^\infty)} \ge ||\phi||_{W^{-1}(\mathbb{R}^\infty)}.$$

Let $L^2(0,T;W^1(\mathbb{R}^\infty))$ be the space of square integrable measurable functions $t\to\phi(t)$ of $]0,T[\to W^1(\mathbb{R}^\infty)$, where the variable t denotes the " time "; $t\in]0,T[$, $T<\infty$. This space is a Hilbert space with respect to the scalar product

$$(\phi,\psi)_{L^2(0,T;W^1(\mathbb{R}^\infty))} = \int_0^T (\phi(t),\psi(t))_{W^1(\mathbb{R}^\infty)}dt,$$

and its dual is the space $L^2(0,T;W^{-1}(\mathbb{R}^\infty))$, analogously, we can define the spaces $L^2(0,T;L^2(\mathbb{R}^\infty))$ which we shall denote by $L^2(Q)$.

Let $\Omega\subset\mathbb{R}^\infty$ is a bounded, open set with boundary Γ, which is a C^∞ manifold of dimension $(n-1)$. Locally, Ω is totally on one side of Γ and denote by $W^1(\Omega,\mathbb{R}^\infty, d\rho(x))$ (briefly $W^1(\Omega,\mathbb{R}^\infty)$) the Sobolev space of vector function $y(x)$ defined on Ω.

The construction of the Cartesian product of n-times to the above Hilbert spaces can be construct, for example

$$(W^1(\Omega,\mathbb{R}^\infty))^n = \underbrace{W^1(\Omega,\mathbb{R}^\infty)\times W^1(\Omega,\mathbb{R}^\infty)\times\cdots\times W^1(\Omega,\mathbb{R}^\infty)}_{n-times} = \prod_{i=1}^{n}(W^1(\Omega,\mathbb{R}^\infty))^i,$$

with norm defined by:

$$||\boldsymbol{\phi}||_{(W^1(\Omega,\mathbb{R}^\infty))^n} = \sum_{i=1}^{n} ||\phi_i||_{W^1(\Omega,\mathbb{R}^\infty)},$$

where $\boldsymbol{\phi} = (\phi_1, \phi_2, ..., \phi_n) = (\phi_i)_{i=1}^n$ is a vector function and $\phi_i \in W^1(\Omega, \mathbb{R}^\infty)$.

Finally, we have the following chain:

$$(L^2(0,T; W^1(\Omega,\mathbb{R}^\infty)))^n \subseteq (L^2(Q))^n \subseteq (L^2(0,T; W^{-1}(\Omega,\mathbb{R}^\infty)))^n,$$

where $(L^2(0,T; W^{-1}(\Omega,\mathbb{R}^\infty)))^n$ are the dual spaces of $(L^2(0,T; W^1(\Omega,\mathbb{R}^\infty)))^n$. The spaces considered in this paper are assumed to be real.

3. Existence and Uniqueness of Solutions

Consider now the distributed-parameter system described by the following parabolic delay equation:

$$\frac{\partial y}{\partial t} + \mathcal{A}(t)y + \int_a^b c(x,t)y(x,t-h)\,dh = u, \quad x \in \Omega,\, t \in (0,T),\, h \in (a,b), \tag{1}$$

$$y(x,t') = \Phi_0(x,t'), \quad x \in \Omega,\, t' \in [-b,0), \tag{2}$$

$$y(x,0) = y_0(x), \quad x \in \Omega, \tag{3}$$

$$\frac{\partial y(x,t)}{\partial \eta_{\mathcal{A}}} = \int_a^b d(x,t)y(x,t-h)\,dh + v, \quad x \in \Gamma,\, t \in (0,T),\, h \in (a,b), \tag{4}$$

$$y(x,t') = \Psi_0(x,t'), \quad x \in \Gamma,\, t' \in [-b,0), \tag{5}$$

where Ω and Γ have the same properties as in Section 2. We have

$$y \equiv y(x,t;u), \quad u \equiv u(x,t), \quad v \equiv v(x,t),$$

$$Q \equiv \Omega \times (0,T), \quad \overline{Q} \equiv \overline{\Omega} \times [0,T], \quad Q_0 \equiv \Omega \times [-b,0)$$

$$\Sigma \equiv \Gamma \times (0,T), \quad \Sigma_0 \equiv \Gamma \times [-b,0),$$

T is a specified positive number representing a time horizon, c is a given real C^∞ function defined on $\overline{Q}$, d is a given real C^∞ function defined on Σ, h is a time lag such that $h \in (a,b)$ and $a > 0$, Φ_0 and Ψ_0 are initial functions defined on Q_0 and Σ_0, respectively.

The parabolic operator $\dfrac{\partial}{\partial t} + \mathcal{A}(t)$ in the state equation (1) is a second order parabolic operator with infinite number of variables and $\mathcal{A}(t)$ (Berezanskii, 1975), (Gali & El-Saify, 1982; 1983) and (Kotarski & El-Saify & Bahaa, 2002b) is given by:

$$\begin{aligned} \mathcal{A}(t)y(x) &= \left(-\sum_{k=1}^{\infty} \frac{1}{\sqrt{p_k(x_k,t)}} \frac{\partial^2}{\partial x_k^2} \sqrt{p_k(x_k,t)} + q(x,t)\right) y(x) \\ &= -\sum_{k=1}^{\infty} D_k^2 y(x) + q(x,t)y(x), \end{aligned} \tag{6}$$

where

$$D_k y_(x) = \frac{1}{\sqrt{p_k(x_k,t)}} \frac{\partial}{\partial x_k} \sqrt{p_k(x_k,t)} y(x), \tag{7}$$

and $q(x,t)$ is a real-valued function in x which is a bounded and measurable on $\Omega \subset \mathbb{R}^\infty$, such that $q(x,t) \geq \xi_0 > 1$, ξ_0 is a constant. The operator $\mathcal{A}(t)$ is a bounded second order self-adjoint elliptic partial differential operator with an infinite number of variables maps $W^1(\Omega, \mathbb{R}^\infty)$ onto $W^{-1}(\Omega, \mathbb{R}^\infty)$.

For this operator we define the bilinear form as follows:

Definition 3.1. For each $t \in (0,T)$, we define a family of bilinear forms on $W^1(\Omega, \mathbb{R}^\infty)$ by:

$$\pi(t; y, \phi) = (\mathcal{A}(t)y, \phi)_{L^2(\Omega,\mathbb{R}^\infty)}, \quad y, \, \phi \in W^1(\Omega, \mathbb{R}^\infty), \tag{8}$$

where $\mathcal{A}(t)$ maps $W^1(\Omega, \mathbb{R}^\infty)$ onto $W^{-1}(\Omega, \mathbb{R}^\infty)$ and takes the above form. Then

$$\begin{aligned}
\pi(t; y, \phi) &= \Big(\mathcal{A}(t)y, \phi \Big)_{L^2(\Omega,\mathbb{R}^\infty)} \\
&= \Big(-\sum_{k=1}^{\infty} D_k^2 y(x) + q(x,t)y(x), \phi(x) \Big)_{L^2(\Omega,\mathbb{R}^\infty)} \\
&= \int_\Omega \sum_{k=1}^{\infty} D_k y(x) D_k \phi(x) \, d\rho(x) + \int_\Omega q(x,t) y(x) \phi(x) \, d\rho(x).
\end{aligned}$$

Lemma 3.1. *The bilinear form $\pi(t; y, \phi)$ is coercive on $W^1(\Omega, \mathbb{R}^\infty)$, that is*

$$\pi(t; y, y) \geq \lambda \, ||y||^2_{W^1(\Omega,\mathbb{R}^\infty)}, \quad \lambda > 0. \tag{9}$$

Proof. It is well known that the ellipticity of $\mathcal{A}(t)$ is sufficient for the coerciveness of $\pi(t; y, \phi)$ on $W^1(\Omega, \mathbb{R}^\infty)$.

$$\pi(t; \phi, \psi) = \int_\Omega \sum_{k=1}^{\infty} D_k \phi(x) D_k \psi(x) \, d\rho + \int_\Omega q(x,t) \phi(x) \psi(x) \, d\rho.$$

Then

$$\begin{aligned}
\pi(t; y, y) &= \int_\Omega \sum_{k=1}^{\infty} |D_k y(x)|^2 \, d\rho(x) + \int_\Omega q(x,t) |y(x)|^2 \, d\rho(x) \\
&\geq \sum_{k=1}^{\infty} ||D_k y(x)||^2_{L^2(\Omega,\mathbb{R}^\infty)} + \xi_0 ||y(x)||^2_{L^2(\Omega,\mathbb{R}^\infty)} \\
&= ||y(x)||^2_{W^1(\Omega,\mathbb{R}^\infty)} + \xi_0 ||y(x)||^2_{L^2(\Omega,\mathbb{R}^\infty)} \\
&\geq ||y(x)||^2_{W^1(\Omega,\mathbb{R}^\infty)} \\
&= \lambda ||y||^2_{W^1(\Omega,\mathbb{R}^\infty)}, \quad \lambda > 0.
\end{aligned}$$

Also we have:

$$\left.\begin{array}{l}\forall y,\phi \in W^1(\Omega,\mathbb{R}^\infty) \text{ the function } t \to \pi(t;y,\phi) \text{ is continuously differentiable in } (0,T) \text{ and} \\ \pi(t;y,\phi)=\pi(t;\phi,y)\end{array}\right\} \tag{10}$$

Equations (1)–(5) constitute a Neumann problem. Then the left-hand side of the boundary condition (4) may be written in the following form:

$$\frac{\partial y(u)}{\partial \eta_{\mathcal{A}}} = \sum_{k=1}^{\infty} (D_k y(u)) \cos(n, x_k) = g(x,t), \tag{11}$$

where $\dfrac{\partial}{\partial \eta_{\mathcal{A}}}$ is a normal derivative at Γ, directed towards the exterior of Ω, $\cos(n, x_k)$ is the $k-th$ direction cosine of n, with n being the normal at Γ exterior to Ω, and

$$g(x,t) = \int_a^b d(x,t) y(x,t-h)\, dh + v(x,t), \quad x \in \Gamma,\ t \in (0,T),\ h \in (a,b). \tag{12}$$

First we shall prove sufficient conditions for the existence of a unique solution of the mixed initial boundary value problem (1)–(5) for the cases where the control u or v belong to $L^2(Q)$ or $L^2(\Sigma)$ respectively.

