Dynamics of Price Model with Nonlinear Demand Function*

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Abstract *In this paper we derive the dynamic price model by introducing a general nonlinear form of the demand function into the traditional Cobweb model, and establish three propositions about the existence and structure stability of limit cycles by applying the qualitative theory of ordinary differential equations. The dynamic characteristics under the specific nonlinear demand function in six individual situations are discussed and illustrated by the petroleum oil daily price data from January 2 to October 27, 2009.*

Keywords *Demand function Limit cycle Dynamic characteristics of price Petroleum oil price*

1. Introduction

The Cobweb Theory proposed by Ragnar Frisch and Jan Tinbergen belongs to the category dynamic equilibrium analysis. It investigates the change of commodity prices in the supply-demand equilibrium and the inherent stability problems. The early Cobweb model assumes that producers have the same expectations, only to produce one commodity or participate in one market, and to adjust the supply by the price of the last period. Besides, the demand and supply functions are of a linear form of the prices, and the market is clear at every period (namely, the overall balance of supply and demand), etc. Recently, the researchers continue to relax these assumptions or add the new ones in order to make the situation studied closer to the real market, and apply the new methods to analyze the complex performances and equilibrium conditions of the price in the nonlinear dynamic Cobweb model. For instant, Chiarella (1988) introduces a general nonlinear supply function into the traditional Cobweb model with adaptive expectations, and finds that the dynamics of the model is driven by a chaotic type of single-hump map; Hommes (1994) studies the dynamics of the price-quantities model derived from the Cobweb model with adaptive expectations and nonlinear supply and demand curves, also find the chaotic dynamic behavior even if both the supply and demand functions are monotone; Brock & Hommes (1997) analyze the nonlinear dynamic equilibrium process by allowing the suppliers with heterogeneous expectations to switch between the naïve and rational ones freely, but restricting the linear function formation of supply and demand in the cobweb model, and dominate that a bifurcation route will change to the chaos and strange attractors when the intensity of choice to switch prediction strategies increases; Goeree & Hommes (2000) develop the Brock & Hommes's evolutionary Cobweb model by expanding the linear supply and demand to the nonlinear ones, while reaching similar conclusions. Particularly, with adding other more realistic hypotheses into the traditional Cobweb model, some recent literatures show that the improved models also generate chaotic dynamic behaviors. Chiarella et al. (2006) discuss the Cobweb model with boundedly rational heterogeneous producers (i.e., risk-averse producers), and conclude that each dimension of heterogeneity enriches the Cobweb dynamics with respect to the case of homogeneous producers; Choudhary & Orszag (2008) study the traditional Cobweb model in one market with local externalities which imply that the

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firms must forecast both price and quantities, and obtain the evidence of clusters of firms whose output behavior is correlated as equilibrium is reached; Dieci & Westerhoff (2009) consider that the producers face two kinds of commodity markets and tend to enter the more profitable one in the recent past, and find such a switching process is a further source of nonlinearity for the dynamics of prices.

In addition, the more widely the new research methods are applied, the more deeply the chaotic behaviors of dynamic Cobweb model are researched, including the model of stability and convergence, unstable conditions and the possibility of internal changes, and the evolutionary steps of the prediction strategy choice. For instance, Ying Yi-Rong (1996) applies the ordinary differential equation qualitative theory, especially the "limit cycle" theory, to explore the existence and uniqueness of stable limit cycles, and the magnitude and period of price shocks in the nonlinear Cobweb model. Onozaki et al. (2000) use the classical homoclinic point theorem to discuss the nonlinear Cobweb model with adaptive production adjustment, and find the observable chaos (strange attractors) as well as topological chaos (saddle point) associated with homoclinic points. (For the mathematical and physical mechanisms underlying the appearance of chaos, please consult with (Lin and OuYang, 1998; Lin, 2008)). Chiarella et al. (2006) construct the geometric decay processes (GDP) of supply quantities to study the dynamic features of the nonlinear Cobweb model while the supply function contains the limited memory or infinite memory. Li et al. (2008) introduce the Gaussian white noise into the demand function, and transform the equilibrium price model into a quasi-integrable Hamiltonian system, then take the random averaging method to discuss the first-pass damage of this system and give the conditional reliability function and the probability distribution function.

