Internal Time of a Dynamical System

Robert Vallée
Professor emeritus Université Paris-Nord, President of WOSC
Email: r.vallee@afscet.asso.fr

Abstract *We have a dynamical system. The evolution of its state, X(t) at instant t, is given by a differential equation dX(t)/dt = f(X(t),t), independent of the environment. We propose to introduce a time s, or internal time, different from time t, or reference time.*

For this purpose we consider duration, or time elapsed between two instants. Reference duration, between instants t_1 and t_2 is obviously given by $dr(t_1,t_2) = t_2 - t_1$. Any duration, for example internal duration $di(t_1,t_2)$, must satisfy certain conditions. Once we have an internal duration $di(t_1,t_2)$, we can generate an internal time $s = di(t_0,t)$.

The choice of $di(t_1,t_2)$ depends upon the "weight" of reference duration t_2-t_1, seen from the internal point of view, or equivalently that of infinitesimal reference duration dt between t and t+dt. We propose that the internal duration corresponding to reference duration dt is equal to $(d(X(t)/dt)^2\,dt$. In a way $(dX(t)/dt)^2$ is an index of the "importance" of instant t.

As an example, we consider an "explosive-implosive" dynamical system described by a certain evolution equation. The corresponding internal time varies from $-\infty$ to $+\infty$ while reference time varies from 0 to $+\infty$. Interpretations (physiology, cosmology) are given.

Keywords *Explosion-implosion Internal duration Cosmological time*

1. Introduction

We want to give a definition of the internal time, or intrinsic time of a dynamical system evolving independently of its environment. We proposed this definition for the first time in 1996. We developed it mainly in an article (Vallée, 2005) which the present text reproduces partly.

The notion of internal timeis opposed to that of external time, or reference time, taken for granted and which is used in the evolution equation. The basic idea is that the internal time does not elapse if the state of the system does not change, a conception close to that of Aristotle for whom time ceases to be known when the "soul" does not vary.

So if X(t), belonging to a finite dimensional linear space, is the state of the system at reference instant t, any real positive and increasing function, null for argument 0, of a norm of dX(t)/dt, is a measure of the intensity of change of the system at instant t .We make the most simple choice, that of the square of the Euclidian norm (or scalar square) $(dX(t)/dt)^2$ which may be seen as an index of "importance" of instant t. So we consider that the internal duration corresponding to reference duration dt (between t and t+dt) is equal to $(dX(t)/dt)^2\,dt$ and we define (Vallée, 1996, 2001) the internal duration $di(t_1, t_2)$ of interval (t_1,t_2), whose reference duration is $t_2 - t_1$ by

$$di(t_1,t_2) = \int_{t_1,t_2} (dX(t)/dt)^2 \, dt \qquad (1)$$

So if $(dX(t)/dt)^2$ is equal to 0 on the interval, the internal duration is 0, and if $(dX(t)/dt)^2$ is equal to1, the internal duration is equal to the reference duration. In short, the higher the values of $(dX(t)/dt)^2$ on the interval, the greater the internal duration.

We can now, define the internal time s(t) by

$$s(t) = di(t_0,t) = \int_{t_0,\, t} (dX(s)/ds)^2 \, ds \qquad (2)$$

where t_0 is any reference instant. So s(t) if determined up to an arbitrary additive constant. Of course we have

$$d_i(t_1,t_2)=s(t_2)-s(t_1). \qquad (3)$$

It is interesting to verify if equation (3) is consistent with the axioms that a « time » must satisfy. If f(a,b) is a duration attached to interval (a,b), we must have, with $a \leq b \leq c$,

$$f(a,b) + f(b,c) \equiv f(a,c), \quad f(a,b) > 0 \text{ for } b > a, \quad f(a,a) \equiv 0 \qquad (4)$$

f (a,b) increasing with b and decreasing with a. With the hypothesis that f is differentiable, it is easy to solve functional equation (4). We have