To this purpose, for any pair of real numbers $r, s \geq 0$, we introduction the Sobolev space $W^{r,s}(Q)$ (Lions and Magenes, 1972, Vol. 2, p. 6) defined by

$$W^{r,s}(Q) = L^2\left(0,T; W^r(\Omega,\mathbb{R}^\infty)\right) \cap W^s\left(0,T; L^2(\Omega,\mathbb{R}^\infty)\right) \tag{13}$$

which is a Hilbert space normed by

$$\left(\int_0^T ||y(t)||^2_{W^r(\Omega,\mathbb{R}^\infty)} dt + ||y||^2_{W^s(0,T;L^2(\Omega,\mathbb{R}^\infty))}\right)^{1/2}, \tag{14}$$

where $W^s\left(0,T;L^2(\Omega,\mathbb{R}^\infty)\right)$ denotes the Sobolev space of order s of functions defined on $(0,T)$ and taking values in $L^2(\Omega,\mathbb{R}^\infty)$.

The existence of a unique solution for the mixed initial-boundary value problem (1)–(5) on the cylinder Q can be proved using a constructive method, i.e., first, solving (1)–(5) on the sub-cylinder Q_1 and in turn on Q_2, and so on, until the procedure covers the whole cylinder Q. In this way, the solution in the previous step determines the next one.

For simplicity, we introduce the following notation:

$$E_j \triangleq ((j-1)a, ja), \quad Q_j = \Omega \times E_j, \quad \Sigma_j = \Gamma \times E_j,\ j = 1,2,\ldots. \tag{15}$$

Case 1: $u \in L^2(Q)$

Using Theorem 6.1 of Lions & Magenes (1972, vol. 2, p. 33), we can prove the following lemma.

Lemma 3.2. *Let*

$$u \in L^2(Q), \tag{16}$$

$$f_j(x,t) \in L^2(Q_j), \tag{17}$$

where

$$f_j = u(x,t) - \int_a^b c(x,t) y_{j-1}(x,t-h)\,dh,$$

$$y_{j-1}(\cdot,(j-1)a) \in W^1(\Omega, \mathbb{R}^\infty), \tag{18}$$

$$g_j \in W^{\frac{1}{2},\frac{1}{4}}(\Sigma_j), \tag{19}$$

where

$$g_j(x,t) = \int_a^b d(x,t) y_{j-1}(x,t-h)\,dh + v(x,t).$$

Then, there exists a unique solution $y_j \in W^{2,1}(Q_j)$ for the mixed initial-boundary value problem (1), (4) and (18).

Proof. We observe that for $j=1$, $y_0|_{Q_0}(x,t-h) = \Phi_0(x,t-h)$ and $y_0|_{\Sigma_0}(x,t-h) = \Psi_0(x,t-h)$. Then the assumptions (17)–(19) are fulfilled if we assume that $\Phi_0 \in W^{2,1}(Q_0)$, $y_0 \in W^1(\Omega,\mathbb{R}^\infty)$, $v \in W^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ and $\Psi_0 \in W^{\frac{1}{2},\frac{1}{4}}(\Sigma_0)$. These assumptions are sufficient to ensure the existence of a unique solution $y_1 \in W^{2,1}(Q_1)$. In order to extend the result to Q_2, we have to prove that $y_1(\cdot,a) \in W^1(\Omega,\mathbb{R}^\infty)$, $g_2 \in W^{\frac{1}{2},\frac{1}{4}}(\Sigma_2)$ and $f_2 \in L^2(Q_2)$. Really, from Theorem 3.1, p.19 of Lions & Magenes vol.1, $y_1 \in W^{2,1}(Q_1)$ implies that the mapping $t \to y_1(\cdot,t)$ is continuous from $[0,a] \to W^1(\Omega,\mathbb{R}^\infty)$. Thus $y_1(\cdot,a) \in W^1(\Omega,\mathbb{R}^\infty)$. Then using the trace theorem of Lions & Magenes (1972, vol. 2, p. 9) we can verify that $y_1 \in W^{2,1}((Q_1)$ implies that $y_1 \to y_1|_{\Sigma_1}$ is a linear, continuous mapping of $W^{2,1}(Q_1) \to W^{\frac{1}{2},\frac{1}{4}}(\Sigma_1)$. Assuming that d is a C^∞ function and $v \in W^{\frac{1}{2},\frac{1}{4}}(\Sigma)$, the condition $g_2 \in W^{\frac{1}{2},\frac{1}{4}}(\Sigma_2)$ is fulfilled. Also it is easy to notice that the assumption (17) follows from the fact that $y_1 \in W^{2,1}(Q_1)$ and $u \in L^2(Q)$. Then, there exists a unique solution $y_2 \in W^{2,1}(Q_2)$. Finally, we can extend our result to any Q_j, $j = 3,4\ldots$.

Theorem 3.3. *Let y_0, Φ_0, Ψ_0, v and u be given with $y_0 \in W^1(\Omega,\mathbb{R}^\infty)$, $\Phi_0 \in W^{2,1}(Q_0)$, $v \in W^{\frac{1}{2},\frac{1}{4}}(\Sigma)$, $\Psi_0 \in W^{\frac{1}{2},\frac{1}{4}}(\Sigma_0)$ and $u \in L^2(Q)$. Then, there exists a unique solution $y \in W^{2,1}(Q)$ for the mixed initial-boundary value problem (1)–(5). Moreover, $y(\cdot,ja) \in W^1(\Omega,\mathbb{R}^\infty)$ for $j = 1,2,\ldots$.*

Case 2: $v \in L^2(\Sigma)$

Using Theorem 15.2 of Lions & Magenes (1972, vol. 2, p. 81), we can prove the following lemma.

Lemma 3.4. *Let*

$$u \in W^{-\frac{1}{2},-\frac{1}{4}}(Q), \quad v \in L^2(\Sigma) \tag{20}$$

$$f_j \in W^{-\frac{1}{2},-\frac{1}{4}}(Q_j), \tag{21}$$

$$y_{j-1}(\cdot,(j-1)a) \in W^{\frac{1}{2}}(\Omega,\mathbb{R}^\infty), \tag{22}$$

$$g_j \in L^2(\Sigma_j). \tag{23}$$

Then, there exists a unique solution $y_j \in W^{\frac{3}{2},\frac{3}{4}}(Q_j)$ for the mixed initial-boundary value problem (1), (4) and (22).

Proof. For $j = 1$, the assumptions (21)–(23) are fulfilled if we assume that $\Phi_0 \in W^{\frac{3}{2},\frac{3}{4}}(Q_0)$, $y_0 \in W^{\frac{1}{2}}(\Omega, \mathbb{R}^\infty)$ and $\Psi_0 \in L^2(\Sigma_0)$. These assumptions are sufficient to ensure the existence of a unique solution $y_1 \in W^{\frac{3}{2},\frac{3}{4}}(Q_1)$. In order to extend the result to Q_2, we have to prove that $y_1(\cdot, a) \in W^{\frac{1}{2}}(\Omega, \mathbb{R}^\infty)$, $y_1|_{\Sigma_1} \in L^2(\Sigma_1)$ and $f_2 \in W^{-\frac{1}{2},-\frac{1}{4}}(Q_2)$. First using Theorem 3.1 of Lions & Magenes (1972, vol. 1, p. 19) we can prove that $y_1 \in W^{\frac{3}{2},\frac{3}{4}}(Q_1)$ implies that the mapping $t \to y_1(\cdot, t)$ is continuous from $[0, a] \to W^{\frac{3}{4}}(\Omega, \mathbb{R}^\infty) \subset W^{\frac{1}{2}}(\Omega, \mathbb{R}^\infty)$. Hence $y_1(\cdot, a) \in W^{\frac{1}{2}}(\Omega, \mathbb{R}^\infty)$. Again, from trace theorem of Lions & Magenes (1972, vol. 2, p. 9), we can verify that $y_1 \in W^{\frac{3}{2},\frac{3}{4}}(Q_1)$ implies that $y_1 \to y_1|_{\Sigma_1}$ is a linear, continuous mapping of $W^{\frac{3}{2},\frac{3}{4}}(Q_1) \to W^{1,\frac{1}{2}}(\Sigma_1)$. Thus $y_1|_{\Sigma_1} \in L^2(\Sigma_1)$. Moreover, it is worth mentioning that the assumption (21) follows from the fact that $y_1 \in W^{\frac{3}{2},\frac{3}{4}}(Q_1)$ and $u \in W^{-\frac{1}{2},-\frac{1}{4}}(Q)$. Then, there exists a unique solution $y_2 \in W^{\frac{3}{2},\frac{3}{4}}(Q_2)$. Finally, we can extend our result to any Q_j, $j = 3, 4 \ldots$.

Theorem 3.5. *Let y_0, Φ_0, Ψ_0, v and u be given with $y_0 \in W^{\frac{1}{2}}(\Omega, \mathbb{R}^\infty)$, $\Phi_0 \in W^{\frac{3}{2},\frac{3}{4}}(Q_0)$, $\Psi_0 \in L^2(\Sigma_0)$, $v \in L^2(\Sigma)$ and $u \in W^{-\frac{1}{2},-\frac{1}{4}}(Q)$. Then, there exists a unique solution $y \in W^{\frac{3}{2},\frac{3}{4}}(Q)$ for the mixed initial-boundary value problem (1)–(5). Moreover, $y(\cdot, ja) \in W^{\frac{1}{2}}(\Omega, \mathbb{R}^\infty)$ for $j = 1, 2, \ldots$.*

Now we shall verify the existence of a unique solution for the problem (1), (2), (3) and (5) with the Dirichlet boundary condition involving a time lag

$$y(x, t) = g(x, t) \tag{24}$$

where g is given by the formula (12).