In short, there is a diversification trend in the study of Dynamic Cobweb Model due to the complexity of economic and social activities. In fact, chaos is everywhere so that different assumptions and research methods are introduced and developed to explore the different sides of the complex phenomena. For enriching the studies of nonlinear cobweb models, this paper focuses on the dynamic equilibrium of price in a single commodity market with a general nonlinear formation of demand function basing on Ying Yi-Rong's earlier study, and derives three propositions about the existence and structure stability of limit cycles from this nonlinear Cobweb model. Furthermore, we discuss the dynamic characteristics under the specific nonlinear demand function in six individual situations, and find that there is the specific demand function in a shorter term illustrated by the petroleum oil daily price data. But there exist many other nonlinear forms of the demand function which are hard to determinate because of the chaotic behaviors existing in the real market.

In the next section we will construct the general nonlinear demand function and the nonlinear dynamic Cobweb model. The third section focuses on the existence and stability of the limit cycles under different conditions; and the fourth section expands the conclusions developed in Section 3 by discussing the dynamic price equilibrium with the special nonlinear demand function form. The final section concludes the presentation of this work.

2. Modeling

2.1 Nonlinear Demand Function

Ying Yi-Rong (1996) assumes that the current demand is a nonlinear function of the price and its growth rate. That is

$$D(t) = \alpha + aP + (c_0 + c_1 P + c_2 P^2)\frac{dP}{dt} \tag{1}$$

where $D(t)$ is demand of the goods at time t; P is price of the goods at time t; α, a, c_0, c_1, c_2 are parameters.

Denote $f(P) = aP^2 + bP + c$, then

$$D(t) = \alpha + aP + f(P)\frac{dP}{dt} \tag{2}$$

In Eq. (2), $D(t)$ is made up of the simple linear impacts of price ($\alpha + aP$) and the nonlinear impact ($f(P)\frac{dP}{dt}$) that explains the consuming attitude respect to the growth rate of price. This assumption makes the model closer to the real market than the simple linear demand function in the traditional Cobweb model.

In accordance with microeconomics views, besides the price of the commodity itself, the other important factors, generally including income, consumption preferences, and other commodity prices, can directly generate either positive or negative impact on the commodity demands. Furthermore, these factors also have some unascertained impact on the consumer price sensitivity and pricing behavior of suppliers. The uncertainty involved eventually leads to the proposed nonlinear demand function. For example, with the growth in income, the consumer increases the demand but reduces the degree of consuming price sensitivity at the same time. So rational price hikes by the supplier might not necessarily lead to declining demand. In addition, in some commodity markets, the short-term change in consumption preferences may greatly improve or reduce the degree of consuming price sensitivity with respect to the price increases, which directly cause volatile fluctuations in demand. Therefore, given the consuming attitude that is mainly influenced by the sensitivity of consumers to price changes, $f(P)$ in Eq. (2) may take any function formation. If still denoted by $f(P)$, then Eq. (2) can be considered as the general form of any nonlinear demand function.

2.2 The Nonlinear Cobweb Equilibrium Model

Basing on Ying Yi-Rong's study (1996), in general, one can assume that the supply at time t is still a linear function of the price. The general form is as follows:

$$S(t) = \beta + bP\,, \quad b > 0 \tag{3}$$

where $S(t)$ stands for the supply of the goods at time t with β, b being the parameters.

Assumed that the suppliers do pricing all the time to make the rate of change in price proportional to the amount of shortage, which is caused by the stock fallen below a certain threshold level. That is,

$$\frac{dP}{dt} = -\lambda(Q(t) - \bar{Q})\,, \quad \lambda > 0 \tag{4}$$

where $Q(t)$ is stock of the goods at time t; $\bar{Q}$ is the threshold level of stock set by suppliers; λ the proportion of the stock shortage.

