Adv Syst Sci Appl 2026; **01**:148–153
Published online at http://ijassa.ipu.ru.

On the Existence of Non-Negative Solution to the Cauchy Problem in Dynamic Economic Models

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Abstract: The paper is devoted to the study of dynamic models of economic processes, which are described by systems of linear differential equations for the existence of a non-negative solution to the Cauchy problem on a segment under the condition of non-negativity of the initial condition. The Leontief model of intersectoral balance, as well as the Evans-Samuelson model of a market economy, were studied. The conditions under which the Cauchy problem has a non-negative solution on the interval [0,T] are obtained.

Keywords: mathematical models, economic processes, differential equations, existence of a solution

1. INTRODUCTION

This paper discusses mathematical models of economic processes that are described by systems of linear differential equations with non-negative initial conditions. Leontief and Evans-Samuelson models is considered.

1.1. Leontief model

Before the advent of Wassily Leontief's Input-Output model in the 1930s [1]- [4], economic analysis often treated industries as relatively independent entities. The profound and complex interdependencies between sectors – how the steel industry relies on coal and iron, how the automobile industry consumes steel, rubber, and glass, and how all industries depend on the output of the energy sector – were acknowledged but not rigorously quantified. The static Leontief model provided the first comprehensive, quantitative framework to map these intricate production relationships across an entire economy. It answered a fundamental, yet operationally complex, question: What must be the total output of each industry in an economy to satisfy a given bundle of final demands? This document explores the history behind this revolutionary model, delves into its core mathematical structure, and summarizes the pivotal results and insights it has generated for economic theory and policy.

The history of the development of modern economic theory dates back to the end of the eighteenth century like the Physiocratic Precedent. The French Physiocrats, particularly François Quesnay with his Tableau Économique (1758) [5], were the first to conceptualize the economy as a circular flow of goods and money between different social classes (farmers, landowners, artisans). While primitive, the Tableau planted the seed of inter-sectoral analysis. Then the Classical and Neoclassical Focus: The subsequent Classical and early Neoclassical schools shifted focus towards macroeconomic aggregates (supply, demand, capital, labor)

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and the determination of prices in individual markets. The detailed, material connections between industries were largely overlooked in favor of more abstract, monetary analyses. The Socialist Calculation Debate (1920s-30s): This debate, concerning whether a socialist state could rationally allocate resources without market prices, highlighted the need for a system to understand production interdependencies. It created a fertile ground for a model that could, in principle, calculate the required outputs for a planned set of final demands. Leontief's Synthesis: Wassily Leontief, a Russian-born economist, synthesized these strands. He was influenced by both the Soviet planning apparatus's need for material balances and the neoclassical tradition he encountered in Europe and America. His genius lay in combining the concept of the circular flow with the empirical rigor of collecting vast amounts of data on inter-industry transactions. His seminal work, *The Structure of American Economy, 1919-1929*, published in 1941, was the culmination of this effort, for which he would later receive the Nobel Memorial Prize in Economic Sciences in 1973.

For any sector i, its total output x_i is divided between what is sold to other industries as intermediate inputs and what is left for final demand y_i.

$$x_i = z_{i1} + z_{i2} + \cdots + z_{in} + y_i,$$

where z_{ij} is the flow of output from industry i to industry j.

Static Leontief model describes as linear equation

$$x = Ax + y.$$

This reads like Total Output = Intermediate Demand + Final Demand. And matrix $A = \{a_{ij}\} \in [0, 1]$. The technical coefficient, a_{ij}, is defined as the amount of input from industry i required to produce one unit of output in industry j: $a_{ij} = z_{ij}/x_j$. These coefficients are assumed to be fixed, reflecting the core assumption of the model. The matrix A collects all these coefficients

1.2. *Evans-Samuelson dynamic model*

Before the mid-20th century, classical economic theory often treated market equilibrium as a static, self-correcting state. The "invisible hand" of supply and demand was understood to guide prices toward a point where markets cleared. However, this static analysis left a crucial question unanswered: What is the precise path that prices and quantities take over time to reach this equilibrium? The dynamic process of adjustment—the wobbles, oscillations, and convergence (or divergence) of prices—remained a theoretical black box. These issues were investigated in particular by G. C. Evans [6] and Paul A. Samuelson [7], [8]. Although developed separately, their models shared a core mathematical structure and insight. Their work synthesized the static theory of market equilibrium with the formal tools of dynamic analysis, specifically difference and differential equations. The model that emerged, now universally known as the Evans-Samuelson model, provided a rigorous framework for analyzing how prices and quantities evolve over time in response to disequilibrium. Its power lay in its generality. While often applied to competitive markets with production lags, the model's structure could be adapted to analyze monopoly behavior, inventory cycles, and even macroeconomic phenomena.