Making use of the results of Lions & Magenes (1972, vol. 2, p. 33 and p. 81) we can prove the following lemmas and theorems.

Case 3: $u \in L^2(Q)$

Lemma 3.6. *Let*

$$u \in L^2(Q), \tag{25}$$

$$f_j \in L^2(Q_j), \tag{26}$$

$$y_{j-1}(\cdot, (j-1)a) \in W^1(\Omega, \mathbb{R}^\infty), \tag{27}$$

$$g_j \in W^{\frac{3}{2},\frac{3}{4}}(\Sigma_j), \tag{28}$$

and the following compatibility relation is fulfilled

$$y_{j-1}(x, (j-1)a) = g_j(x, (j-1)a), \quad \text{on } \Gamma. \tag{29}$$

Then, there exists a unique solution $y_j \in W^{2,1}(Q_j)$ for the mixed initial-boundary value problem (1), (24) and (27).

Proof. For $j = 1$, the assumptions (26)–(28) can be satisfied if we assume that $\Phi_0 \in W^{2,1}(Q_0)$, $v \in W^{\frac{3}{2},\frac{3}{4}}(\Sigma)$ and $\Psi_0 \in W^{\frac{3}{2},\frac{3}{4}}(\Sigma_0)$. These assumptions are sufficient to ensure the existence of a unique solution $y_1 \in W^{2,1}(Q_1)$ if $y_0 \in W^1(\Omega, \mathbb{R}^\infty)$ and the following compatibility relation is satisfied

$$y_0(x, 0) = g_1(x, 0), \quad \text{on } \Gamma. \tag{30}$$

In order to extend the result to Q_2, we have to prove that $y_1 \in W^{2,1}(Q_1)$ and it is necessary to impose the compatibility relation

$$y_1(x, a) = g_2(x, a), \quad \text{on } \Gamma \tag{31}$$

and it is sufficient to verify that

$$f_2 \in L^2(Q_2), \tag{32}$$

$$y_1(\cdot, a) \in W^1(\Omega, \mathbb{R}^\infty), \tag{33}$$

$$g_2 \in W^{\frac{3}{2},\frac{3}{4}}(\Sigma_2). \tag{34}$$

First using the solution in the previous step and the condition (25) we can prove immediately the condition (32). To verify (33), we use the fact that $y_1 \in W^{2,1}(Q_1)$ implies that the mapping $t \to y_1(\cdot, t)$ is continuous from $[0, a] \to W^1(\Omega, \mathbb{R}^\infty)$ (by Theorem 3.1 of Lions & Magenes (1972, vol. 1, p. 19)), hence $y_1(\cdot, a) \in W^1(\Omega, \mathbb{R}^\infty)$. From the trace theorem of Lions & Magenes (1972, vol. 2, p. 9) $y_1 \in W^{2,1}((Q_1)$ implies that $y_1 \to y_1|_{\Sigma_1}$ is a linear, continuous mapping of $W^{2,1}(Q_1) \to W^{\frac{3}{2},\frac{3}{4}}(\Sigma_1)$. Assuming that d is a C^∞ function and $v \in W^{\frac{3}{2},\frac{3}{4}}(\Sigma)$, the condition (34) is fulfilled. Then, there exists a unique solution $y_2 \in W^{2,1}(Q_2)$. Finally, we can extend our result to any Q_j, $j = 3, 4 \ldots$.

Theorem 3.7. *Let y_0, Φ_0, Ψ_0, v and u be given with $y_0 \in W^1(\Omega, \mathbb{R}^\infty)$, $\Phi_0 \in W^{2,1}(Q_0)$, $v \in W^{\frac{3}{2},\frac{3}{4}}(\Sigma)$, $\Psi_0 \in W^{\frac{3}{2},\frac{3}{4}}(\Sigma_0)$, $u \in L^2(Q)$ and the compatibility relation (29) is fulfilled. Then, there exists a unique solution $y \in W^{2,1}(Q)$ for the mixed initial-boundary value problem (1), (2), (3) (5) and (24) with $y(\cdot, ja) \in W^1(\Omega, \mathbb{R}^\infty)$ for $j = 1, 2, \ldots$.*

Case 4: $v \in L^2(\Sigma)$

Lemma 3.8. *Let*

$$u \in W^{-\frac{3}{2},-\frac{3}{4}}(Q), \quad v \in L^2(\Sigma) \tag{35}$$

$$f_j \in W^{-\frac{3}{2},-\frac{3}{4}}(Q_j), \tag{36}$$

$$y_{j-1}(\cdot, (j-1)a) \in W^{-\frac{1}{2}}(\Omega, \mathbb{R}^\infty), \tag{37}$$

$$g_j \in L^2(\Sigma_j). \tag{38}$$

Then, there exists a unique solution $y_j \in W^{\frac{1}{2},\frac{1}{4}}(Q_j)$ for the mixed initial-boundary value problem (1), (24) and (37).

Proof. We observe that for $j = 1$, the assumptions (36)–(38) are satisfied if we assume that $\Phi_0 \in W^{\frac{1}{2},\frac{1}{4}}(Q_0)$, $y_0 \in W^{-\frac{1}{2}}(\Omega, \mathbb{R}^\infty)$ and $\Psi_0 \in L^2(\Sigma_0)$. These assumptions are sufficient to ensure the existence of a unique solution $y_1 \in W^{\frac{1}{2},\frac{1}{4}}(Q_1)$. Next for $j = 2$, using the solution in the first step, it is sufficient to verify that $f_2 \in W^{-\frac{3}{2},-\frac{3}{4}}(Q_2)$, $y_1(\cdot, a) \in W^{-\frac{1}{2}}(\Omega, \mathbb{R}^\infty)$ and $y_1|_{\Sigma_1} \in L^2(\Sigma_1)$. Then it is worth mentioning that the condition $f_2 \in W^{-\frac{3}{2},-\frac{3}{4}}(Q_2)$ follows from the fact that $y_1 \in W^{\frac{1}{2},\frac{1}{4}}(Q_1)$ and $u \in W^{-\frac{3}{2},-\frac{3}{4}}(Q)$. Since $y_1 \in W^{\frac{1}{2},\frac{1}{4}}(Q_1)$ implies that the mapping $t \to y_1(\cdot, t)$ is continuous from $[0, a] \to W^{\frac{1}{4}}(\Omega, \mathbb{R}^\infty)$ (by Theorem 3.1 of Lions & Magenes (1972, vol. 1, p. 19)), hence $y_1(\cdot, a) \in W^{\frac{1}{4}}(\Omega, \mathbb{R}^\infty) \subset L^2(\Omega, \mathbb{R}^\infty) \subset W^{-\frac{1}{4}}(\Omega, \mathbb{R}^\infty) \subset W^{-\frac{1}{2}}(\Omega, \mathbb{R}^\infty)$. We shall prove that $y_1|_{\Sigma_1} \in L^2(\Sigma_1)$. We must notice that for proving $y_1|_{\Sigma_1} \in L^2(\Sigma_1)$ we cannot use the trace theorem of Lions & Magenes (1972, vol. 2, p. 9), since $y_1 \in W^{\frac{1}{2},\frac{1}{4}}(Q_1)$. It is worth mentioning that this difficulty can be avoided by using the condition $y_1|_{\Sigma_1} = g_1 \in L^2(\Sigma_1)$. This implies that $y_1|_{\Sigma_1} \in L^2(\Sigma_1)$. Then, there exists a unique solution $y_2 \in W^{\frac{1}{2},\frac{1}{4}}(Q_2)$. Finally, we can extend our result to any Q_j, $j = 3, 4 \ldots$.

Theorem 3.9. *Let y_0, Φ_0, Ψ_0, v and u be given with $y_0 \in W^{-\frac{1}{2}}(\Omega, \mathbb{R}^\infty)$, $\Phi_0 \in W^{\frac{1}{2},\frac{1}{4}}(Q_0)$, $\Psi_0 \in L^2(\Sigma_0)$, $v \in L^2(\Sigma)$ and $u \in W^{-\frac{3}{2},-\frac{3}{4}}(Q)$. Then, there exists a unique solution $y \in W^{\frac{1}{2},\frac{1}{4}}(Q)$ for the mixed initial-boundary value problem (1), (2), (3), (5) and (24). Moreover, $y(\cdot, ja) \in W^{-\frac{1}{2}}(\Omega, \mathbb{R}^\infty)$ for $j = 1, 2, \ldots$.*

4. Optimal Distributed Control

Now, we shall restrict our considerations to the case of the distribute control for the Neumann problem. Therefore, we shall formulate the minimum-time problem for (1)–(5) in the context of the Theorem 3.3, i.e.,

$$u \in U = \{u \in L^2(Q) : |u(x, t)| \leq 1\}. \tag{39}$$

We shall define the reachable set H such that

$$H = \{y \in L^2(\Omega, \mathbb{R}^\infty) : ||y - z_d||_{L^2(\Omega, \mathbb{R}^\infty)} \leq \varepsilon\} \tag{40}$$

where $z_d \in L^2(\Omega, \mathbb{R}^\infty)$ and $\varepsilon > 0$.