Let $Q(t) = Q(0) + \int_0^t (S(t) - D(t))dt$ with $Q(0)$ being the initial level of $Q(t)$. Then

$$\frac{dP}{dt} = -\lambda[Q(0) - \bar{Q} + \int_0^t (S(t) - D(t))dt]\,, \quad \lambda > 0 \tag{5}$$

Substituting Eqs. (2), (3) into Eq. (5), one can easily obtain the following second order differential equation

$$\frac{d^2 p}{dt^2} - \lambda f(P)\frac{dp}{dt} + \lambda(b-a)P = \lambda(\alpha - \beta)\quad , \quad \lambda > 0 \tag{6}$$

When $P = \bar{P}$, then $S(t) = D(t)$, $\frac{dP}{dt} = 0$. So one can get $\bar{P} = \frac{\alpha - \beta}{b - a}$ ($b \neq a$), where $\bar{P}$ is the equilibrium price.

Let $p = P(t) - \bar{P}$ and substitute it into Eq. (6), we have

$$\frac{d^2 p}{dt^2} - \lambda f(p+\bar{P})\frac{dp}{dt} + \lambda(b-a)p = 0 \tag{7}$$

Without loss of generality, we still denote $f(p+\bar{P})$ by $f(p)$, meanwhile let $t = \frac{\tau}{\sqrt{\lambda\alpha}}$, $\mu = -\sqrt{\frac{\lambda}{b-a}} > 0$, we can get

$$\frac{d^2 p}{d\tau^2} + \mu f(p)\frac{dp}{d\tau} + p = 0 \tag{8}$$

Let $p = x$, $\frac{dy}{d\tau} = -x$. Then Eq. (8) can be rewritten as the Rayleigh equations by using Lienard transformation. That is

$$\begin{cases} \frac{dx}{d\tau} = y - F(x) \\ \frac{dy}{d\tau} = -x \end{cases} \tag{9}$$

where $F(x) = \int_0^x \mu f(p)dp$.

3. Analysis

According to the qualitative theory of ordinary differential equations, there are three conclusions in terms of Eq. (9).

Proposition 1 If $f(0) > 0$, the singular point (0,0) is an unstable focus or unstable node of Eq. (9); if $f(0) < 0$, the singular point (0,0) is a stable focus or stable node of Eq. (9).

Proof. The characteristic equation of the linear part in Eq. (9) at point (0,0) is

$$\lambda^2 + \mu f(0)\lambda + 1 = 0$$

Therefore the Proposition 1 must be true according to the qualitative theory of ordinary differential equations.

Proposition 2 If the divergence of Eq. (9) is constant within a simply connected domain area G, then there does not exist any limit cycle in Eq.(9) so that all closed orbits are located within the area G.

Proof. Let us compute the divergence of Eq. (9) as follows:

$$div|_{(9)} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y - F(x)) = 2 > 0$$

Hence Proposition 2 can be shown to hold true by applying Theorem 1.10 derived by Ye Yan-Qian (1984).

Proposition 3 If Eq. (9) satisfies the following two conditions:

(a) $xF(x) < 0$, when $x \neq 0$ and $|x|$ is sufficiently small;

(b) there exist a positive constant M and constants K and K' with $K > K'$ such that $F(x) \geq K$, when $x > M$; $F(x) \leq K'$, when $x < -M$. Then Eq. (9) has the stable limit cycle.

Proof. The function in Eq. (9) shows that there exist $g(x) \equiv x$. Then the three conditions in Theorem 5.1 in the monograph of Ye Yan-Qian (1984) can be satisfied. Thus considering the conclusions of Proposition 1, Proposition 3 must be true.

