Extensions of Liouville's Theorem

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Abstract *In this paper, we first generalize Liouville's theorem into the general forms based on power series representations for analytic functions. Second, in simply connected domains harmonic functions can be identified as real parts of analytic functions. Observing the relations between analytic functions and harmonic functions, we extend Liouville's theorem to harmonic functions by the Harnack's inequality. The generalized Liouville's theorems obtained in this paper will help us to further study the properties of entire functions and harmonic functions.*

Keywords *Liouville's theorem Entire function Extension Harmonic function Harnack's inequality Analytic*

1. Introduction

If $f(z)$ is analytic on the whole complex plane, then it is said to be an entire function. For entire functions, there exists a beautiful theorem, known as Liouville's theorem. It gives many important properties of entire functions, these properties have been widely applied to Complex Analysis. And it enables us to prove the Fundamental Theorem of Algebra. Therefore, it is necessary for us to further discuss Liouville's theorem. In this paper, we first derive two extension forms of Liouville's theorem by simplifying some conditions of classical Liouville's theorem. Meanwhile we get two results for analytic functions. Second, in simply connected domains harmonic functions can be identified as real parts of analytic functions. And there are many consequences for analytic functions. Some of these are the infinite differentiability of analytic functions, Liouville's theorem, and the maximum modulus theorem. Hence we think that these results have analogues for harmonic functions. In terms of the ideas, we extend Liouville's theorem to harmonic functions. The generalized Liouville's theorems provide theoretical basis for us to further study the properties of entire functions and harmonic functions.

2. Inequalities and Lemmas

We list some useful inequalities and lemmas before giving to the generalized Liouville's theorems.

Theorem 2.1 (Liouville's Theorem) [1] The only bounded entire functions are the constant functions.

Lemma 2.1 [2] Let φ be harmonic on a simply connected domain D. Then there is an analytic function f such that $\varphi = \operatorname{Re} f$ on D.

Lemma 2.2 (mean-value theorem for harmonic function) [3] Let φ be harmonic in a domain containing the dish $|z| \le R$. Then

$$\varphi(0) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(Re^{it})dt \tag{1}$$

Lemma 2.3 (Poisson integral formula) [2] Let φ be harmonic in a domain containing the dish $|z| \le R$. Then for $0 \le r < R$, we have

$$\varphi\left(re^{i\theta}\right)=\frac{R^2-r^2}{2\pi}\int_0^{2\pi}\frac{\varphi\left(Re^{it}\right)}{R^2+r^2-2rR\cos\left(t-\theta\right)}dt \tag{2}$$

Proof: Using **Lemma 2.1**, we have

$\varphi=\operatorname{Re} f$

Here f is analytic on a simply connected domain D.

(Assuming the domain D includes the circle $C_R:|z|=R$ as well as its interior)

Applying Cauchy integral formula, we obtain

$$f(z)=\frac{1}{2\pi i}\int_{C_R}\frac{f(\zeta)}{\zeta-z}d\zeta \quad \left(|z|<R\right) \tag{3}$$

For fixed z, with $|z|<R$, the function $\dfrac{f(\zeta)\bar{z}}{R^2-\zeta\bar{z}}$ is an analytic function of ζ inside and on C_R. Hence by Cauchy theorem

$$\frac{1}{2\pi i}\int_{C_R}\frac{f(\zeta)\bar{z}}{R^2-\zeta\bar{z}}d\zeta=0 \tag{4}$$

We add it to Equation (Eq.) (3):

$$f(z)=\frac{1}{2\pi i}\int_{C_R}\left(\frac{1}{\zeta-z}+\frac{\bar{z}}{R^2-\zeta\bar{z}}\right)f(\zeta)d\zeta$$

$$=\frac{1}{2\pi i}\int_{C_R}\frac{R^2-|z|^2}{(\zeta-z)\left(R^2-\zeta\bar{z}\right)}f(\zeta)d\zeta \tag{5}$$

If we parameterize C_R by $\zeta=Re^{it}$, $0\le t\le 2\pi$, Eq. (5) becomes

$$f(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-|z|^2}{\left(Re^{it}-z\right)\left(R^2-Re^{it}\bar{z}\right)}f\left(Re^{it}\right)Re^{it}dt \tag{6}$$

$$=\frac{R^2-|z|^2}{2\pi}\int_0^{2\pi}\frac{f\left(Re^{it}\right)}{\left(Re^{it}-z\right)\left(Re^{-it}-\bar{z}\right)}dt \tag{7}$$

$$=\frac{R^2-|z|^2}{2\pi}\int_0^{2\pi}\frac{f\left(Re^{it}\right)}{\left|Re^{it}-z\right|^2}dt \tag{8}$$

