

Block Coordinate Descent Method in the Problem of the Fastest Transition into a Given Area for Linear Discrete Systems

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Abstract: In the paper, we construct a numerical solution to the problem of the fastest possible transfer of the state of a linear discrete-time system from a given initial position into a pre-determined ellipsoid around the origin. Our algorithmic approach is based on the method of block coordinate descent. As the main result we formulate and prove the convergence theorem for the proposed algorithm.

Keywords: block coordinate descent, linear discrete systems, time optimization problem

1. INTRODUCTION

The problem of the fastest transition into a given area naturally arises in the context of mathematical modeling the processes of landing aircrafts and docking spacecrafts and underwater vehicles. In these problems the target area can often be described quite accurately as some ellipsoid around the origin in a vector space. For example, it can be an elongated and axes-oriented ellipsoid in $\mathbb{R}^{2n}$, where the first n coordinates correspond to the real phase space, and the second n coordinates correspond to the space of velocities. Such areas allow simple mathematical description as the unit balls in a proper finite-dimensional normed space. Now, using the fact that in applications time is discrete, we derive the following reformulation of the problem: find the minimal value N of discrete time such that the state vector $x(N)$ of the system satisfy the condition $\|x(N)\| \leq 1$, where symbol $\|\cdot\|$ denotes the norm corresponding to the target area.

This problem is very similar to the classical time-optimization problem: transfer the state of the system from a given point exactly into the origin as fast as possible. Any numerical approach to the time-optimization problem transforms it into our statement with the ε -Euclidean ball as the target area (for a small value $\varepsilon > 0$). In [1] we construct an algorithm for approximate solving time-optimization problem based on the block coordinate descent method [2] applied to the smooth functions of Euclidean norm squared values for the state vector in different moments of discrete time. In this paper, we apply the same approach to the problem under consideration.

2. PROBLEM STATEMENT

Consider a linear nonstationary dynamical system with discrete time of the form

$$x(k+1) = A(k)x(k) + B(k)u(k), \quad k = 0, 1, \dots, \quad (2.1)$$

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where $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in U(k)$ is the control, $U(k)$ are convex and compact sets in $\mathbb{R}^m$ containing the origin, $A(k) \in \mathbb{R}^{n \times n}$ and $B(k) \in \mathbb{R}^{n \times m}$ are arbitrary given matrices. Here $\mathbb{R}^n$ and $\mathbb{R}^m$ denote vector spaces of dimensions n and m respectively, $\mathbb{R}^{n \times n}$ and $\mathbb{R}^{n \times m}$ denote vector spaces of real-valued matrices of sizes $n \times n$ and $n \times m$. Compactness is understood in the sense of any of the equivalent norms.

The initial condition for system (2.1) is fixed:

$$x(0) = x_0 \in \mathbb{R}^n. \quad (2.2)$$

Let Q be a given positive definite real valued matrix of sizes $n \times n$. Let $\|\cdot\|_Q$ denote the norm

$$\|x\|_Q := \langle x, Qx \rangle^{1/2}, \quad x \in \mathbb{R}^n,$$

throughout the paper, where $\langle \cdot, \cdot \rangle$ denotes standard inner product in $\mathbb{R}^n$. Define the target area

$$T := \{z \in \mathbb{R}^n \mid \|z\|_Q \leq 1\}. \quad (2.3)$$

For example, if Q is the identity matrix, then T is the unit ball centered at the origin in $\mathbb{R}^n$.

Consider the problem: find minimal $N > 0$ such that there exists a control sequence $\{u^*(k)\}$ producing the trajectory $\{x^*(k)\}$ by (2.1) – (2.2) with the property $x^*(N) \in T$. Here we assume that this problem is solvable. To solve the problem means to find such $N =: N_{\min}$ and a corresponding control sequence $\{u^*(k)\}$.

3. AUXILIARY PROBLEM

Let us write an auxiliary problem of terminal stabilization at a given moment of discrete time. Let $N > 0$ be fixed. Consider the problem

$$\|x(N)\|_Q \rightarrow \min_{u \in \mathcal{U}_N}, \quad (3.4)$$

where $\mathcal{U}_N := \{k \mapsto u(k) : \{0, \dots, N-1\} \rightarrow \mathbb{R}^m \mid u(k) \in U(k)\}$, and $x(N) \in \mathbb{R}^n$ is defined by (2.1) – (2.2) for all $u \in \mathcal{U}_N$. We can rewrite the problem (2.1) – (3.4) explicitly:

$$J_N(u) := \left\| \mathcal{A}_N(0)x_0 + \sum_{k=0}^{N-1} \mathcal{A}_N(k+1)B(k)u(k) \right\|_Q \rightarrow \min_{u \in \mathcal{U}_N}, \quad (3.5)$$

where $\mathcal{A}_N(k) := A(N-1) \dots A(k)$, $\mathcal{A}_N(N) := I$, I is the identity matrix of sizes $n \times n$. Now, the set $\mathcal{U}_N$ can be interpreted as the Cartesian product $U(0) \times \dots \times U(N-1)$ of the initially given nonempty convex and compact sets $U(k) \subset \mathbb{R}^m$. Therefore, $\mathcal{U}_N$ is a nonempty convex and compact set in $\mathbb{R}^{mN}$. So, the problem (3.5) has at least one solution, thanks to continuity of function J_N .