4. Applications

4.1 Case 1: $f(p) = const$

Let $f(p) = c_0$, c_0 is a none-zero constant. Substituting $f(p)$ into Eq. (8) leads to

$$\frac{d^2 p}{d\tau^2} + \mu c_0 \frac{dp}{d\tau} + p = 0 \tag{10}$$

(1) If $\Delta = (\mu c_0)^2 - 4 > 0$, then $c_0 > 2(b-a)$, and $p = A_1 e^{p_1 t} + A_2 e^{p_2 t}$, where A_1, A_2 are any constants, p_1, p_2 the two distinct real roots of the characteristic equation of Eq. (10). When $p_1 > 0$ and $|p_1| > |p_2|$, the price keeps on moving further and further away from the equilibrium price $\overline{P}$ (for the convenience of drawing the figure, assume $\overline{P} = 0$); when $p_1 < 0$ and $p_2 < 0$, the price gradually moves toward the equilibrium price. Figure 1 and 2 show the two kinds of price movements at some specific values $c_0 = 15$ and $\mu = -0.2$. In the same fashion, we can produce Figure 3 – 6.

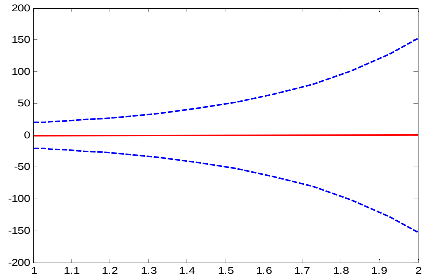


Figure 1 $p(0) = \pm 20, p'(0) = \pm 30$

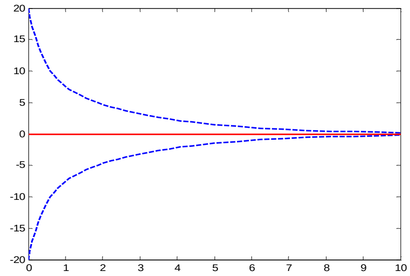


Figure 2 $p(0) = \pm 20, p'(0) = \pm 30$

(2) If $\Delta = (\mu c_0)^2 - 4 = 0$, then $p = (A_1 + A_2 t)e^{-\frac{\mu c_0}{2}t}$, where $p_1 = p_2$. When $c_0 > 0$, after reaching the equilibrium price, the price continues to move further and further away from it, see Figure 3 ($c_0 = 10$, $\mu = -0.2$) for more details. When $c_0 < 0$, the price firstly experiences larger fluctuations in the vicinity of the equilibrium price, then moves closer to it, Figure 4 ($c_0 = 10$, $\mu = -0.2$).

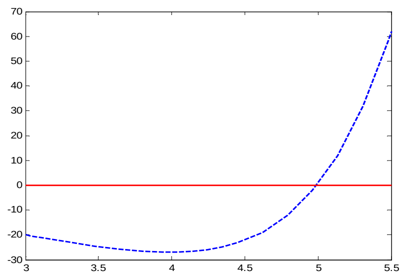


Figure 3 $p(0) = -20, p'(0) = -10$

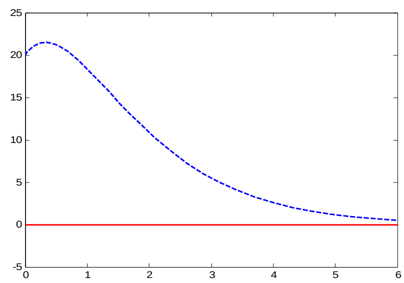


Figure 4 $p(0) = 20, p'(0) = 10$

(3) If $\Delta = (\mu c_0)^2 - 4 < 0$, then $c_0 < 2(b-a)$, and $p = Ae^{-\frac{\mu c_0}{2}} \cos(\omega t + \psi)$, where $p_1, p_2 = -\dfrac{\mu c_0}{2} \pm \omega i$, ψ stands for the initial phase. When $c_0 > 0$, the price contains a period of oscillations with an increasing amplitude around the equilibrium price, and then moves away

from it, Figure 5. However, when $c_0 < 0$, the price contains a period of oscillations with a decreasing amplitude around the equilibrium price, and finally close in to it, Figure 6.