Writing z in the polar form $z=re^{i\theta}$, we have

$$f\left(re^{i\theta}\right)=\frac{R^2-r^2}{2\pi}\int_0^{2\pi}\frac{f\left(Re^{it}\right)}{\left|Re^{it}-re^{i\theta}\right|^2}dt \tag{9}$$

$$=\frac{R^2-r^2}{2\pi}\int_0^{2\pi}\frac{f\left(Re^{it}\right)}{R^2+r^2-2rR\cos\left(t-\theta\right)}dt \tag{10}$$

Finally, by taking the real part of this equation, we arrive at Poisson integral formula

$$\varphi(r,\theta)=\frac{R^2-r^2}{2\pi}\int_0^{2\pi}\frac{\varphi(R,t)}{R^2+r^2-2rR\cos\left(t-\theta\right)}dt \tag{11}$$

or, equivalently,

$$\varphi\left(re^{i\theta}\right)=\frac{R^2-r^2}{2\pi}\int_0^{2\pi}\frac{\varphi\left(Re^{it}\right)}{R^2+r^2-2rR\cos\left(t-\theta\right)}dt \tag{12}$$

Poisson integral formula expresses the values of a harmonic function in a region is completely determined by its values on the boundary. Using Poisson integral formula and mean - value theorem for harmonic function, we may derive the following important inequality.

Theorem 2.2 (Harnack's inequality) [2] Let φ be harmonic and nonnegative in a domain containing the dish $|z| \le R$. Then for $0 \le r < R$, we have

$$\varphi(0)\frac{R-r}{R+r} \le \varphi\left(re^{i\theta}\right) \le \varphi(0)\frac{R+r}{R-r} \tag{13}$$

Proof: Applying **Lemma 2.3** to harmonic function φ. Then for $0 \le r < R$,

$$\varphi\left(re^{i\theta}\right) = \frac{R^2 - r^2}{2\pi}\int_0^{2\pi} \frac{\varphi\left(Re^{it}\right)}{R^2 + r^2 - 2rR\cos\left(t-\theta\right)}dt \tag{14}$$

Observing that

$$\left(R-r\right)^2 \le R^2 + r^2 - 2rR\cos\left(t-\theta\right) \le \left(R+r\right)^2 \tag{15}$$

Since φ is nonnegative, we have

$$\int_0^{2\pi} \frac{\varphi\left(Re^{it}\right)}{\left(R+r\right)^2}dt \le \int_0^{2\pi} \frac{\varphi\left(Re^{it}\right)}{R^2 + r^2 - 2rR\cos\left(t-\theta\right)}dt \le \int_0^{2\pi} \frac{\varphi\left(Re^{it}\right)}{\left(R-r\right)^2}dt \tag{16}$$

Hence

$$\frac{R-r}{R+r}\cdot\frac{1}{2\pi}\int_0^{2\pi}\varphi\left(Re^{it}\right)dt \le \varphi\left(re^{i\theta}\right) \le \frac{R+r}{R-r}\cdot\frac{1}{2\pi}\int_0^{2\pi}\varphi\left(Re^{it}\right)dt \tag{17}$$

Finally, by **Lemma 2.2**, the proof is completed.

3. The Generalized Liouville's Theorems

In this section, we consider the extension problem of classical Liouville's theorem. This is the main work of this paper.

Firstly, we obtain two extension forms of Liouville's theorem by reducing the condition of classical Liouville's theorem. These theorems are stated below.

Theorem 3.1 I f f is an entire function, and suppose there are a nonnegative integer n, and two positive constants R, M such that $|f(z)| \le M|z|^n$ when $|z| \ge R$, then f is a polynomial with $\deg(f) \le n$ or constant.

Proof: The case $n = 0$ will be treated first.

In this case, $|f(z)| \le M$ when $|z| \ge R$. Since f is an entire function, it must be continuous on the closed domain $|z| \le R$.Under such circumstances it is known from calculus that the function must be bounded there. In other words, there exists a positive constant G such that $|f(z)| \le G$ when $|z| \le R$. By taking $N = \max\{M, G\} > 0$, we have

$$|f(z)| \le N \tag{18}$$

when $|z| < +\infty$. Thus from **Theorem 2.1** there follows f is constant. This completes the proof.