Problem (3.5) has very intuitive geometric sense. We have a number of nonempty convex and compact sets $K_i = N^{-1}\mathcal{A}_N(0)x_0 + \mathcal{A}_N(i)B(i-1)U(i-1)$, $i = 1, \dots, N$, in vector space $\mathbb{R}^n$, and we need to choose exactly one point from each K_i such that their sum appears to be the closest to the origin among all other possible choices with respect to a given distance determined by the norm $\|\cdot\|_Q$. If many points from $U(i-1)$ give a desired point in K_i , then we take any one of them.

4. BLOCK COORDINATE DESCENT

Let us present the formulation of an iterative procedure, proposed for use in order to solve the initial and auxiliary problems. Consider problem (3.5). Fix arbitrary point $u^{(0)} \in \mathcal{U}_N$. Assume

that for $l \geq 0$ we already have some point $u^{(l)} \in \mathcal{U}_N$. Then we define a new point $u^{(l+1)} \in \mathcal{U}_N$ by the rule

$$u^{(l+1)}(k) \in \operatorname{Arg} \min_{\mu \in U(k)} \|\mathcal{A}_N(k+1)B(k)\mu + \mathcal{A}_N(k)x^{(l+1)}(k) - \xi^{(l)}(k+1)\|_Q, \quad (4.6)$$

where $k = 0, \dots, N-1$,

$$x^{(l+1)}(k+1) = A(k)x^{(l+1)}(k) + B(k)u^{(l+1)}(k), \quad x^{(l+1)}(0) = x_0,$$

$$\xi^{(l)}(k) = \xi^{(l)}(k+1) - \mathcal{A}_N(k+1)B(k)u^{(l)}(k), \quad \xi^{(l)}(N) = 0.$$

It is directly verified (see [1] for the details) that the value under the Arg min sign in (4.6) equals to the value

$$J_N(u^{(l+1)}(0), \dots, u^{(l+1)}(k-1), \mu, u^{(l)}(k+1), \dots, u^{(l)}(N-1)).$$

Therefore, rule (4.6) gives us

$$J_N(u^{(l+1)}) \leq J_N(u^{(l)}), \quad l = 0, 1, 2, \dots$$

for any initial point $u^{(0)} \in \mathcal{U}_N$. In the next section we prove the convergence of the sequence $\{u^{(l)}\}$ to the set of solutions in problem (3.5).

5. MAIN RESULT

Our main result writes as follows.

Theorem 5.1:

If the point $\bar{u} \in \mathcal{U}_N$ is a limit point in $\mathbb{R}^{mN}$ for the sequence $\{u^{(l)}\}$ constructed according to the rule (4.6) from an arbitrary initial point $u^{(0)} \in \mathcal{U}_N$, then

$$J_N(\bar{u}) = \min_{u \in \mathcal{U}_N} J_N(u).$$

For the proof we need some preliminary constructions.

Consider the function $\varphi : \mathbb{R}^{nN} \rightarrow \mathbb{R}$ of the form

$$\varphi(y) = \varphi(y_1, \dots, y_N) := \left\| \sum_{i=1}^N y_i \right\|_Q^2. \quad (5.7)$$

We will use the following properties of the function φ .

Lemma 5.1:

The function φ is continuously differentiable on $\mathbb{R}^{nN}$, and each of the functions

$$\mu \mapsto \varphi(y_1, \dots, y_{i-1}, \mu, y_{i+1}, \dots, y_N) = \left\| \mu + \sum_{j \neq i} y_j \right\|_Q^2$$

is strictly convex on $\mathbb{R}^n$ for any fixed values $y_j \in \mathbb{R}^n$, $j \neq i$, where $i = 1, \dots, N$.