$$f(a+da,b) + f(b,c+dc) \equiv f(a+da, c+dc)$$

Replacing f(a+da,b) by $f(a,b) + \partial f(a,b)/\partial a \, da$ and the like for f(b,c+dc) and f(a+da, c+dc), we obtain

$$\partial f(a,b)/\partial a \equiv \partial f(a,c)/\partial a$$

So $\partial f(a,b)/\partial a$ is independent of b. Consequently, by integration with respect to a, we have

$$f(a,b) = F(a) + G(b)$$

But since

$$f(a,a) \equiv 0 \equiv F(a) + G(a)$$

we have G = -F

so

$$f(a,b) = G(b) - G(a)$$

which is consistent with (3).

2. Examples

We call *explosion* the evolution of a system whose state vector has a modulus starting with value 0 at t = 0 then increasing with t, and such that the modulus of its speed vector starts with value $+\infty$ at t = 0. The first instants of the evolution of the system have an exceptional importance since $(dX(t)/dt)^2$ tends to $+\infty$ when t tends to 0. We have here an idealisation as well as in the case of what we call *implosion* where X(t) decreases with t and attains value 0 at the final instant while $(dX(t)/dt)^2$ tends to $+\infty$ system may be explosive at the beginning and implosive at the end, then we say that we have an *explosion-implosion*. For the sake of simplicity we shall suppose now that X(t) is a mere scalar.

We start with an *explosion-implosion* (Vallée, 1996, 2001) defined by the differential equation

$$X(t)/dt = q/p \operatorname{sgn}(p-t) \, (q^2 - X^2(t))^{1/2} / X(t), \quad X(0) = 0, \text{ p et q} > 0, \quad 0 \leq t \leq p \qquad (5)$$

where sgn(p-t) is the sign of p-t. The solution of this equation is given by function

$$X(t) = q/p \, (p^2 - (p-t)^2)^{1/2} \qquad (6)$$

whose graph is an half-ellipse (great axis 2p, small axis 2q). We say that we have an *elliptic explosion-implosion.* When t varies from 0 to 2p, X(t) increases from 0 to q, then decreases from q to 0,with a speed of infinite absolute value at t = 0 and t = 2p.The square of the speed of evolution is given by

$$(dX(t)/dt)^2 = q^2/p^2 \, (p-t)^2 / t(2p-t) = q^2/2p \, (1/t - 2/p + 1/2p-t)$$

which shows that the "*importance" of instant* t is infinite at the beginning (t = 0) and at the end (t=2p). If we integrate $(dX(t)/dt)^2$ from t_1 to t_2 we obtain the *internal duration* of the reference time interval (t_1,t_2)

$$di(t_1,t_2) = q^2/2p \, (\text{Log } (t_2/t_1) - 2(t_2-t_1)/p - \text{Log } (2p-t_2/2p-t_1)) \qquad (7)$$

Now, choosing $t_1 = p$ and $t_2 = t$ we have an *internal time* s(t) defined by

$$s(t) = di(p,t) = q^2/2p \, (\text{Log } t - 2t/p + 2 - \text{Log } (2p-t)) \qquad (8)$$

Of course constant $q^2/2p$ may be supressed, then we have another *internal time*

$$\sigma(t) = q^2/2p \, (\text{Log } t - 2t/p - \text{Log}(2p-t))$$

We see that when the *reference time* t varies from 0 to 2p, generating a finite *reference duration* of the evolution equal to 2p, the internal time s varies from $-\infty$ to $+\infty$ generating an infinite *internal duration* of the evolution. The initial instant 0 is pushed back to $-\infty$ and the final instant 2p is pushed forward to $+\infty$.The internal duration of any interval (0, t) is infinite as well as the internal duration of any interval (t, 2p).