Solving the stated minimum-time problem is equivalent to hitting the target set H in minimum time, that is, minimizing the time t, for which $y(t; u) \in H$ and $u \in U$. Moreover, we assume that

$$\text{there exists a } T > 0 \text{ and } u \in U \text{ with } y(T; u) \in H \tag{41}$$

then we have the following theorem

Theorem 4.1. *If the assumption (41) holds, then the set H is reached in minimum time t^* by an admissible control $u^* \in U$. Moreover*

$$\int_\Omega [z_d - y(t^*; u^*)]\, [y(t^*; u) - y(t^*; u^*)]\; d\rho \leq 0, \quad \forall\, u \in U. \tag{42}$$

Proof. Let us define the following set

$$t^* := \inf\{t : y(t;u) \in H \text{ for some } u \in U\} \tag{43}$$

The minimum is well defined, as (41) guarantees that this set is nonempty. By definition, we can choose $t_n \downarrow t^*$ and admissible controls $\{u_n\}$ such that

$$y(t_n;u_n) \in H, \quad n = 1, 2, 3, \ldots. \tag{44}$$

Each u_n is defined on $\Omega \times (0, t_n) \supset \Omega \times (0, t^*)$. To simplify the notation, we denote the restriction of u_n to $\Omega \times (0, t^*)$ again by u_n. The set of admissible controls then forms a weakly compact, convex set in $L^2(\Omega \times (0, t^*))$, and so we can extract a weakly convergent subset $\{u_m\}$, which converges weakly to some admissible control u^*.

Consequently, Theorem 3.3 implies that $y(t;u) \in W^1(\Omega, \mathbb{R}^\infty) \subset L^2(\Omega, \mathbb{R}^\infty)$ for each $u \in L^2(Q)$ and $t > 0$. Then using Theorem 1.2 of (Lions, 1971, p. 102) and Theorem 3.3 it is easy to verify that the mapping $u \to y(t^*;u)$ from $L^2(\Omega \times (0, t^*))$ into $L^2(\Omega, \mathbb{R}^\infty)$, is continuous. Since any continuous linear mapping between Banach spaces is also weakly continuous (Dunford and Schwartz, 1958), Theorem V. 3.15, the affine mapping $u \to y(t^*;u)$ must also be weakly continuous. Hence,

$$y(t^*;u_m) \to y(t^*;u^*) \text{ weakly in } L^2(\Omega, \mathbb{R}^\infty). \tag{45}$$

Moreover,

$$\frac{dy(u)}{dt} \in L^2\left([0, t^*]; L^2(\Omega, \mathbb{R}^\infty)\right), \tag{46}$$

for each $u \in U$, by definition of $W^{2,1}(\Omega \times (0, t^*))$ and

$$||y(t_m;u_m) - y(t^*;u_m)||_{L^2(\Omega,\mathbb{R}^\infty)} = \left|\left|\int_{t^*}^{t_m} \dot{y}(\sigma;u_m)\, d\sigma\right|\right|_{L^2(\Omega,\mathbb{R}^\infty)} \tag{47}$$

$$\leq \sqrt{t_m - t^*}\left(\int_{t^*}^{t_m} ||\dot{y}(\sigma;u_m)||^2_{L^2(\Omega,\mathbb{R}^\infty)}\, d\sigma\right)^{1/2} \tag{48}$$

Applying Theorem 1.2 of (Lions, 1971) and Theorem 3.3 again, the set $\{\dot{y}(u_m)\}$ must be bounded in $L^2(0, t^*; L^2(\Omega, \mathbb{R}^\infty))$, and so

$$||y(t_m;u_m) - y(t^*;u_m)||_{L^2(\Omega,\mathbb{R}^\infty)} \leq M\sqrt{t_m - t^*}. \tag{49}$$

Combining (45) and (49) shows that

$$y(t_m;u_m) - y(t^*;u^*) = (y(t_m;u_m) - y(t^*;u_m)) + (y(t^*;u_m) - y(t^*;u^*)), \tag{50}$$

converges weakly to zero in $L^2(\Omega, \mathbb{R}^\infty)$, and therefore $y(t^*;u^*) \in H$ as H is closed and convex, hence weakly closed. This shows that H is reached in time t^* by an admissible control accordingly, t^* must be the minimum time and u^* an optimal control.

We shall now prove the second part of our theorem. Indeed, from Theorem 3.1 (Lions and Magenes, 1972, Vol. 1, p. 19) $y(u) \in W^{2,1}(Q)$ implies that the mapping $t \to y(t;u)$ is continuous from $[0,T] \to W^1(\Omega,\mathbb{R}^\infty) \subset L^2(\Omega,\mathbb{R}^\infty)$ is continuous for each fixed u, and so $y(t^*;u) \notin \text{int}H$, for any $u \in U$, by the minimality of t^*.

From our earlier remarks, the set

$$\mathcal{A}(t^*) = \{y(t^*;u_x) : u_x \in U\}, \tag{51}$$

is weakly compact and convex in $L^2(\Omega,\mathbb{R}^\infty)$. Applying Theorem 21.11 of (Choquet, 1969) to the sets $\mathcal{A}(t^*)$ and H shows that there exists a nontrivial hyperplane $z \in L^2(\Omega,\mathbb{R}^\infty)$ separating these sets, that is,

$$\int_\Omega zy(t^*;u)\,d\rho \le \int_\Omega zy(t^*;u^*)\,d\rho \le \int_\Omega zy\,d\rho \tag{52}$$

for all $u \in U$ and $y \in L^2(\Omega,\mathbb{R}^\infty)$ with $||y - z_d||_{L^2(\Omega,\mathbb{R}^\infty)} \le \varepsilon$.

From the second inequality in (52), z must support the set H at $y(t^*;u^*)$. Since $L^2(\Omega,\mathbb{R}^\infty)$ is a Hilbert space, z must be of the form

$$z = \mu(z_d - y(t^*;u^*)) \text{ for some } \mu > 0. \tag{53}$$

Subsequently, dividing (52) by μ gives the desired result (42).

We shall apply Theorem 4.1 to the control problem of (1)–(5).

To simplify (42), we introduce the adjoint equation, and for every $u \in U$ we define the adjoint variable $p = p(u) = p(x,t;u)$ as the solution of the following system

$$-\frac{\partial p(u)}{\partial t} + \mathcal{A}^*(t)p(u) + \int_a^b c(x,t+h)p(x,t+h;u)\,dh = 0, \quad x \in \Omega,\, t \in (0,t^*-b), \tag{54}$$

$$-\frac{\partial p(u)}{\partial t} + \mathcal{A}^*(t)p(u) + \int_a^{t^*-t} c(x,t+h)p(x,t+h;u)\,dh = 0, \quad x \in \Omega,\, t \in (t^*-b,t^*-a), \tag{55}$$

$$-\frac{\partial p(u)}{\partial t} + \mathcal{A}^*(t)p(u) = 0, \quad x \in \Omega,\, t \in (t^*-a,t^*), \tag{56}$$

$$p(x,t^*;u) = z_d(x) - y(x,t^*;u), \quad x \in \Omega, \tag{57}$$

$$\frac{\partial p(u)}{\partial \eta_{\mathcal{A}^*}}(x,t) = \int_a^b d(x,t+h)p(x,t+h;u)\,dh, \quad x \in \Gamma,\, t \in (0,t^*-b), \tag{58}$$

$$\frac{\partial p(u)}{\partial \eta_{\mathcal{A}^*}}(x,t) = \int_a^{t^*-t} d(x,t+h)p(x,t+h;u)\,dh, \quad x \in \Gamma,\, t \in (t^*-b,t^*-a), \tag{59}$$

$$\frac{\partial p(u)}{\partial \eta_{\mathcal{A}^*}}(x,t) = 0, \quad x \in \Gamma,\, t \in (t^*-a,t^*), \tag{60}$$

where

$$\frac{\partial p(u)}{\partial \eta_{\mathcal{A}^*}}(x,t) = \sum_{k=1}^{\infty}(D_k p(u))\cos(n,x_k), \tag{61}$$

$$\mathcal{A}^*(t)p(u) = \Big(-\sum_{k=1}^{\infty} D_k^2 + q(x,t)\Big)p(u). \tag{62}$$

Remark 4.2. If $t^* < b$, then we consider (55) and (59) on $\Omega\times(0,t^*-a)$ and $\Gamma\times(0,t^*-a)$, respectively.

The existence of a unique solution to the problem (54)–(60) on the cylinder $\Omega\times(0,t^*)$ can be proved using a constructive method. It is easy to notice that for given z_d and u, the problem (54)–(60) can be solved backwards in time starting from $t = t^*$, i.e., first, solving (54)–(60) on the sub-cylinder Q_k and in turn on Q_{k-1}, and so on, until the procedure covers the whole cylinder $\Omega\times(0,t^*)$. For this purpose, we may apply Theorem 3.3 (with an obvious change of variables).

Hence, using Theorem 3.3, the following result can be proved.

Theorem 4.3. *Let the hypothesis of Theorem 3.3 be satisfied. Then for given $z_d \in L^2(\Omega,\mathbb{R}^\infty)$ and any $u \in L^2(Q)$, there exists a unique solution $p(u) \in W^{2,1}(\Omega\times(0,t^*))$ for the adjoint problem (54)–(60).*

Now, we have the main result.

Theorem 4.4. *If the assumptions concerning system (1)–(5) and controllability condition (41) are satisfied, then the time-optimal control u^* exists and is characterized by the following condition*

$$\int_0^{t^*}\int_\Omega p(u^*)(u-u^*)\,d\rho\,dt \le 0, \quad \forall\, u\in U, \tag{63}$$

where $p(u^*)$ is the solution of the adjoint system (54)–(60).