2. DESCRIPTION OF THE LEONTIEF DYNAMIC MODEL

This model is described by a system of linear differential equations

$$x(t) = Ax(t) + B\dot{x}(t) + y \tag{2.1}$$

with initial conditions

$$x(0) = x_0, \tag{2.2}$$

where $x(t) = (x_1(t), \dots, x_n(t)) \in \mathbb{R}^n_+; x_0 = (x_{01}, \dots, x_{0n}) \in \mathbb{R}^n_+; y = (y_1, \dots, y_n) \in \mathbb{R}^n_+; \dot{x}(t) = (\dot{x}_1(t), \dots, \dot{x}_n(t)) \in \mathbb{R}^n; A = \{a_{ij}\}, a_{ij} \in [0, 1), B = \{b_{ij}\}, b_{ij} \in [0, 1), i, j = 1, \dots, n \quad \forall t \in [0, T]$. Require that the matrix A be a ”productive” matrix, meaning that all the principal minors of A being positive and $\det B \neq 0$.

3. RESEARCH OF THE LEONTIEF MODEL

The task was to obtain sufficient conditions for the existence of non-negative solutions to the Cauchy problem of the system 2.1, 2.2 under the condition that initial conditions are nonnegative. System 2.1, 2.2 can be given as

$$\dot{x}(t) = B^{-1}(E - A)x(t) - B^{-1}y.$$

Let $M = E - A$ then the homogeneous equation

$$\dot{x} = B^{-1}Mx(t)$$

has the solution

$$x_h(t) = e^{(B^{-1}M)t}C,$$

where C is a constant vector. y is independent of t, constant vector. We are looking for a particular solution in the form of a constant vector

$$x_p = M^{-1}y$$

from initial conditions $x(0) = x_0 = C + M^{-1}y$. Then

$$C = x_0 - M^{-1}y.$$

And

$$x(t) = e^{(B^{-1}M)t}(x_0 - M^{-1}y) + M^{-1}y.$$

To obtain the conditions for the existence of non-negative solutions in the model 2.1, 2.2 for $n = 2$, we prove the following theorem.

Theorem 3.1:
For $n = 2$, the eigenvalues of the matrix $B^{-1}(E - A)$ in model 2.1, 2.2 are positive numbers.

Proof
The characteristic polynomial of the matrix

$$B^{-1}(E - A) = \begin{pmatrix} b^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} 1 - a_{11} & -a_{12} \\ -a_{21} & 1 - a_{22} \end{pmatrix} = \begin{pmatrix} (1 - a_{11})b_{11}^{-1} & -a_{12}b_{11}^{-1} \\ -a_{21}b_{22}^{-1} & (1 - a_{22})b_{22}^{-1} \end{pmatrix}$$

is equal

$$\det(\lambda E - B^{-1}(E - A)) = \begin{vmatrix} \lambda - (1 - a_{11})b_{11}^{-1} & a_{12}b_{11}^{-1} \\ a_{21}b^{-1}b_{22}^{-1} & \lambda - (1 - a_{22})b_{22}^{-1} \end{vmatrix} =$$

$$= \lambda^2 - \frac{(1 - a_{22})b_{22}^{-1} + (1 - a_{11})b_{11}^{-1}}{b_{11}b_{22}}\lambda + \frac{(1 - a_{11})(1 - a_{22}) - a_{12}a_{21}}{b_{11}b_{22}}.$$