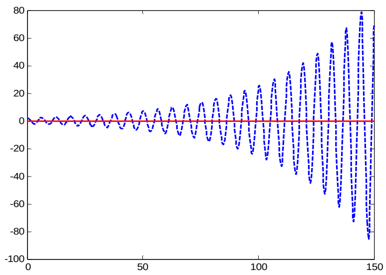


Figure 5 $c_0 = 2.5$, $\mu = -0.2$, $p(0) = 2, p'(0) = 0$

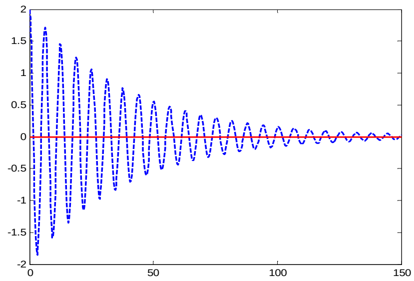
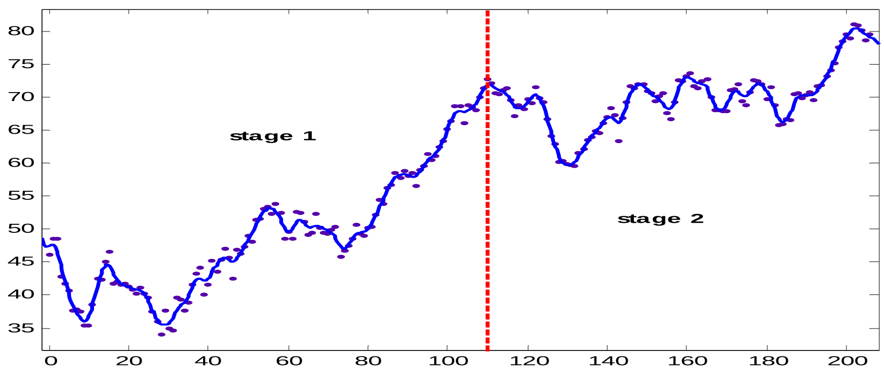


Figure 6 $c_0 = -2.5$, $\mu = -0.2$, $p(0) = 2, p'(0) = 0$

4.2 Case 2: The Short-term Fluctuations of Petroleum Oil Price

According to the U.S. government's official energy statistics, the daily data of petroleum oil prices (FOB, except weekends and holidays) obviously has a growing uptrend during January 2 to October 27 in 2009 (see Figure 7). What's shown is that petroleum oil equilibrium price points were continuously broken within the interval of the 207 said days due to changes in supply and demand, and gradually went upward from the initial equilibrium price point with the short-term irregular fluctuations or a random walk. However, in a smaller time interval, the price fluctuations show the similar characteristics of dynamic changes as in Figure 1– 6. For example, the data set is simply divided into two stages at the price point of June 11, 2009. In stage 1, although the petroleum oil price irregularly moves without a clear direction in the first 40 days, it goes upward and moves far away from the initial price after that; eventually it reaches the highest price on June 11. It has the similar characteristics as described in Figure 3. In the stage 2, before October 10, the petroleum oil price seems to keep moving around a certain equilibrium price with decreasing amplitude. It is similar in principle to what is shown in Figure 6. But it is not easy to find whether its movement within the rest of the time period would have the similar performance as those in the previous figures.



Data source： http://tonto.eia.doe.gov/dnav/pet/hist/rwtcd.htm；

Figure 7 01/02/2009-10/27/2009 petroleum oil price trend.

Some evidences can be found from this real market experience. Generally speaking, if these factors, such as price, income, preferences, and the prices of other commodities, change regularly without a significant fluctuation, or no war, no pestilence, no earthquakes, etc. to occur, both suppliers and consumers' price sensitivity may vary little within a shorter term or a sufficiently shorter interval. That is, the price sensitivity can be assumed to be constant theoretically so that $f(p)$ in Eq. (9) can be roughly seen as a constant in a short time period.