Proof. We simplify the left-hand side of the inequality (42) using the adjoint equation (54)–(60). For this purpose, setting $u = u^*$ in (54)–(60), multiplying both sides of (54), (55) and (56) by $y(u)-y(u^*)$, then integrating over $\Omega\times(0,t^*-b)$, $\Omega\times(t^*-b,t^*-a)$ and $\Omega\times(t^*-a,t^*)$ respectively and then adding both sides of (54), (55) and (56), we get

$$\begin{aligned}
&\int_0^{t^*}\int_\Omega\left(-\frac{\partial p(u^*)}{\partial t}+\mathcal{A}^*(t)p(u^*)\right)(y(u)-y(u^*))\,d\rho\,dt\\
&+\int_0^{t^*-b}\int_\Omega\left(\int_a^b c(x,t+h)p(x,t+h;u^*)\,dh\right)\times(y(x,t;u)-y(x,t;u^*))\,d\rho\,dt\\
&+\int_{t^*-b}^{t^*-a}\int_\Omega\left(\int_a^{t^*-t} c(x,t+h)p(x,t+h;u^*)\,dh\right)(y(x,t;u)-y(x,t;u^*))\,d\rho\,dt\\
=&-\int_\Omega p(x,t^*;u^*)(y(x,t^*;u)-y(x,t^*;u^*))\,d\rho+\int_0^{t^*}\int_\Omega p(u^*)\frac{\partial}{\partial t}(y(u)-y(u^*))\,d\rho\,dt\\
&+\int_0^{t^*}\int_\Omega \mathcal{A}^*p(u^*)(y(u)-y(u^*))\,d\rho\,dt\\
&+\int_0^{t^*-b}\int_\Omega\int_a^b c(x,t+h)p(x,t+h;u^*)\,(y(x,t;u)-y(x,t;u^*))\,dh\,d\rho\,dt\\
&+\int_{t^*-b}^{t^*-a}\int_\Omega\int_a^{t^*-t} c(x,t+h)p(x,t+h;u^*)\,(y(x,t;u)-y(x,t;u^*))\,dh\,d\rho\,dt=0.
\end{aligned} \tag{64}$$

Then, applying (57), the equation (64) can be expressed as

$$\begin{aligned}
&\int_\Omega (z_d - y(t^*;u^*))(y(x,t^*;u) - y(x,t^*;u^*))\,d\rho \\
&= \int_0^{t^*}\int_\Omega p(u^*)\frac{\partial}{\partial t}(y(u)-y(u^*))\,d\rho\,dt + \int_0^{t^*}\int_\Omega \mathcal{A}^* p(u^*)(y(u)-y(u^*))\,d\rho\,dt \\
&\quad + \int_a^b\int_\Omega\int_0^{t^*-b} c(x,t+h)p(x,t+h;u^*)\,(y(x,t;u)-y(x,t;u^*))\,dt\,d\rho\,dh \\
&\quad + \int_a^{t^*-t}\int_\Omega\int_{t^*-b}^{t^*-a} c(x,t+h)p(x,t+h;u^*)\,(y(x,t;u)-y(x,t;u^*))\,dt\,d\rho\,dh.
\end{aligned} \tag{65}$$

Using (1), the first integral on the right-hand side of (65) can be rewritten as

$$\begin{aligned}
&\int_0^{t^*}\int_\Omega p(u^*)\frac{\partial}{\partial t}(y(u)-y(u^*))\,d\rho\,dt \\
&= -\int_0^{t^*}\int_\Omega p(u^*)\mathcal{A}(y(u)-y(u^*))\,d\rho\,dt \\
&\quad - \int_0^{t^*}\int_\Omega p(x,t;u^*)\left(\int_a^b c(x,t)(y(x,t-h;u)-y(x,t-h;u^*))\,dh\right)\,d\rho\,dt \\
&\quad + \int_0^{t^*}\int_\Omega p(x,t;u^*)(u-u^*)\,d\rho\,dt
\end{aligned}$$

$$
\begin{aligned}
&= -\int_0^{t^*}\int_\Omega p(u^*)\mathcal{A}(y(u)-y(u^*))\,d\rho\,dt \\
&\quad -\int_0^{t^*}\int_\Omega\int_a^b p(x,t;u^*)c(x,t)(y(x,t-h;u)-y(x,t-h;u^*))\,dh\,d\rho\,dt \\
&\qquad +\int_0^{t^*}\int_\Omega p(x,t;u^*)(u-u^*)\,d\rho\,dt \\
&= -\int_0^{t^*}\int_\Omega p(u^*)\mathcal{A}(y(u)-y(u^*))\,d\rho\,dt \\
&\quad -\int_a^b\int_\Omega\int_0^{t^*} p(x,t;u^*)c(x,t)(y(x,t-h;u)-y(x,t-h;u^*))\,dt\,d\rho\,dh \\
&\qquad +\int_0^{t^*}\int_\Omega p(x,t;u^*)(u-u^*)\,d\rho\,dt \\
&= -\int_0^{t^*}\int_\Omega p(u^*)\mathcal{A}(y(u)-y(u^*))\,d\rho\,dt \\
&\quad -\int_a^b\int_\Omega\int_{-h}^{t^*-h} p(x,t'+h;u^*)c(x,t'+h)(y(x,t';u)-y(x,t';u^*))\,dt'\,d\rho\,dh \\
&\quad +\int_0^{t^*}\int_\Omega p(x,t;u^*)(u-u^*)\,d\rho\,dt \\
&= -\int_0^{t^*}\int_\Omega p(u^*)\mathcal{A}(y(u)-y(u^*))\,d\rho\,dt \qquad (66)\\
&\quad -\int_a^b\int_\Omega\int_{-h}^{0} p(x,t'+h;u^*)c(x,t'+h)(y(x,t';u)-y(x,t';u^*))\,dt'\,d\rho\,dh \\
&\quad -\int_a^b\int_\Omega\int_{0}^{t^*-b} p(x,t'+h;u^*)c(x,t'+h)(y(x,t';u)-y(x,t';u^*))\,dt'\,d\rho\,dh \\
&\quad -\int_a^b\int_\Omega\int_{t^*-b}^{t^*-h} p(x,t'+h;u^*)c(x,t'+h)(y(x,t';u)-y(x,t';u^*))\,dt'\,d\rho\,dh \\
&\quad +\int_0^{t^*}\int_\Omega p(x,t;u^*)(u-u^*)\,d\rho\,dt \\
&= -\int_0^{t^*}\int_\Omega p(u^*)\mathcal{A}(y(u)-y(u^*))\,d\rho\,dt \\
&\quad -\int_a^b\int_\Omega\int_{-h}^{0} p(x,t'+h;u^*)c(x,t'+h)(y(x,t';u)-y(x,t';u^*))\,dt'\,d\rho\,dh \\
&\quad -\int_a^b\int_\Omega\int_{0}^{t^*-b} p(x,t'+h;u^*)c(x,t'+h)(y(x,t';u)-y(x,t';u^*))\,dt'\,d\rho\,dh \\
&\quad -\int_a^{t^*-t}\int_\Omega\int_{t^*-b}^{t^*-a} p(x,t'+h;u^*)c(x,t'+h)(y(x,t';u)-y(x,t';u^*))\,dt'\,d\rho\,dh \\
&\quad +\int_0^{t^*}\int_\Omega p(x,t;u^*)(u-u^*)\,d\rho\,dt.
\end{aligned}
$$

The second integral on the right-hand side of (65), in view of Green formula, can be expressed as

$$\int_0^{t^*}\int_\Omega \mathcal{A}^* p(u^*)(y(u)-y(u^*))\,d\rho\,dt = \int_0^{t^*}\int_\Omega p(u^*)\mathcal{A}(y(u)-y(u^*))\,d\rho\,dt$$
$$+\int_0^{t^*}\int_\Gamma p(u^*)\left(\frac{\partial y(u)}{\partial\eta_\mathcal{A}}-\frac{\partial y(u^*)}{\partial\eta_\mathcal{A}}\right)d\Gamma\,dt - \int_0^{t^*}\int_\Gamma \frac{\partial p(u^*)}{\partial\eta_{\mathcal{A}^*}}(y(u)-y(u^*))\,d\Gamma\,dt. \tag{67}$$

Using the boundary condition (4), the second component on the right-hand side of (67) can be written as

$$\begin{aligned}
&\int_0^{t^*}\int_\Gamma p(u^*)\left(\frac{\partial y(u)}{\partial\eta_\mathcal{A}}-\frac{\partial y(u^*)}{\partial\eta_\mathcal{A}}\right)d\Gamma\,dt\\
&=\int_0^{t^*}\int_\Gamma p(x,t;u^*)\left(\int_a^b d(x,t)(y(x,t-h;u)-y(x,t-h;u^*))\,dh\right)d\Gamma\,dt\\
&=\int_0^{t^*}\int_\Gamma\int_a^b p(x,t;u^*)d(x,t)(y(x,t-h;u)-y(x,t-h;u^*))\,dh\,d\Gamma\,dt\\
&=\int_a^b\int_\Gamma\int_0^{t^*} p(x,t;u^*)d(x,t)(y(x,t-h;u)-y(x,t-h;u^*))\,dt\,d\Gamma\,dh\\
&=\int_a^b\int_\Gamma\int_{-h}^{t^*-h} p(x,t'+h;u^*)d(x,t'+h)(y(x,t';u)-y(x,t';u^*))\,dt'\,d\Gamma\,dh\\
&=\int_a^b\int_\Gamma\int_{-h}^{0} p(x,t'+h;u^*)d(x,t'+h)(y(x,t';u)-y(x,t';u^*))\,dt'\,d\Gamma\,dh\\
&\quad+\int_a^b\int_\Gamma\int_{0}^{t^*-b} p(x,t'+h;u^*)d(x,t'+h)(y(x,t';u)-y(x,t';u^*))\,dt'\,d\Gamma\,dh\\
&\quad+\int_a^b\int_\Gamma\int_{t^*-b}^{t^*-h} p(x,t'+h;u^*)d(x,t'+h)(y(x,t';u)-y(x,t';u^*))\,dt'\,d\Gamma\,dh\\
&=\int_a^b\int_\Gamma\int_{-h}^{0} p(x,t'+h;u^*)d(x,t'+h)(y(x,t';u)-y(x,t';u^*))\,dt'\,d\Gamma\,dh\\
&\quad+\int_a^b\int_\Gamma\int_{0}^{t^*-b} p(x,t'+h;u^*)d(x,t'+h)(y(x,t';u)-y(x,t';u^*))\,dt'\,d\Gamma\,dh\\
&\quad+\int_a^{t^*-t}\int_\Gamma\int_{t^*-b}^{t^*-a} p(x,t'+h;u^*)d(x,t'+h)(y(x,t';u)-y(x,t';u^*))\,dt'\,d\Gamma\,dh.
\end{aligned}\tag{68}$$

