# Dynamic Models of Mob Excitation Control[1]

Ivan N. Barabanov[a] and Dmitry A. Novikov[a,b]

[a)] *V.A. Trapeznikov Institute of Control Sciences of Russian Academy of Sciences, 117997, Profsoyuznaya street, 65, Moscow, Russia;*

[b)] *Moscow Institute of Physics and Technology, 141700, Institutskiy per., 9, Dolgoprudny, Moscow Region, Russia*

*E-mail: ivbar@ipu.ru, novikov@ipu.ru*

*Abstract.* This paper formulates and solves the mob excitation control problem in the continuous-time setting by introducing an appropriate number of "provokers" at each moment of control.

*Key words*: collective behavior, Granovetter's model, stochastic models of mob control.

## 1. INTRODUCTION

In the case of *conformity decision-making* of agents that perform binary choice between "passivity" and "activity," the associated control problems involve the models of *collective behavior* based on classical Granovetter's model [1] (see the surveys in [2, 3]). Within this model, each *agent* is characterized by his *threshold*, a number from the interval [0; 1], in the following way. He decides to be active if the fraction of active agents in his neighborhood exceeds the threshold; otherwise, the agent prefers passivity. The dynamics of the fraction of active agents depends on its initial value and the distribution function of the agents' thresholds. Hence, a goal-directed (exogenous) change in the number of agents at the initial and/or subsequent moments affects the behavioral dynamics of the whole group.

Consider the collective behavior of agents forming a *mob* [4]. In this case, control consists in choosing a fraction of always active agents to-be-introduced to the mob (the so-called "*provokers*"). The problems belonging to this class can be classified by several bases such as *discrete-time setting* or *continuous-time setting*, single or multiple application of the control actions (*constant controls* and *time-varying controls*, respectively), and *open-loop* or *feedback control*.

The discrete-time optimal choice problem for a single constant control applied by a control subject (*Principal*) to a mob was formulated and solved in [4]. In [5], this problem was generalized to the case of multiple open-loop controls in the discrete-time setting.

The present paper is dedicated to the continuous-time mob control models. Further exposition follows the paper [6], in particular, the proofs of Propositions 2-6 can be taken there.

The mob model proper is imported from [7, 8], actually representing a generalization of Granovetter's model to the continuous-time case as follows. Suppose that we know the initial *fraction* $x_0 \in [0; 1]$ of active agents at the zero moment. Then the evolution of this fraction $x(t)$ in the continuous time $t \geq 0$ is governed by the equation

$$\dot{x} = F(x) - x \ , \tag{1}$$

where $F(\cdot)$ is a known continuous function possessing the properties of a distribution function, $F(0) = 0$ and $F(1) = 1$. Actually, this is the distribution function of the agents' thresholds [2, 4]. Similar to [4, 5], by applying a control action $u(t) \in [0; 1]$ (introducing provokers), we obtain the controlled dynamic system

$$\dot{x} = u(t) + (1 - u(t))F(x) - x \ . \tag{2}$$

This paper is organized in the following way. Section 2 studies the reachability set and the monotonicity of the system trajectories in the control action. Next, section 3 is focused on the case of constant controls according to the above classification. Section 4 considers the models

[1] This work was partially supported by RSF grant 16-19-10609.

where control excites the whole mob. And finally, section 5 deals with the case of feedback control.

## 2. REACHABILITY SET AND MONOTONICITY

First, we formulate a lemma required for further analysis. Consider functions $G_1(x,t)$ and $G_2(x,t)$: $R\times[t_0,+\infty)\to R$ that are continuously differentiable with respect to $x$ and continuous in $t$. By assumption, the functions $G_1$ and $G_2$ are such that the solutions to the Cauchy problems for the differential equations $\dot{x}=G_i(x,t)$, $i=1,2$, with initial conditions $(t_0,x_0)$, $x_0\in R$, admit infinite extension in $t$. Denote by $x_i(t,(t_0,x_0)), i=1,2$, the solutions of the corresponding Cauchy problems.

***Lemma* [6].** Let $\forall x\in R, \forall t\geq t_0 \to G_1(x,t)>G_2(x,t)$. Then $\forall t>t_0\to x_1(t,(t_0,x_0))$ $>x_2(t,(t_0,x_0))$.

Note that, for validity of this lemma, one should not consider the inequality $G_1(x,t)>G_2(x,t)$ for all $x\in R$. It suffices to take the union of the reachability sets of the equations $\dot{x}=G_i(x,t)$, $i=1,2$, with the chosen initial conditions $(t_0,x_0)$.