However, if those factors change chaotically, or one or some of the events mentioned above suddenly occur, then the price sensitivity is easily changed drastically even within a sufficiently shorter interval. So $f(p)$ may be a linear or even nonlinear function instead of a constant. Therefore, the price movements of the petroleum oil always stand for a combined effect of two individual performances even in a sufficiently small interval. One is similar to what are described in the cases of Figure 1– 6, another represents a complex process.

Given 10 days as a sufficiently short interval for the petroleum oil price data, then there are 198 samples by sampling randomly 10 consecutive days. All the price movement may be roughly divided into two categories with $f(p)$ being constant or non-constant. So, let us simply choose two special samples for easy comparative analysis. Additionally, assume the lag one of the initial price in the sample is the equilibrium price under the adaptive expectations, which is estimated by smoothing spline interpolation method (with the smoothing coefficient being 0.5). Figure 8 shows that during April 27 to May 8, the fitting curve by Fourier Interpolation method goes up to cross the equilibrium price line on April 28, and keeps on moving further upward. If excluding the irregular change, and contrasting with Figure 3, this curve implies that there exists a constant $f(p)$ greater than zero in the petroleum oil price dynamics of this sample. Whereas the fitting curve obtained in the same way in Figure 9 is not similar to any of those in Figure 1– 6 during the 10th to 24th of February. It has a period of almost six trading days of oscillation with the same amplitude underneath the equilibrium price line, and hovers over the line in the last four days. It can be inferred that $f(p)$ in Figure 9 is no longer a constant but a nonlinear function. Unfortunately, it is not easy to find its specific functional expression. Because it is more difficult to determine the existence and stability conditions of limit cycles according to Proposition 3, more in depth study will be needed.

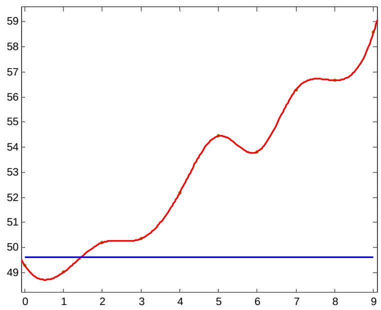


Figure 8 Petroleum oil price trend in 04/27-05/08/2009

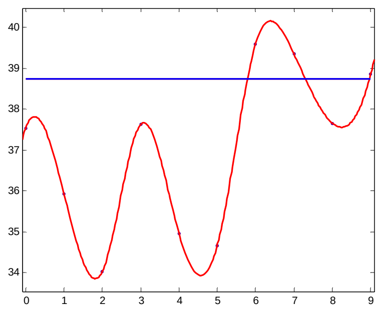


Figure 9 petroleum oil price trend in 02/10-02/24/2009

5. Conclusions

Based on the studies of Ying Yi-Rong (1996), the hypothesis of a linear demand function in the traditional Cobweb model is extended to a nonlinear form. Then for the given nonlinear demand function of the general form $D(t) = \alpha + aP + f(P)\dfrac{dP}{dt}$, of which $f(p)$ could be any function, the nonlinear dynamic price model is derived and transformed into the Rayleigh equations, and three propositions on the limit cycles are proved to hold true by applying the qualitative theory of ordinary differential equations. On this basis, two specific cases are discussed. One is the study of the dynamics of the nonlinear price model with a constant $f(p)$, where six situations of price fluctuations are graphed. Another is the analysis of the petroleum oil price movement from January 2 to October 27, 2009; it is found that there may appear two kinds of situations where $f(p)$ is either a constant or non-constant in a sufficiently short term by contrasting to what are shown in Figure 1– 6. For non-constant $f(p)$, it might be caused by significant changes in price, income, preferences, and prices of other commodities, or by the appearance of one or more uncontrollable events. So, $f(p)$ might be a nonlinear function (To this end, please consult with (Lin, 2008), where it is shown that the study of economics should be mainly about interactions of nonlinear forces). But its specific functional expression is difficult to find, constituting an open problem for future studies.

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