The last component in (67) can be rewritten as

$$\int_0^{t^*}\int_\Gamma \frac{\partial p(u^*)}{\partial\eta_{\mathcal{A}^*}}(y(u)-y(u^*))\,d\Gamma\,dt = \int_0^{t^*-b}\int_\Gamma \frac{\partial p(u^*)}{\partial\eta_{\mathcal{A}^*}}(y(u)-y(u^*))\,d\Gamma\,dt$$
$$+\int_{t^*-b}^{t^*-a}\int_\Gamma \frac{\partial p(u^*)}{\partial\eta_{\mathcal{A}^*}}(y(u)-y(u^*))\,d\Gamma\,dt+\int_{t^*-a}^{t^*}\int_\Gamma \frac{\partial p(u^*)}{\partial\eta_{\mathcal{A}^*}}(y(u)-y(u^*))\,d\Gamma\,dt. \tag{69}$$

Substituting (68) and (69) into (67) and then (66) and (67) into (65), we obtain

$$
\begin{aligned}
&\int_\Omega (z_d - y(t^*;u^*))(y(x,t^*;u) - y(x,t^*;u^*))\,d\rho \\
&= -\int_0^{t^*}\int_\Omega p(u^*)\mathcal{A}(y(u)-y(u^*))\,d\rho\,dt \\
&\quad -\int_a^b\int_\Omega\int_{-h}^0 c(x,t+h)p(x,t+h;u^*)(y(x,t;u)-y(x,t;u^*))\,dt\,d\rho\,dh \\
&\quad -\int_a^b\int_\Omega\int_0^{t^*-b} c(x,t+h)p(x,t+h;u^*)(y(x,t;u)-y(x,t;u^*))\,dt\,d\rho\,dh \\
&\quad -\int_a^{t^*-t}\int_\Omega\int_{t^*-b}^{t^*-a} c(x,t+h)p(x,t+h;u^*)(y(x,t;u)-y(x,t;u^*))\,dt\,d\rho\,dh \\
&\quad +\int_0^{t^*}\int_\Omega p(u^*)\mathcal{A}(y(u)-y(u^*))\,d\rho\,dt \\
&\quad +\int_a^b\int_\Gamma\int_{-h}^0 d(x,t+h)p(x,t+h;u^*)(y(x,t;u)-y(x,t;u^*))\,dt\,d\Gamma\,dh \\
&\quad +\int_a^b\int_\Gamma\int_0^{t^*-b} d(x,t+h)p(x,t+h;u^*)(y(x,t;u)-y(x,t;u^*))\,dt\,d\Gamma\,dh \\
&\quad +\int_a^{t^*-t}\int_\Gamma\int_{t^*-b}^{t^*-a} d(x,t+h)p(x,t+h;u^*)(y(x,t;u)-y(x,t;u^*))\,dt\,d\Gamma\,dh \\
&\quad +\int_0^{t^*}\int_\Omega p(x,t;u^*)(u-u^*)\,d\rho\,dt \\
&-\int_0^{t^*-b}\int_\Gamma \frac{\partial p(u^*)}{\partial\eta_{\mathcal{A}^*}}(y(u)-y(u^*))\,d\Gamma\,dt \\
&-\int_{t^*-b}^{t^*-a}\int_\Gamma \frac{\partial p(u^*)}{\partial\eta_{\mathcal{A}^*}}(y(u)-y(u^*))\,d\Gamma\,dt - \int_{t^*-a}^{t^*}\int_\Gamma \frac{\partial p(u^*)}{\partial\eta_{\mathcal{A}^*}}(y(u)-y(u^*))\,d\Gamma\,dt \\
&+\int_a^b\int_\Omega\int_0^{t^*-b} c(x,t+h)p(x,t+h;u^*)(y(x,t;u)-y(x,t;u^*))\,dt\,d\rho\,dh \\
&+\int_a^{t^*-t}\int_\Omega\int_{t^*-b}^{t^*-a} c(x,t+h)p(x,t+h;u^*)(y(x,t;u)-y(x,t;u^*))\,dt\,d\rho\,dh.
\end{aligned}
\tag{70}
$$

Then, using the fact that $y(x,t;u) = y(x,t;u^*) = \Phi_0(x,t)$ for $x \in \Omega$ and $t \in [-b,0)$, and $y(x,t;u) = y(x,t;u^*) = \Psi_0(x,t)$ for $x \in \Gamma$ and $t \in [-b,0)$, we obtain

$$
\int_\Omega (z_d - y(t^*;u^*))(y(x,t^*;u) - y(x,t^*;u^*))\,d\rho = \int_0^{t^*}\int_\Omega p(x,t;u^*)(u-u^*)\,d\rho\,dt, \tag{71}
$$

then, substituting (71) into (42), we get (63) and this finishes proof of the theorem.

5. Generalization

Time-optimal control problem presented her can be extended to certain different two cases. Case 1: Time-optimal control problem for (2×2) coupled system of parabolic equations with infinite number of variables, in which time lags appear in integral form in both the state equation and the boundary condition. Case 2: Time-optimal control problem for $(n \times n)$ coupled system of parabolic equations with infinite number of variables, in which time lags appear in integral form in both the state equation and the boundary condition.

5.1 Time-optimal Control Problem for (2×2) Coupled System of Parabolic Equations with Infinite Number of Variables.

We can extend the discussions to study the time-optimal control problem for 2×2 coupled system of parabolic equations with infinite number of variables, in which time lags appear in integral form in both the state equation and the boundary condition.

Consider now the distributed-parameter system described by the following (2×2) coupled system of parabolic equations with infinite number of variables, for $i = 1, 2$,

$$\frac{\partial y_i}{\partial t} + \mathbf{A}(t) y_i + \int_a^b c_i(x,t) y(x, t-h)\, dh = u_i, \quad x \in \Omega,\ t \in (0,T),\ h \in (a,b), \tag{72}$$

$$y_i(x,t') = \Phi_{i,0}(x,t'), \quad x \in \Omega,\ t' \in [-b, 0), \tag{73}$$

$$y_i(x,0) = y_{i,0}(x), \quad x \in \Omega, \tag{74}$$

$$\frac{\partial y_i(x,t)}{\partial \eta_{\mathbf{A}}} = \int_a^b d_i(x,t) y_i(x,t-h)\, dh + v_i, \quad x \in \Gamma,\ t \in (0,T),\ h \in (a,b), \tag{75}$$

$$y_i(x,t') = \Psi_{i,0}(x,t'), \quad x \in \Gamma,\ t' \in [-b, 0), \tag{76}$$

where

$$\mathbf{A}(t) y_i(x) = \left(-\sum_{k=1}^{\infty} D_k^2 + q(x,t) \right) y_i(x) + \sum_{j=1}^{2} a_{ij} y_j(x) \quad \forall\, i = 1, 2, \tag{77}$$

$$a_{ij} = \begin{cases} 1, & i \geq j; \\ -1, & i < j. \end{cases} \tag{78}$$

It is easy to see that $\mathbf{A}(t)$ is (2×2) matrix which takes the form

$$\mathbf{A}(t) = \begin{pmatrix} -\sum_{k=1}^{\infty} D_k^2 + q + 1 & -1 \\ 1 & -\sum_{k=1}^{\infty} D_k^2 + q + 1 \end{pmatrix}_{2 \times 2}. \tag{79}$$

Also we have

$$y_i \equiv y_i(x,t;\mathbf{u}), \quad u_i \equiv u_i(x,t), \quad v_i \equiv v_i(x,t), \quad \mathbf{u} \equiv (u_1, u_2),$$

c_i and d_i, $i = 1, 2$, are real C^∞ functions defined on $\overline{Q}$ and Σ, respectively, $\Phi_{i,0}$ and $\Psi_{i,0}$, $i = 1, 2$, are initial functions defined on Q_0 and Σ_0, respectively.

Now we discuss the case of the distributed control for the Neumann problem. Then, as Section 3, for $\mathbf{u} = (u_1, u_2) \in (L^2(Q))^2$, we can obtain the following results.