Denote by $x_t(u)$ the fraction of active agents at the moment $t$ under the control action $u(\cdot)$. The right-hand side of the expression (1) increases monotonically in $u$ for each $t$ and $\forall\, x \in [0; 1]$: $F(x) \leq 1$. Hence, we have the following result.

***Proposition 1.*** Let the function $F(x)$ be such that $F(x) < 1$ for $x \in [0; 1)$. If $\forall t\geq t_0\to u_1(t)>u_2(t)$ and $x_0(u_1) = x_0(u_2)$ ($x_0<1$), then $\forall t>t_0$: $x_t(u_1) > x_t(u_2)$.

Indeed, by the premises, for all $t$ and $x<1$ we have the inequality $u_1(t)+(1-u_1(t))F(x)-x>u_2(t)+(1-u_2(t))F(x)-x$, as the convex combination of different numbers (1 and $F(x)$) is strictly monotonic. The point $x=1$ forms the equilibrium of the system (1) under any control actions $u(t)$. And so, it is unreachable for any finite $t$. Using the above lemma, we find that $x_t(u_1) > x_t(u_2)$ under same initial conditions.

Suppose that the control actions are subjected to the *constraint*

$$u(t)\leq\Delta,\ t\geq t_0, \tag{3}$$

where $\Delta \in [0; 1]$ means some constant.

We believe that $t_0=0$, $x(t_0)=x(0)=0$, i.e., initially the mob is in the nonexcited state. If the *efficiency criterion* is defined as the fraction of active agents at a given moment $T > 0$, then the corresponding *terminal control problem* takes the form

$$\begin{cases} x_T(u)\to \max\limits_{u(\cdot)}, \\ \quad (2), (3). \end{cases} \tag{4}$$

Here is a series of results (Propositions 2-4) representing the analogs of the corresponding Propositions from [5].

***Proposition 2.*** The solution of the problem (4) is given by $u(t) = \Delta$, $t \in [0; T]$.

Denote by $\tau(\hat{x},u)=\min\{t\geq 0 \mid x_t(u)\geq \hat{x}\}$ the first moment when the fraction of active agents achieves a required value $\hat{x}$ (if the set $\{t\geq 0 \mid x_t(u)\geq\hat{x}\}$ is empty, just specify $\tau(\hat{x},u)$ $= +\infty$). Within the current model, one can pose the following *time-optimal problem*:

$$\begin{cases} \tau(\hat{x},u)\to \min\limits_{u(\cdot)}, \\ \quad (2), (3). \end{cases} \tag{5}$$

***Proposition 3.*** The solution of the problem (5) is given by $u(t) = \Delta$, $t \in [0; \tau]$.

By analogy with the discrete-time models [5], the problem (4) or (5) has the following practical interpretation. The Principal benefits most from introducing the maximum admissible number of provokers in the mob at the initial moment, doing nothing after that (e.g., instead of first decreasing and then again increasing the number of introduced provokers). This structure of the optimal solution can be easily explained, as in the models (4) and (5) the Principal incurs no costs to introduce and/or keep the provokers.

What are the properties of the *reachability set* $D = \bigcup_{u(t)\in[0;\Delta]} x_T(u)$? Clearly, $D \subseteq [0;1]$, since the right-hand side of the dynamic system (2) vanishes for $x = 1$.

In the sense of potential applications, a major interest is attracted by the case of *constant controls* ($u(t) = v$, $t \geq 0$). Here the Principal chooses the same fraction $v \in [0; \Delta]$ of provokers at all moments. Let $x_T(\Delta) = x_T(u(t) \equiv \Delta)$, $t \in [0; T]$, and denote by $D_0 = \bigcup_{v\in[0;\Delta]} x_T(v) \subseteq [0;1]$ the reachability set under constant control actions. According to Proposition 1, $x_T(v)$ represents a monotonic continuous mapping of $[0; \Delta]$ into $[0; 1]$ such that $x_T(0) = 0$. This leads to the following.