Due to the assumptions of the model 2.1, 2.2 with respect to the elements of the matrices A and B

$$(1 - a_{11})b_{22} + (1 - a_{22})b_{11} > 0.$$

And

$$\det(E - A) = (1 - a_{11})(1 - a_{22}) - a_{12}a_{21} > 0.$$

From equation 3, we can see that for the non-negativity of the solution of the Cauchy problem of system 2.1, 2.2 it is necessary that

$$x_0 - (E - A)^{-1} y \geq 0, \tag{3.3}$$

$$(E - A)^{-1} y \geq 0. \tag{3.4}$$

Condition 3.4 fulfilled due to the properties of the matrix $(E - A)$. This matrix named as Leontief matrix. The matrix $(E - A)$ is non-negative. The inverse matrix $(E - A)^{-1}$ consists of only non-negative elements. And vector y has non-negative constant elements. Then we need the conditions of inequality 3.3 to be met. Let $\exists Y = (Y_1, Y_2) : 0 \geq y \geq Y$ and

$$x_0 \geq (E - A)^{-1} Y \tag{3.5}$$

Condition 3.5 is a sufficient condition for the existence of a non-negative solution to the Cauchy problem in the model 2.1, 2.2 for $n = 2$. $\square$

4. DESCRIPTION OF THE EVANS-SAMUELSON DYNAMIC MODEL

This model is described by a system of linear differential equations

$$S_1 \dot{x} + S_2 x + S_3 = D_1 \dot{x} + D_2 x + D_3 \tag{4.6}$$

with initial conditions

$$x(0) = x_0. \tag{4.7}$$

Here $x(t) = (x_1(t), \dots, x_n(t)) \in \mathbb{R}^n_+; x_0 = (x_{01}, \dots, x_{0n}) \in \mathbb{R}^n_+; S_1 = (s_{1ij}), s_{1ij} \in \mathbb{R}, S_2 = (s_{2ij}), s_{2ij} \in \mathbb{R}, D_1 = (d_{1ij}), d_{1ij} \in \mathbb{R}, D_2 = (d_{2ij}), d_{2ij} \in \mathbb{R}, i, j = 1, \dots, n. S_3 = (S_{31}, \dots, S_{3n}) \in \mathbb{R}, D_3 = (D_{31}, \dots, D_{3n}) \in \mathbb{R}$. And $\det(S_1 - D_1) \neq 0$.

5. RESEARCH OF THE EVANS-SAMUELSON MODEL

The task was to obtain sufficient conditions for the existence of non-negative solutions to the Cauchy problem of the system (4.6), (4.7) under the condition that initial conditions are nonnegative. System (4.6), (4.7) can be given as

$$\dot{x}(t) = (S_1 - D_1)^{-1}(D_2 - S_2)x(t) + (S_1 - D_1)^{-1}(D_3 - S_3),$$

with initial condition

$$x(0) = x_0.$$

Let $(S_1 - D_1)^{-1} = K$, $(D_2 - S_2) = H$ and $(D_3 - S_3) = v$.

Then

$$\dot{x}(t) = K^{-1} H x(t) + K^{-1} v.$$

Set $K^{-1} H = Z$ and $K^{-1} v = q$. Then

$$\dot{x}(t) = Z x(t) + q. \tag{5.8}$$

Let's consider the homogeneous equation

$$\dot{x}(t) = Z x(t), \qquad x(0) = x_0.$$

The solution is $x_h(t) = e^{Zt}C$, where C is a constant vector. Particular solution $x_p(t)$ is

$$\dot{x}_p = Zx_p + q$$

Since q is constant, a natural guess is a constant particular solution. Take $x_p(t) = x_{eq}$ constant, then $\dot{x}_p = 0$ and

$$0 = Zx_{eq} + q \ \rightarrow \ x_{eq} = x_p = -Z^{-1}q.$$

So we have general solution for 5.8

$$x(t) = e^{Zt}C - Z^{-1}q.$$

Apply $x(0) = x_0$ Then at $t = 0$

$$x(0) = e^0 C - Z^{-1}v = C - H^{-1}v = x_0$$

Thus

$$C = x_0 + H^{-1}v.$$

We'll make a reverse replacement

$$x(t) = e^{(K^{-1}H)t}(x_0 + H^{-1}v) - H^{-1}v.$$

And

$$x(t) = e^{(S_1-D_1)(D_2-S_2)t}(x_0 + (D_2 - S_2)^{-1}(D_3 - S_3)) - (D_2 - S_2)^{-1}(D_3 - S_3)$$

or

$$x(t) = e^{(S_1-D_1)(D_2-S_2)t}(x_0 - (D_2 - S_2)^{-1}(S_3 - D_3)) + (D_2 - S_2)^{-1}(S_3 - D_3). \quad (5.9)$$

To obtain the conditions for the existence of non-negative solutions in the model 4.6, 4.7 for $n = 2$, we prove the following theorem.