Lemma 5.1. *Let*

$$\mathbf{u} \in \left(L^2(Q)\right)^2, \tag{80}$$

$$\mathbf{f}_j = (f_{1,j}, f_{2,j}) \in \left(L^2(Q_j)\right)^2, \tag{81}$$

where

$$f_{i,j}(x,t) = u_i(x,t) - \int_a^b c_i(x,t) y_{i,j-1}(x,t-h)\, dh, \quad i = 1, 2,$$

$$\mathbf{y}_{j-1}(\cdot,(j-1)a) = (y_{1,j-1}(\cdot,(j-1)a), y_{2,j-1}(\cdot,(j-1)a)) \in \left(W^1(\Omega,\mathbb{R}^\infty)\right)^2, \tag{82}$$

$$\mathbf{g}_j = (g_{1,j}, g_{2,j}) \in \left(W^{\frac{1}{2},\frac{1}{4}}(\Sigma_j)\right)^2, \tag{83}$$

where

$$g_{i,j}(x,t) = \int_a^b d_i(x,t) y_{i,j-1}(x,t-h)\, dh + v_i(x,t), \quad i = 1, 2.$$

Then, there exists a unique solution $\mathbf{y}_j \in \left(W^{2,1}(Q_j)\right)^2$ *for the mixed initial-boundary value problem (72), (75) and (82).*

Theorem 5.2. *Let* $y_{i,0}$, $\Phi_{i,0}$, $\Psi_{i,0}$, v *and* u *be given with* $y_{i,0} \in W^1(\Omega,\mathbb{R}^\infty)$, $\Phi_{i,0} \in W^{2,1}(Q_0)$, $v_i \in W^{\frac{1}{2},\frac{1}{4}}(\Sigma)$, $\Psi_{i,0} \in W^{\frac{1}{2},\frac{1}{4}}(\Sigma_0)$ *and* $u_i \in L^2(Q)$, $i = 1, 2$. *Then, there exists a unique solution* $\mathbf{y} \in \left(W^{2,1}(Q)\right)^2$ *for the mixed initial-boundary value problem (72)–(76). Moreover,* $\mathbf{y}(\cdot, ja) \in \left(W^1(\Omega,\mathbb{R}^\infty)\right)^2$ *for* $j = 1, 2, \ldots$.

Now, we shall formulate the minimum-time problem for (72)–(76) in the context of the Theorem 5.2, i.e.,

$$\mathbf{u} \in \mathbf{U} = \{\mathbf{u} \in \left(L^2(Q)\right)^2 : |u_i(x,t)| \le 1,\ i = 1, 2\}. \tag{84}$$

We shall define the reachable set $\mathbf{H}$ such that

$$\mathbf{H} = \{\mathbf{y} \in \left(L^2(\Omega,\mathbb{R}^\infty)\right)^2 : \sum_{i=1}^{2} ||y_i - z_{i,d}||_{L^2(\Omega,\mathbb{R}^\infty)} \le \varepsilon\} \tag{85}$$

where $z_{i,d} \in L^2(\Omega,\mathbb{R}^\infty)$, $i = 1, 2$, and $\varepsilon > 0$.

Solving the stated minimum-time problem is equivalent to hitting the target set $\mathbf{H}$ in minimum time, that is, minimizing the time t, for which $\mathbf{y}(t;\mathbf{u}) \in \mathbf{H}$ and $\mathbf{u} \in \mathbf{U}$. Moreover, we assume that

$$\text{there exists a } T > 0 \text{ and } \mathbf{u} \in \mathbf{U} \text{ with } \mathbf{y}(T;\mathbf{u}) \in \mathbf{H} \tag{86}$$

then, as Section 4, we can prove the following theorem

Theorem 5.3. *If the assumption (86) holds, then the set* $\mathbf{H}$ *is reached in minimum time* t^* *by an admissible control* $\mathbf{u}^* \in \mathbf{U}$. *Moreover*

$$\sum_{i=1}^{2} \int_\Omega \left[z_{i,d} - y_i(t^*;\mathbf{u}^*)\right]\left[y_i(t^*;\mathbf{u}) - y_i(t^*;\mathbf{u}^*)\right] d\rho \le 0, \quad \forall\, \mathbf{u} \in \mathbf{U}. \tag{87}$$

Now, we introduce the adjoint equation, and for every $\mathbf{u} \in \mathbf{U}$ we define the adjoint variable $\mathbf{p} = \mathbf{p}(\mathbf{u}) = \mathbf{p}(x, t; \mathbf{u})$ as the solution of the following (2×2) system

$$-\frac{\partial p_i(\mathbf{u})}{\partial t} + \mathbf{A}^*(t) p_i(\mathbf{u}) + \int_a^b c_i(x, t+h) p_i(x, t+h; \mathbf{u})\, dh = 0, \quad x \in \Omega,\, t \in (0, t^* - b), \quad (88)$$

$$-\frac{\partial p_i(\mathbf{u})}{\partial t} + \mathbf{A}^*(t) p_i(\mathbf{u}) + \int_a^{t^*-t} c_i(x, t+h) p_i(x, t+h; \mathbf{u})\, dh = 0, \quad x \in \Omega,\, t \in (t^* - b, t^* - a), \quad (89)$$

$$-\frac{\partial p_i(\mathbf{u})}{\partial t} + \mathbf{A}^*(t) p_i(\mathbf{u}) = 0, \quad x \in \Omega,\, t \in (t^* - a, t^*), \quad (90)$$

$$p_i(x, t^*; \mathbf{u}) = z_{i,d}(x) - y_i(x, t^*; \mathbf{u}), \quad x \in \Omega, \quad (91)$$

$$\frac{\partial p_i(\mathbf{u})}{\partial \eta_{\mathbf{A}^*}}(x, t) = \int_a^b d_i(x, t+h) p_i(x, t+h; \mathbf{u})\, dh, \quad x \in \Gamma,\, t \in (0, t^* - b), \quad (92)$$

$$\frac{\partial p_i(\mathbf{u})}{\partial \eta_{\mathbf{A}^*}}(x, t) = \int_a^{t^*-t} d_i(x, t+h) p_i(x, t+h; \mathbf{u})\, dh, \quad x \in \Gamma,\, t \in (t^* - b, t^* - a), \quad (93)$$

$$\frac{\partial p_i(\mathbf{u})}{\partial \eta_{\mathbf{A}^*}}(x, t) = 0, \quad x \in \Gamma,\, t \in (t^* - a, t^*), \quad (94)$$

where

$$\frac{\partial p_i(\mathbf{u})}{\partial \eta_{\mathbf{A}^*}}(x, t) = \sum_{k=1}^{\infty} (D_k p_i(\mathbf{u})) \cos(n, x_k), \quad (95)$$

$$\mathbf{A}^*(t) p_i(\mathbf{u}) = \Big(-\sum_{k=1}^{\infty} D_k^2 + q(x, t) \Big) p_i(\mathbf{u}) + \sum_{j=1}^{2} a_{ji} p_j(\mathbf{u}), \quad (96)$$

and a_{ji} is the transpose of a_{ij}.

Hence, as the proof of Theorem 4.4 in Section 4, we can prove the following theorem.

Theorem 5.4. *If the assumptions concerning system (72)–(76) and controllability condition (86) are satisfied, then the time-optimal control* $\mathbf{u}^* = (u_1^*, u_2^*)$ *exists and is characterized by the following condition*

$$\sum_{i=1}^{2} \int_0^{t^*} \int_\Omega p_i(u^*)(u_i - u_i^*)\, d\rho\, dt \le 0, \quad \forall\, \mathbf{u} \in \mathbf{U}, \quad (97)$$

where $\mathbf{p}(\mathbf{u}^*)$ *is the solution of the adjoint system (88)–(94).*

5.2 Time-optimal Control Problem for $(n \times n)$ Coupled System of Parabolic Equations with Infinite Number of Variables.

We can extend the discussions to study the time-optimal control problem for $n \times n$ coupled system of parabolic equations with infinite number of variables, in which time lags appear in integral form in both the state equation and the boundary condition.

Consider now the distributed-parameter system described by the following $(n \times n)$ coupled system of parabolic equations with infinite number of variables, for $i = 1, 2, \ldots, n$,

$$\frac{\partial y_i}{\partial t} + \boldsymbol{\mathcal{A}}(t) y_i + \int_a^b c_i(x,t) y(x, t-h)\, dh = u_i, \quad x \in \Omega,\, t \in (0,T),\, h \in (a,b), \tag{98}$$

$$y_i(x, t') = \Phi_{i,0}(x, t'), \quad x \in \Omega,\, t' \in [-b, 0), \tag{99}$$

$$y_i(x, 0) = y_{i,0}(x), \quad x \in \Omega, \tag{100}$$

$$\frac{\partial y_i(x,t)}{\partial \eta_{\boldsymbol{\mathcal{A}}}} = \int_a^b d_i(x,t) y_i(x, t-h)\, dh + v_i, \quad x \in \Gamma,\, t \in (0,T),\, h \in (a,b), \tag{101}$$

$$y_i(x, t') = \Psi_{i,0}(x, t'), \quad x \in \Gamma,\, t' \in [-b, 0), \tag{102}$$

where

$$\boldsymbol{\mathcal{A}}(t) y_i(x) = \left(-\sum_{k=1}^{\infty} D_k^2 + q(x,t) \right) y_i(x) + \sum_{j=1}^{n} a_{ij} y_j(x) \quad \forall\, i = 1, 2, \ldots, n, \tag{103}$$

$$a_{ij} = \begin{cases} 1, & i \geq j; \\ -1, & i < j. \end{cases} \tag{104}$$

It is easy to see that $\boldsymbol{\mathcal{A}}(t)$ is $(n \times n)$ matrix which takes the form

$$\boldsymbol{\mathcal{A}}(t) = \begin{pmatrix} -\sum\limits_{k=1}^{\infty} D_k^2 + q + 1 & -1 & \cdots & -1 \\ 1 & -\sum\limits_{k=1}^{\infty} D_k^2 + q + 1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & -\sum\limits_{k=1}^{\infty} D_k^2 + q + 1 \end{pmatrix}_{n \times n}. \tag{105}$$

Also we have

$$y_i \equiv y_i(x, t; \mathbf{u}), \quad u_i \equiv u_i(x,t), \quad v_i \equiv v_i(x,t), \quad \mathbf{u} \equiv (u_1, u_2, \ldots, u_n),$$

c_i and d_i, $i = 1, 2, \ldots, n$, are real C^∞ functions defined on $\overline{Q}$ and Σ, respectively, $\Phi_{i,0}$ and $\Psi_{i,0}$, $i = 1, 2, \ldots, n$, are initial functions defined on Q_0 and Σ_0, respectively.

Now we discuss the case of the distributed control for the Neumann problem. Then, as Section 3, for $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \left(L^2(Q)\right)^n$, we can obtain the following results.