Proof

By definition of the derivative in Euclidean space $\mathbb{R}^{nN}$, we have

$$\varphi'(y) = \left(\frac{\partial \varphi(y)}{\partial y_1}, \dots, \frac{\partial \varphi(y)}{\partial y_N} \right) \in \mathbb{R}^{nN}, \quad y \in \mathbb{R}^{nN},$$

where

$$\frac{\partial \varphi(y)}{\partial y_j} = \frac{\partial}{\partial y_j} \left\| \sum_{i=1}^N y_i \right\|_Q^2 = \frac{\partial}{\partial y_j} \left\langle \sum_{i=1}^N y_i, Q \sum_{i=1}^N y_i \right\rangle = 2Q \sum_{i=1}^N y_i.$$

This mapping $y \mapsto \varphi'(y) : \mathbb{R}^{nN} \rightarrow \mathbb{R}^{nN}$ is continuous everywhere. Now, let us fix some $i \in \{1, \dots, N\}$ and $y_j \in \mathbb{R}^n, j \neq i$, and consider the function

$$\mu \mapsto \phi^i(\mu) := \left\| \mu + \sum_{j \neq i} y_j \right\|_Q^2 : \mathbb{R}^n \rightarrow \mathbb{R}.$$

For any $\mu, \mu' \in \mathbb{R}^n, \mu \neq \mu'$, and $\lambda \in (0, 1)$ we have $\lambda(\lambda - 1)\|\mu - \mu'\|_Q^2 < 0$. It follows that

$$\begin{aligned} \phi^i(\lambda\mu + (1-\lambda)\mu') &= \left\| \lambda\mu + (1-\lambda)\mu' + \sum_{j \neq i} y_j \right\|_Q^2 = \\ &= \left\| \lambda \left(\mu + \sum_{j \neq i} y_j \right) + (1-\lambda) \left(\mu' + \sum_{j \neq i} y_j \right) \right\|_Q^2 < \\ &< \lambda \left\| \mu + \sum_{j \neq i} y_j \right\|_Q^2 + (1-\lambda) \left\| \mu' + \sum_{j \neq i} y_j \right\|_Q^2 = \lambda\phi^i(\mu) + (1-\lambda)\phi^i(\mu'). \end{aligned}$$

Therefore, the function ϕ^i is strictly convex on $\mathbb{R}^n$. □

Let $K_1, \dots, K_N$ be nonempty convex and compact sets in $\mathbb{R}^n$, and $K = K_1 \times \dots \times K_N$. We apply the following result ensured by [3, Proposition 2.7.1] and [3, Proposition 2.1.2].

Proposition 5.1:

Let a continuous function $f : \mathbb{R}^{nN} \rightarrow \mathbb{R}$ be convex on the set $K = K_1 \times \dots \times K_N$. Assume that the following conditions hold:

- (i) function f is continuously differentiable on the set K ;
- (ii) function $\mu \mapsto f(y_1, \dots, y_{i-1}, \mu, y_{i+1}, \dots, y_N)$ has a unique minimum on the set K_i for each i and arbitrary values $y_j \in K_j, j \neq i$.

Then the following is true. If the point $\bar{y}$ is a limit point for the sequence $\{y^{(l)}\}$ constructed according to the rule

$$y_i^{(l+1)} \in \text{Arg} \min_{\mu \in K_i} f(y_1^{(l+1)}, \dots, y_{i-1}^{(l+1)}, \mu, y_{i+1}^{(l)}, \dots, y_N^{(l)}), \quad i = 1, \dots, N, \quad (5.8)$$

from an arbitrary initial point $y^{(0)} \in K$, then

$$f(\bar{y}) = \min_{y \in K} f(y).$$

Now we prove Theorem 5.1.

Proof of Theorem 5.1

In Proposition 5.1 put $f = \varphi$, where φ is the function given by (5.7), and $K_i = N^{-1}\mathcal{A}_N(0)x_0 + \mathcal{A}_N(i)B(i-1)U(i-1)$, $i = 1, \dots, N$. By Lemma 5.1, the conditions (i) and (ii) in Proposition 5.1 are satisfied. Consider the mapping $\gamma : \mathcal{U}_N \rightarrow K$ of the form

$$\gamma(u)_i = N^{-1}\mathcal{A}_N(0)x_0 + \mathcal{A}_N(i)B(i-1)u(i-1), \quad i = 1, \dots, N. \quad (5.9)$$

Then $f(\gamma(u)) = \varphi(\gamma(u)) = J_N(u)^2$ for all $u \in \mathcal{U}$. For the given sequence $\{u^{(l)}\}$ constructed according to the rule (4.6) from a point $u^{(0)}$, and for the limit point $\bar{u}$, we have that the sequence $\{y^{(l)}\} = \{\gamma(u^{(l)})\}$ satisfy the rule (5.8) and $\bar{y} = \gamma(\bar{u})$ is a limit point of $\{y^{(l)}\}$. Therefore, by Proposition 5.1, $J_N(\bar{u})^2 = f(\bar{y}) = \min_{y \in K} f(y) = \min_{u \in \mathcal{U}} J_N(u)^2$. $\square$

6. SOLVING THE PROBLEMS

By Theorem 5.1, any sequence $\{u^{(l)}\}$ constructed according to the rule (4.6) from any initial point $u^{(0)} \in \mathcal{U}_N$ is a minimizing sequence in auxiliary problem (3.5). Thus, finding minimal N with $x^*(N) \in T$ is equivalent to finding minimal N such that the optimal value in problem (3.5) does not exceed 1. Fortunately, we have a way to estimate this optimal value having constructed only a part of our minimizing sequence, thanks to the following results.