We consider now the differential equation

$$dX(t)/dt = q/p \quad (q^2 + X^2(t))^{1/2} / X(t)\,, \quad X(0) = 0, \text{ p et q} >0, \quad 0\leq\tau< +\infty \qquad (9)$$

Its solution is given by function

$$X(t) = q/p \ ((p+t)^2 - p^2)^{1/2} \qquad (10)$$

whose graph is the right part of an half-hyperbola (great axis 2p,"small axis" 2q).We say that we have an *hyperbolic explosion* : X(t) increases from 0 to $+\infty$ with an infinite speed at instant 0 and, for the great values of t, X(t) behaves like (t-p) a/p. The square of the speed is given by

$$(dX(t)/dt)^2 = q^2/p^2\ (p+t)^2 / t(t+2p) = q^2/2p \quad (1/t + 2/p - 1/2p+t)$$

It shows that the *« importance » of instant* t is infinite at t = 0. Calculations, similar to those for the elliptic case, give the internal duration of interval (t_1,t_2) and consequently an *internal time* such as

$$s(t) = q^2 /2p \quad (\text{Log } t + 2t/p - \text{Log } (2p+t)) \qquad (11)$$

When t varies from 0 to $+\infty$, s(t) varies from $-\infty$ to $+\infty$, the initial instant 0 being pushed back to $-\infty$. Any interval (0, t), of reference duration t, has an infinite internal duration. Moreover s(t) behaves as $q^2/2p$ Log t for small values of t and as $q^2/p^2 \quad t$ for great values.

An intermediary case, which we call *parabolic explosion* (Vallée, 1996, 2001), is obtained when p and q tend to infinity while q^2/p keeps a constant value 2h. Starting indifferently from equation (6) or (9), we obtain

$$dX(t)/dt = 2h / X(t), \ X(0) = 0, \quad h>0 \qquad (12)$$

The solution is given by function

$$X(t) = 2 \ (ht)^{1/2} \qquad (13)$$

whose graph is an half-parabola of parameter h. We see that Xt) increases from 0 to $+\infty$ with an infinite initial speed. Then we have

$$(dX(t)/dt)^2 = h/t$$

The "*importance" of instant* t is infinite at t = 0 and tends to 0 when t tends to $+\infty$. The calculation of *internal duration* generates an *internal time*

$$s(t) = h \text{ Log } t, \qquad (14)$$

the initial instant 0 is pushed back to $-\infty$ and any interval (0,t) has an infinite *internal duration.*

3. Interpretations

In the first interpretation we consider an *elliptic explosion-implosion* as the evolution of a living being whose birth may be compared to a kind of explosion and the end of life as an involution, or a kind of implosion, more or less quick. In our model the implosive part is symmetrical with the explosive one. This is not very realistic, since it seems that the implosive part must be shorter. Nevertheless if we consider only the qualitative aspect of the conclusions we can say that, from the internal time point of view, the initial instant (conception) is pushed back to $-\infty$ and the final instant (death) is pushed forward to $+\infty$ (Vallée,1991,1996). The first part of this qualitative conclusion is in accordance with the natural feeling of a human being (if we consider this case) of having no beginning.

We can also consider the case of a *parabolic explosion* limited at the instant of death. The initial instant is pushed back to $-\infty$ and the *internal time* is proportional to the logarithm of the elapsed reference time. It elapses slowly at the beginning and more and more quickly near the end. This is close to the ideas of Lecomte du Noüy (Lecomte du Noüy, 1936). For him the physiological duration of an astronomical time interval (reference time interval in our terminology), of given length, is proportional to the speed of healing of wounds. This speed varying roughly as the inverse of age, a logarithmic physiological time is generated. But a remark seems necessary in order to avoid apparent paradoxes: *the internal time* of a conscious being may be different from the *perceived internal time*.

The second interpretation concerns *internal duration* in cosmology. The cases of elliptic explosion-implosion, parabolic or hyperbolic explosion have common traits with certain cosmological models with primordial explosion followed by final implosion or with primordial explosion only.