Lemma 5.5. *Let*

$$\mathbf{u} \in \left(L^2(Q)\right)^n, \tag{106}$$

$$\mathbf{f}_j = (f_{1,j}, f_{2,j}, \ldots, f_{n,j}) \in \left(L^2(Q_j)\right)^n, \tag{107}$$

where

$$f_{i,j}(x,t) = u_i(x,t) - \int_a^b c_i(x,t) y_{i,j-1}(x, t-h)\, dh, \quad i = 1, 2, \ldots, n,$$

$$\mathbf{y}_{j-1}(\cdot,(j-1)a) =$$
$$(y_{1,j-1}(\cdot,(j-1)a), y_{2,j-1}(\cdot,(j-1)a), \dots, y_{n,j-1}(\cdot,(j-1)a)) \in \left(W^1(\Omega,\mathbb{R}^\infty)\right)^n, \quad (108)$$

$$\mathbf{g}_j = (g_{1,j}, g_{2,j}, \dots, g_{n,j}) \in \left(W^{\frac{1}{2},\frac{1}{4}}(\Sigma_j)\right)^n, \quad (109)$$

where

$$g_{i,j}(x,t) = \int_a^b d_i(x,t) y_{i,j-1}(x,t-h)\, dh + v_i(x,t), \quad i = 1,2,\dots n.$$

Then, there exists a unique solution $\mathbf{y}_j \in \left(W^{2,1}(Q_j)\right)^n$ *for the mixed initial-boundary value problem (98), (101) and (108).*

Theorem 5.6. *Let* $y_{i,0}$, $\Phi_{i,0}$, $\Psi_{i,0}$, v *and* u *be given with* $y_{i,0} \in W^1(\Omega,\mathbb{R}^\infty)$, $\Phi_{i,0} \in W^{2,1}(Q_0)$, $v_i \in W^{\frac{1}{2},\frac{1}{4}}(\Sigma)$, $\Psi_{i,0} \in W^{\frac{1}{2},\frac{1}{4}}(\Sigma_0)$ *and* $u_i \in L^2(Q)$, $i = 1,2,\dots,n$. *Then, there exists a unique solution* $\mathbf{y} \in \left(W^{2,1}(Q)\right)^n$ *for the mixed initial-boundary value problem (98)–(102). Moreover,* $\mathbf{y}(\cdot, ja) \in \left(W^1(\Omega,\mathbb{R}^\infty)\right)^n$ *for* $j = 1,2,\dots$.

Now, we shall formulate the minimum-time problem for (98)–(102) in the context of the Theorem 5.6, i.e.,

$$\mathbf{u} \in \boldsymbol{\mathcal{U}} = \{\mathbf{u} \in \left(L^2(Q)\right)^n : |u_i(x,t)| \le 1,\ i = 1,2,\dots,n\}. \quad (110)$$

We shall define the reachable set $\boldsymbol{\mathcal{H}}$ such that

$$\boldsymbol{\mathcal{H}} = \{\mathbf{y} \in \left(L^2(\Omega,\mathbb{R}^\infty)\right)^n : \sum_{i=1}^n ||y_i - z_{i,d}||_{L^2(\Omega,\mathbb{R}^\infty)} \le \varepsilon\} \quad (111)$$

where $z_{i,d} \in L^2(\Omega,\mathbb{R}^\infty)$, $i = 1,2,\dots,n$, and $\varepsilon > 0$.

Solving the stated minimum-time problem is equivalent to hitting the target set $\boldsymbol{\mathcal{H}}$ in minimum time, that is, minimizing the time t, for which $\mathbf{y}(t;\mathbf{u}) \in \boldsymbol{\mathcal{H}}$ and $\mathbf{u} \in \boldsymbol{\mathcal{U}}$. Moreover, we assume that

$$\text{there exists a } T > 0 \text{ and } \mathbf{u} \in \boldsymbol{\mathcal{U}} \text{ with } \mathbf{y}(T;\mathbf{u}) \in \boldsymbol{\mathcal{H}} \quad (112)$$

then, as Section 4, we can prove the following theorem

Theorem 5.7. *If the assumption (112) holds, then the set* $\boldsymbol{\mathcal{H}}$ *is reached in minimum time* t^* *by an admissible control* $\mathbf{u}^* \in \boldsymbol{\mathcal{U}}$. *Moreover*

$$\sum_{i=1}^n \int_\Omega [z_{i,d} - y_i(t^*;\mathbf{u}^*)]\,[y_i(t^*;\mathbf{u}) - y_i(t^*;\mathbf{u}^*)]\ d\rho \le 0, \quad \forall\, \mathbf{u} \in \boldsymbol{\mathcal{U}}. \quad (113)$$

Now, we introduce the adjoint equation, and for every $\mathbf{u} \in \boldsymbol{\mathcal{U}}$ we define the adjoint variable $\mathbf{p} = \mathbf{p}(\mathbf{u}) = \mathbf{p}(x,t;\mathbf{u})$ as the solution of the following $(n \times n)$ system, $i = 1,2,\dots,n$,

$$-\frac{\partial p_i(\mathbf{u})}{\partial t} + \boldsymbol{\mathcal{A}}^*(t) p_i(\mathbf{u}) + \int_a^b c_i(x,t+h) p_i(x,t+h;\mathbf{u})\, dh = 0, \quad x \in \Omega,\ t \in (0, t^* - b), \quad (114)$$

$$-\frac{\partial p_i(\mathbf{u})}{\partial t}+\boldsymbol{\mathcal{A}}^*(t)p_i(\mathbf{u})+\int_a^{t^*-t} c_i(x,t+h)p_i(x,t+h;\mathbf{u})\,dh=0, \quad x\in\Omega,\, t\in(t^*-b,t^*-a), \tag{115}$$

$$-\frac{\partial p_i(\mathbf{u})}{\partial t}+\boldsymbol{\mathcal{A}}^*(t)p_i(\mathbf{u})=0, \quad x\in\Omega,\, t\in(t^*-a,t^*), \tag{116}$$

$$p_i(x,t^*;\mathbf{u})=z_{i,d}(x)-y_i(x,t^*;\mathbf{u}), \quad x\in\Omega, \tag{117}$$

$$\frac{\partial p_i(\mathbf{u})}{\partial \eta_{\boldsymbol{\mathcal{A}}^*}}(x,t)=\int_a^b d_i(x,t+h)p_i(x,t+h;\mathbf{u})\,dh, \quad x\in\Gamma,\, t\in(0,t^*-b), \tag{118}$$

$$\frac{\partial p_i(\mathbf{u})}{\partial \eta_{\boldsymbol{\mathcal{A}}^*}}(x,t)=\int_a^{t^*-t} d_i(x,t+h)p_i(x,t+h;\mathbf{u})\,dh, \quad x\in\Gamma,\, t\in(t^*-b,t^*-a), \tag{119}$$

$$\frac{\partial p_i(\mathbf{u})}{\partial \eta_{\boldsymbol{\mathcal{A}}^*}}(x,t)=0, \quad x\in\Gamma,\, t\in(t^*-a,t^*), \tag{120}$$

where

$$\frac{\partial p_i(\mathbf{u})}{\partial \eta_{\boldsymbol{\mathcal{A}}^*}}(x,t)=\sum_{k=1}^{\infty}(D_k p_i(\mathbf{u}))\cos(n,x_k), \tag{121}$$

$$\boldsymbol{\mathcal{A}}^*(t)p_i(\mathbf{u})=\Big(-\sum_{k=1}^{\infty}D_k^2+q(x,t)\Big)p_i(\mathbf{u})+\sum_{j=1}^{n}a_{ji}p_j(\mathbf{u}), \tag{122}$$

and a_{ji} is the transpose of a_{ij}.

Hence, as the proof of Theorem 4.4 in Section 4, we can prove the following theorem.

Theorem 5.8. *If the assumptions concerning system (98)–(102) and controllability condition (112) are satisfied, then the time-optimal control* $\mathbf{u}^*=(u_1^*,u_2^*,\ldots,u_n^*)$ *exists and is characterized by the following condition*

$$\sum_{i=1}^{n}\int_0^{t^*}\int_\Omega p_i(u^*)(u_i-u_i^*)\,d\rho\,dt\le 0, \quad \forall\,\mathbf{u}\in\boldsymbol{\mathcal{U}}, \tag{123}$$

where $\mathbf{p}(\mathbf{u}^*)$ *is the solution of the adjoint system (114)–(120).*

6. Conclusions and Perspectives

The results presented in the paper can be treated as a generalization of the results obtained by Knowles (1978), and Kowalewski and Krakowiak (2006; 2008) onto the case of time optimal distributed and boundary control of second order infinite variables parabolic systems with deviating arguments appearing in the integral form both in state equations and in boundary conditions. We considered a different type of control, namely, the control function defined in the distributed and boundary of the spatial domain. Sufficient conditions for the existence of a unique solution of such parabolic equations with Neumann boundary conditions are proved (Lemmas 3.2; 3.4; 3.6; 3.8; 5.1 and 5.5) and (Theorems 3.3; 3.5; 3.7; 3.9; 5.2 and 5.6). The optimal control is characterized by using the adjoint equations (Theorems 4.3; 4.4; 5.4 and 5.8). The conditions (41; 86 and; 112) plays a fundamental role in controllability problems for time-delay parabolic systems. With regard to the controllability assumption (41; 86 and 112), we can investigate the exact controllability problem for the parabolic system (1)-(5).

In this paper, we considered the time-optimal distributed and boundary control problem for infinite variables parabolic systems with non-homogeneous Neumann and Dirichlet boundary conditions. We can also consider an analogous minimum time problem for hyperbolic systems with non-homogeneous Neumann and Dirichlet boundary conditions. Finally, we can consider the time-optimal control problem for discrete time delay distributed and boundary parameter systems. The ideas mentioned above will be developed in forthcoming papers.

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