***Proposition 4.*** $D_0 = [0; x_T(\Delta)]$.

Consider models taking into account the *Principal's control costs*. Given a fixed "price" $\lambda \geq 0$ of one provoker per unit time, the Principal's costs over a period $\tau \geq 0$ are defined by

$$c_\tau(u) = \lambda \int_0^\tau u(t)dt. \tag{6}$$

Suppose that we know a pair of monotonic functions characterizing the Principal's terminal payoff $H(\cdot)$ from the fraction of active agents and his current payoff $h(\cdot)$. Then the problem (4) can be "generalized" to

$$\begin{cases} H(x_T(u)) + \int_0^T h(x(t))dt - c_T(u) \to \max_u, \\ \qquad (2), (3). \end{cases} \tag{7}$$

Under existing constraints on the Principal's "total" costs $C$, the problem (7) acquires the form

$$\begin{cases} H(x_T(u)) + \int_0^T h(x(t))dt \to \max_u, \\ \qquad (2), c_T(u) \leq C. \end{cases} \tag{8}$$

A possible modification of the problems (4), (5), (7), (8) is the one where the Principal achieves a required fraction $\hat{x}$ of active agents by the moment $T$ (the *cost minimization problem*):

$$\begin{cases} c_T(u) \to \min_u, \\ x_T(u) \geq \hat{x}, \\ \qquad (2). \end{cases} \tag{9}$$

The problems of the form (7)-(9) can be easily reduced to standard optimal control problems.

***Example 1.*** Consider the problem (9), where $F(x) = x$, $x_0 = 0$ and the Principal's costs defined by (6) with $\lambda_0 = 1$. This yields the following optimal open-loop control problem with fixed bounds:

$$\begin{aligned} &\dot{x} = u(1-x), \\ &x(0) = 0,\ x(T) = \hat{x}, \\ &0 \leq u \leq \Delta, \\ &\int_0^T u(t)dt \to \min_{u\in[0,\Delta]}. \end{aligned} \tag{10}$$

For the problem (10), construct the Hamilton-Pontryagin function $H = \psi\left(u(1-x)\right) - u$. By the maximum principle, this function takes the maximum values in $u$. As $H$ is linear in $u$, its maximum is achieved at an end of the interval $[0, \Delta]$ depending on the sign of the factor at $u$, i.e.,

$$u = \frac{\Delta}{2}\left(\text{sign}\left(\psi\left(1-x\right)-1\right)+1\right). \tag{11}$$

The fact that the Hamilton-Pontryagin function is linear in control actions actually follows from the same property of the right-hand side of the dynamic system (2) and the functional (6). In other words, we have the result below.

***Proposition 5.*** If the constraints in the optimal control problems (7)-(9) are linear in control actions, then the optimal open-loop control possesses the structure described by (11). That is, at each moment the control action takes either the maximum or the minimum admissible value.

The Hamilton equations acquire the form

$$\dot{x} = \frac{\partial H}{\partial \psi} = u(1-x),$$

$$\dot{\psi} = -\frac{\partial H}{\partial x} = u\psi.$$

The boundary conditions are imposed on the first equation only. For $u = 0$, its solution is a constant; for $u = \Delta$, the solution is

$$x(t) = 1 - \left(1 - x(t_0)\right) e^{-\Delta(t - t_0)}.$$

The last expression restricts the maximum number of provokers required for mob transfer from the zero state to $\hat{x}$: $\Delta \geq \frac{1}{T}\log\frac{1}{1-\hat{x}}$.

And there exists the minimum time $t_{\min} = \frac{1}{\Delta}\log\frac{1}{1-\hat{x}}$, during which control actions take the maximum value $\Delta$, being 0 at the rest moment. Particularly, a solution of the problem (10) has the form

$$u = \begin{cases} \Delta, & t \leq t_{\min} \\ 0, & t_{\min} < t \leq T \end{cases}, \tag{12}$$

when the Principal introduces the maximum number of provokers from the very beginning, maintaining it during the time $t_{\min}$.

The structure of the optimal solution to this problem (a piecewise constant function taking the value of 0 or $\Delta$) possibly requires minimizing the number of control switchovers (discontinuity points). Such an additional constraint reflects situations when the Principal incurs extra costs to introduce or withdraw provokers. If this constraint appears in the problem, the best control actions in the optimal control set are either (12) or $u = \begin{cases} \Delta, & t \in [T - t_{\min}, T], \\ 0, & t < T - t_{\min}. \end{cases}$

## 3. CONSTANT CONTROL

In the class of the constant control actions, we obtain $c_\tau(v) = \lambda\, v\, \tau$ from formula (6). Under given functions $F(\cdot)$, i.e., a known relationship $x_t(v)$, the problems (7)-(9) are reduced to standard scalar optimization problems.

***Example 2.*** Choose $F(x) = x$, $T = 1$, $x_0 = 0$, $H(x) = x$, and $h(x) = \gamma\, x$, where $\gamma \geq 0$ is a known constant. It follows from (2) that

$$x_t\left(u\right) = 1 - \exp\left(-\int_0^t u(y)dy\right). \tag{13}$$

For the constant control actions, $x_t(v) = 1 - e^{-vt}$.