Generally speaking, the differential equation giving the evolution of the universe, whose state at instant t is described by the cosmological scale factor R(t), is according to Lemaître, Friedmann, Robertson (Berry, 1989)

$$(dR(t)/dt)^2 = 8\pi G/3\rho(t)\, R^2(t) - kc^2 + \Lambda/3\, R^2(t),\ R(0) = 0 \qquad (15)$$

where G is the gravitational constant, c the speed of light, k the index of curvature (k = -1, we have a space with negative curvature; k = 0, we have a flat space; k = +1, the space has a positive curvature (then R(t) may be considered as the radius of the universe), Λ the cosmic constant or cosmic repulsion term, $\rho(t)$ the density of matter or its material equivalent in case of pure radiation. In the material case $\rho(t) = a/R^3(t)$ and in the case of pure radiation it is equal to $b/R^4(t)$, a and b being constants.

Equation (5), corresponding to an *elliptic explosion-implosion,* gives if we consider the square of its two members

$$(dX(t)/dt)^2 = q^4/p^2 \ / X^2(t) - q^2/p^2$$

If we substitute R(t) to X(t), the above equation takes one of the possible forms of (15) if $\rho(t) = b/R^4(t)$, $k = +1$, $\Lambda = 0$. We have a case of pure radiation with positive curvature and null cosmic constant. More precisely $q = c\, p$ and $p = 1/c^2 \ (b\, 8\pi G/3)^{1/2}$. Then

$$(dR(t)/dt)^2 = 8\pi G/3\ b/R^2(t) - c^2 \qquad (16)$$

The *internal time* of this system, which we propose to call *generalized cosmological time* (Vallée, 1995, 1996, 2005) is then given, according to (8), by

$$s(t) = c^2p/2 \quad (\text{Log}\ t - 2t/p - \text{Log}\ (2p-t)) \qquad (17)$$

The initial reference instant t = 0 ("big bang") is pushed back to $-\infty$ and the final reference instant t = 2p ("big crunch") is pushed forward to $+\infty$.

But classically a cosmological model with pure radiation is accepted mainly as an approximation valid when the density of matter $(a/R^3(t))$ is negligible compared to the (equivalent) density of matter of pure radiation $(b/R^4(t))$. This happens when R(t) is small, so when t is close to 0. In that case $-kc^2$ is negligible as well as $\Lambda/3R^2$ and it is not even necessary to suppose that $\Lambda = 0$. We then have

$$(dR(t)/dt)^2 = 8\pi G/3\ b/R^2(t) \qquad (18)$$

This equation is that of the radiation-dominated era at the beginning of the universe, or that of an evolution with pure radiation, null cosmic constant and flat universe. It corresponds to a *parabolic explosion* whose differential equation, after taking the square of its two members, gives

$$(dX(t)/dt) = 4h^2 / X^2(t)$$

We have just to substitute R(t) to X(t) and $(b\, 2\pi G/3)^{1/2}$ to h. According to (14), the *internal time* of this universe is

$$s(t) = (b\, 2\pi G/3)^{1/2}\ \text{Log}\ t \qquad (19)$$

The initial instant t = 0 ("big bang") is pushed back to $-\infty$. We recognize here what Milne has called cosmological time (Milne, 1948).

References

[1] Berry, M. Principles of Cosmology and Gravitation. Institute of Physics Publishing, Bristol & Philadelphia, 1989.

[2] Lecomte du Noüy. P. Le temps et la vie. Gallimard, Paris, 1936.

[3] Milne, E. A. Kinematic Relativiy. Clarendon Press, Oxford, 1948.

[4] Vallée, R. Perception, memorisation and multidimensional time, *Kybernetes,* 1991, vol.20, 15-28.

[5] Vallée, R. Cognition et système. Essai d'épistémo-praxéologie. L'Interdisciplinaire, Limonest (France), 1995.

[6] Vallée, R. Temps propre d'un système dynamique, cas d'un système explosif-implosif, in *Actes du 3ème Congrès International de Systémique,* (E. Pessa, M.P. Penna, dirs), Edizioni Kappa, Rome, pp.967-970, 1996.
[7] Vallée, R. Time and dynamical systems. *Systems Science* vol.27, 97-100, 2001.
[8] Vallée, R. Time and systems, *Kybernetes,* vol.34, 9-10, 1563-1569, 2005.