Theorem 5.1:
For $n = 2$, if $S_1 > D_1$, $(D_2 - S_2)$ is a symmetric matrix and $D_2 \geq S_2$, then the eigenvalues of the matrix $((D_2 - S_2)^{-1}(D_3 - S_3))$ in model (4.6), (4.7) are positive numbers.

Proof
Let $(S_1 - D_1)^{-1} = K$, $(D_2 - S_2) = H$ and $(S_3 - D_3) = v$
Then equation 4.6 can be written as

$$\dot{x}(t) = K^{-1}Hx(t) + K^{-1}v.$$

Multiply

$$K^{-1}H = \begin{pmatrix} k_{11}^{-1} & k_{12}^{-1} \\ k_{21}^{-1} & k_{22}^{-1} \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} k_{11}^{-1}h_{11} + k_{12}^{-1}h_{21} & k_{11}^{-1}h_{12} + k_{12}^{-1}h_{22} \\ k_{21}^{-1}h_{11} + k_{22}^{-1}h_{21} & k_{21}^{-1}h_{12} + k_{22}^{-1}h_{22} \end{pmatrix} = G$$

The characteristic polynomial of the matrix G

$$\begin{aligned} p_G(\lambda) &= \det(\lambda E - G) = \\ &= \begin{vmatrix} \lambda - k_{11}^{-1}h_{11} + k_{12}^{-1}h_{21} & k_{11}^{-1}h_{12} + k_{12}^{-1}h_{22} \\ k_{21}^{-1}h_{11} + k_{22}^{-1}h_{21} & \lambda - k_{21}^{-1}h_{12} + k_{22}^{-1}h_{22} \end{vmatrix} = \\ &= \lambda^2 - (G_{11} + G_{22})\lambda + (G_{11} - G_{21}) = \\ &= \lambda^2 - (k_{11}^{-1}h_{11} + k_{12}^{-1}h_{21} + k_{21}^{-1}h_{12} + k_{22}^{-1}h_{22})\lambda + \frac{h_{11}h_{22} - h_{12}h_{21}}{k_{11}k_{22} - k_{22}k_{21}}. \end{aligned}$$

Also $K^{-1} = \frac{1}{\det K}\begin{pmatrix} k_{11} & -k_{12} \\ -k_{21} & k_{22} \end{pmatrix}$, and $h_{11}h_{22} - h_{12}h_{21} = \det H, k_{11}k_{22} - k_{22}k_{21} = \det K$
So

$$p_G(\lambda) = \lambda^2 - (k_{11}^{-1}h_{11} + k_{12}^{-1}h_{21} + k_{21}^{-1}h_{12} + k_{22}^{-1}h_{22})\lambda + \frac{\det H}{\det K}.$$

and

$$p_G(\lambda) = \lambda^2 - \frac{\operatorname{tr}(K^{-1}H)}{\det K}\lambda + \frac{\det H}{\det K}.$$

Due to the assumptions of the model (4.6), (4.7) and the theorem with respect to the elements of the matrices $(D_2 - S_2) > 0$ and $\det(D_3 - S_3) \geq 0$, the claim follows for $n = 2$. □

From equation (5.9), we can see that for the non-negativity of the solution of the Cauchy problem of system (4.6), (4.7) it is necessary that

$$x_0 - H^{-1}y \geq 0, \tag{5.10}$$

$$H^{-1}y \geq 0. \tag{5.11}$$

Then we need the conditions of inequality 5.10 to be met. Let $\exists M = (M_1, M_2) : 0 \geq y \geq M$ and

$$x_0 \geq (D_3 - S_3)^{-1}M \tag{5.12}$$

Condition (5.12) is a sufficient condition for the existence of a non-negative solution to the Cauchy problem in the model (4.6), (4.7) with $(D_3 - S_3)$ a symmetric matrix and $(S_1 > D_1)$.

CONCLUSION

The obtained conditions for the existence of a non-negative solution to the Cauchy problem make it possible to understand the correctness of the compiled model. These results are applicable to Leontief's intersectoral balance production model, as well as to linear continuous dynamic market models.

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