Then the general case $n \ge 1$ will be dealt with.

Since f is an entire function, it must have a Maclaurin series representation, namely

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n + \cdots \quad \left(|z| < +\infty\right) \tag{19}$$

where

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i}\int_{|z|=r}\frac{f(\zeta)}{\zeta^{n+1}}d\zeta \quad \left(r > 0;\ n = 0, 1, 2, \cdots\right) \tag{20}$$

For an arbitrary integer $p \ge 1$ and a large enough $R_0 > R$, we consider that

$$\left|c_{n+p}\right| = \left|\frac{1}{2\pi i}\int_{|z|=R_0} \frac{f(\zeta)}{\zeta^{n+P+1}} d\zeta\right| \tag{21}$$

Notice that

$$\begin{aligned}\left|c_{n+p}\right| &\le \frac{1}{2\pi}\int_{|z|=R_0} \frac{|f(\zeta)|}{|\zeta|^{n+p+1}} ds \\ &\le \frac{1}{2\pi}\int_{|z|=R_0} \frac{M|\zeta|^n}{|\zeta|^{n+p+1}} ds \\ &= \frac{M}{2\pi}\cdot\frac{1}{R_0^{\,p+1}}\cdot 2\pi R_0 = \frac{M}{R_0^{\,p}}\end{aligned} \tag{22}$$

we can get $c_{n+p} = 0\left(p = 1, 2, \cdots\right)$. Thus

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n \quad \left(|z| < +\infty\right) \tag{23}$$

It says that f is a polynomial with $\deg(f) \le n$.

By modifying the condition of **Theorem 3.1**, we easily obtain the following results.

Corollary 3.1 I f f is an entire function, and suppose there are a nonnegative integer n, and three positive constants R, M and N such that $|f(z)| \le N + M|z|^n$ when $|z| \ge R$, then f is a polynomial with $\deg(f) \le n$ or constant.

Corollary 3.2 I f f is an entire function, and there is a positive integer n such that $\lim_{z\to\infty} \frac{f(z)}{z^n} = k > 0$, then f is a polynomial with $\deg(f) \le n$.

Theorem 3.2 Let f be analytic in the extended complex plane. Then f is constant.

Proof: Since f is analytic in the extended complex plane, it must have a Maclaurin series representation, namely

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \qquad \left(|z| < +\infty\right) \tag{24}$$

Meanwhile, $z = \infty$ is a removable singularity of f. Hence $f(z)$ has a finite limit as z approaches z_0. Recalling (24), the conclusion is obtained.

Secondly, in simply connected domains harmonic functions can be identified as real parts of analytic functions. Based on the relations between analytic functions and harmonic functions, harmonic function in R^2 is considered [4] in generalizing Liouville's theorem. We state Liouville's theorem for harmonic functions as follows.

Theorem 3.3 (Liouville's theorem for harmonic functions) [5] Let φ be harmonic in the whole real plane R^2 and bounded from above or below there. Then φ is constant.

Proof: Assume that φ is bounded from above there, namely, there exists a constant M such that $\varphi \le M$ for any $z \in R^2$.

Clearly, $\psi = M - \varphi$ is harmonic and nonnegative in the whole real plane R^2. Using **Theorem 2.2**, for $0 \le r < R < +\infty$, we have

$$\psi(0)\frac{R-r}{R+r} \le \psi\left(re^{i\theta}\right) \le \psi(0)\frac{R+r}{R-r} \tag{25}$$

Letting $R \to +\infty$, deduce that

$$\psi\left(re^{i\theta}\right) = \psi(0) \tag{26}$$

where $r \in [0, +\infty)$.

Notice that r is an arbitrary nonnegative real number hence $\psi = M - \varphi$ is constant. It follows immediately that φ is constant.

4. Conclusions

Summing up, we first transform some useful inequalities and lemmas to generalize Liouville's theorem. Then using power series representations for analytic functions [6], we derive two extension forms of Liouville's theorem by simplifying some conditions of classical Liouville's theorem. Finally, observing the relations between analytic functions and harmonic functions, we extend Liouville's theorem to harmonic functions by Harnack's inequality. In this course, we know that the generalized Liouville's theorems will help us to further study the properties of entire functions. Meanwhile, we can get some important conclusions for harmonic functions [7] based on the consequences for analytic functions in the future.

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