The problem (7) becomes the scalar optimization problem

$$e^{-v}\left(\frac{\gamma}{v}-1\right)-\frac{\gamma}{v}-\lambda v \to \max_{v\in[0;\Delta]} . \qquad (14)$$

Next, the problem (8) becomes the scalar optimization problem

$$e^{-v}\left(\frac{\gamma}{v}-1\right)-\frac{\gamma}{v} \to \max_{v\in[0;\Delta]} . \qquad (15)$$

And finally, the problem (9) acquires the form $\begin{cases} v \to \min\limits_{v\in[0;1]}, \\ 1-e^{-v}=\hat{x}. \end{cases}$ Its solution is described by $v = \log\left(\frac{1}{1-\hat{x}}\right)$.

## 4. EXCITATION OF WHOLE MOB

Consider the "asymptote" of the problems as $T \to +\infty$. Similarly to the corresponding model in [5], suppose that (a) the function $F(\cdot)$ has a unique inflection point and $F(0) = 0$, (b) the equation $F(x) = x$ has a unique solution $q > 0$ on the interval (0; 1) so that $\forall x \in (0;q)\ F(x) < x$, $\forall x \in (q;1)\ F(x) > x$. Several examples of the functions $F(\cdot)$ satisfying these assumptions are provided in [5]. The Principal seeks to excite all agents with the minimum costs.

By the above assumptions on $F(\cdot)$, if for some moment $\tau$ we have $x(\tau) > q$, then the trajectory $x_t(u)$ is nonincreasing and $\lim\limits_{t\to+\infty} x_t(u) = 1$ even under $u(t) \equiv 0\ \forall t > \tau$. As mentioned in [5], this property of the mob admits the following interpretation. The *domain of attraction* of the zero equilibrium without control (without introduced provokers) is the half-interval $[0; q)$. In other words, it takes the Principal only to excite more than the fraction $q$ of the agents; subsequently, the mob itself surely converges to the unit equilibrium even without control.

Denote by $u^\tau$ the solution of the problem

$$\int_0^\tau u(t)dt \to \min_{u:\, u(t)\in[0;\Delta],\, x_\tau(u)>q} \qquad (16)$$

Calculate $Q_\tau = \int_0^\tau u^\tau(t)dt$ and find $\tau^* = \arg\min\limits_{\tau\geq 0} Q_\tau$.

The solution to the problem (16) exists under the condition

$$\Delta > \Delta^* = \max_{x\in[0,q]} \frac{x-F(x)}{1-F(x)} \qquad (17)$$

For practical interpretations, we refer to [5].

Owing to the above assumptions on the properties of the distribution function, the optimal solution to the problem is characterized as follows.

***Proposition 6.*** If the condition (17) holds, then $u^\tau(t) \equiv 0$ for $t > \tau$.

***Example 3.*** The paper [9] constructed the two-parameter function $F(\cdot)$ describing in the best way the evolvement of active users in the Russian-language segments of online social networks *LiveJournal*, *FaceBook* and *Twitter*. The role of the parameters is player by $a$ и $b$. This function has the form

$$F_{a,b}(x) = \frac{\arctan(a(x-b)) + \arctan(ab)}{\arctan(a(1-b)) + \arctan(ab)}, \quad a \in [7;15], b \in [0;1].$$

Choose $a = 13$ that corresponds to *Facebook* and $b = 0.4$. In this case, $q \approx 0.375$ and $\Delta^* \approx 0.169$; the details can be found in [5].

## 5. FEEDBACK CONTROL

In the previous sections, we have considered the optimal open-loop control problem arising in mob excitation. An alternative approach is to use feedback control. Consider two possible statements having transparent practical interpretations.

Within the first statement, the problem is to find a feedback control law $\tilde{u}(x):[0;1]\to[0;1]$ ensuring maximum mob excitation (in the sense of (4) or (5)) under certain constraints imposed on the system trajectory and/or control actions.

By analogy with the expression (3), suppose that the control actions are bounded:

$$\tilde{u}(x) \le \Delta, \quad x \in [0;1], \tag{19}$$

and there exists an additional constraint on the system trajectory in the form

$$\dot{x}(t) \le \delta, \quad t \ge 0, \tag{20}$$

where $\delta > 0$ is a known constant. The condition (20) means that, e.g., a very fast growth of the fraction of excited agents (increment per unit time) is detected by appropriate authorities banning further control. Hence, trying to control mob excitation, the Principal has to maximize the fraction of excited agents subject to the conditions (19) and (20). The corresponding problem possesses the simple solution

$$\tilde{u}^*(x) = \min\left\{\Delta; \max\left\{0; \frac{x+\delta-F(x)}{1-F(x)}\right\}\right\}, \tag{21}$$

owing to the properties of the dynamic system (2), see the lemma. The fraction in (21) results from making the right-hand side of (1) equal to the constant $\delta$. Note that, under small values of $\delta$, the nonnegative control action satisfying (20) may cease to exist.

The second statement of feedback control relates to the so-called *network immunization problem* [4]; here the Principal seeks to reduce the fraction of active agents by introducing an appropriate number (or fraction) of *immunizers*–agents that always prefer passivity.

Denote by $w \in [0; 1]$ the fraction of immunizers. As shown in the paper [4], the fraction of active agents evolves according to the equation

$$\dot{x} = (1-w)F(x) - x, \quad x \in [0;1). \tag{22}$$

Let $\tilde{w}(x):[0;1]\to[0;1]$ be a feedback control. If the Principal is interested in reducing the fraction of active agents, i.e.,

$$\dot{x}(t) \le 0, \quad t \ge 0, \tag{23}$$

then the control actions must satisfy the inequality

$$\tilde{w}(x) \ge 1 - \frac{x}{F(x)} \ . \tag{24}$$

The quantity $\Delta_{\min} = \max_{x\in[0;1]}\left(1-\frac{x}{F(x)}\right)$ characterizes the bottom restrictions on the control actions at each moment when the system (22) is "controllable" in the sense of (23).

## 6. CONCLUSION

This paper has described the continuous-time problems of mob excitation control using introduction of provokers or immunizers.

A promising line of future research is analysis of a differential game describing informational opposition of two control subjects (Principals) that choose in continuous time the fractions (or numbers) of introduced provokers *u* and immunizers *w*, respectively. The corresponding static problem [7] can be a "reference model" here. The controlled object is defined by the dynamic system [4, 10] $\dot{x} = u(1-w) + (1-u-w+2uw)F(x) - x$.

Another line of interesting investigations concerns the mob excitation problems with dynamic (open-loop and/or feedback) control, where the mob dynamics is modeled by the transfer equation [7] of the form $\frac{\partial}{\partial t} p(x,t) + \frac{\partial}{\partial x}\left(\left[u+(1-u)F(x)-x\right] p(x,t)\right) = 0$. In this model, the mob state at each moment is described by a probability distribution function $p(x, t)$, instead of the scalar fraction of active agents.

## References


[1] Granovetter, M., “Threshold Models of Collective Behavior”, *AJS*, vol. 83, no 6, pp. 1420-1443, 1978.

[2] Breer, V. V., “Models of Conformity Behavior: A Survey”, *Control Sciences*, vol. 1, pp. 2-13, 2014 [in Russian].

[3] Slovokhotov, Yu. L., “Physics and Sociophysics. Part 1”, *Control Sciences*, vol. 1, pp. 2-20, 2012 [in Russian].

[4] Breer, V. V., Novikov, D. A., and Rogatkin, A. D., “Stochastic Models of Mob Control”, *Automation and Remote Control*, vol. 77, no. 5, pp. 895-913, 2016

[5] Barabanov, I. N., and Novikov, D. A., “Dynamic Models of Mob Excitation Control in Discrete Time”, *Automation and Remote Control*, vol. 77, no. 10, pp. 1792-1804, 2016.

[6] Barabanov, I. N., and Novikov, D. A., “Dynamic Models of Mob Excitation Control in Continuous Time”, *Upravlenie bol'simi sistemami* (*Large Scale Systems Control*), vol. 63, pp. 71-68, 2016 [in Russian].

[7] Rogatkin, A. D., “Continuous-time Granovetter’s Model”, *Upravlenie bol'simi sistemami* (*Large Scale Systems Control*), vol. 60, pp. 139-160, 2016 [in Russian].

[8] Akhmetzhanov, A.R., Worden, L., and Dushoff, J., “Effects of Mixing in Threshold Models of Social Behavior”, *Phys. Rev. E*, vol. 88, no. 1, 2013, article number 012816.

[9] Batov, A. V., Breer, V. V., Novikov, D. A., and Rogatkin, A. D., “Micro- and Macromodels of Social Networks. II. Identification and Simulation Experiments”, *Automation and Remote Control*, vol. 77, no. 2, pp. 321-331, 2016.

[10] Novikov, D. A., “Models of Informational Confrontation in Mob Control”, *Automation and Remote Control*, vol.77, no.7, pp. 1259-1274, 2016.