Lemma 6.1:

Let $\hat{u} \in \mathcal{U}_N$ and $\bar{J}_N$ denote optimal value of the objective in problem (3.5). Then

$$\|\hat{x}(N)\|_Q^2 - 2 \sum_{j=0}^{N-1} \max_{\mu \in U(j)} \langle \hat{x}(N), Q\mathcal{A}_N(j+1)B(j)(\hat{u}(j) - \mu) \rangle \leq \bar{J}_N^2, \quad (6.10)$$

where $\hat{x}(N)$ is defined by (2.1) – (2.2) with $u = \hat{u}$. The equality in (6.10) holds iff $\hat{u} \in \mathcal{U}_N$ is a solution to problem (3.5).

Proof

Take the function φ of the form (5.7). In proof of Lemma 5.1 we showed that

$$\langle \varphi'(y), h \rangle = \sum_{j=1}^N \left\langle 2Q \sum_{i=1}^N y_i, h_j \right\rangle = 2 \sum_{j=1}^N \left\langle \sum_{i=1}^N y_i, Qh_j \right\rangle \quad \forall y, h \in \mathbb{R}^{nN}.$$

For the convex function φ holds the inequality

$$\varphi(y) \geq \varphi(\hat{y}) + \langle \varphi'(\hat{y}), y - \hat{y} \rangle \quad \forall y, \hat{y} \in \mathbb{R}^{nN}.$$

In particular, for any nonempty compact set K in $\mathbb{R}^{nN}$ and $\hat{y} \in K$ we have

$$\min_{y \in K} \varphi(y) \geq \varphi(\hat{y}) + \min_{y \in K} \langle \varphi'(\hat{y}), y - \hat{y} \rangle = \varphi(\hat{y}) - \max_{y \in K} \langle \varphi'(\hat{y}), \hat{y} - y \rangle. \quad (6.11)$$

Moreover, by [3, Proposition 2.1.2], for the convex set K the last term in (6.11) equals to zero exactly when $\hat{y}$ is a solution to the problem $\varphi(y) \rightarrow \min_{y \in K}$.

Put $K_i = N^{-1}\mathcal{A}_N(0)x_0 + \mathcal{A}_N(i)B(i-1)U(i-1)$ and $K = K_1 \times \dots \times K_N$. Now take $\hat{y} = \gamma(\hat{u})$, where γ is the mapping given by (5.9). Then

$$\min_{y \in K} \varphi(y) = \bar{J}_N^2, \quad \varphi(\hat{y}) = \|\hat{x}(N)\|_Q^2,$$

$$\max_{y \in K} \langle \varphi'(\hat{y}), \hat{y} - y \rangle = 2 \sum_{j=0}^{N-1} \max_{\mu \in U(j)} \langle \hat{x}(N), Q\mathcal{A}_N(j+1)B(j)(\hat{u}(j) - \mu) \rangle.$$

Using these formulas in (6.11), we get (6.10) and the last assertion in the lemma. $\square$

If we combine the results of Theorem 5.1 and Lemma 6.1, then we derive the following assertion.

Corollary 6.1:

The optimal value $\bar{J}_N$ in problem (3.5) satisfies the condition $\bar{J}_N > 1$ iff for any sequence $\{u^{(l)}\}$ constructed according to the rule (4.6) from any initial point $u^{(0)} \in \mathcal{U}_N$ there is a finite number $l^* \geq 0$ such that

$$\|x^{(l^*)}(N)\|_Q^2 - 2 \sum_{j=0}^{N-1} \max_{\mu \in U(j)} \langle x^{(l^*)}(N), Q\mathcal{A}_N(j+1)B(j)(u^{(l^*)}(j) - \mu) \rangle > 1, \quad (6.12)$$

where $x^{(l^*)}(N)$ is defined by (2.1) – (2.2) with $u = u^{(l^*)}$.

Proof

If inequality (6.12) holds for some $\{u^{(l)}\}$ and $l^* \geq 0$, then, by Lemma 6.1, we have $\bar{J}_N > 1$ as well. Conversely, by Theorem 5.1, the distance between $u^{(l)}$ and $\mathcal{U}_N^*$ tends to zero as $l \rightarrow \infty$. Indeed, if $\text{dist}(u^{(l)}, \mathcal{U}_N^*) \not\rightarrow 0$, then we can find a subsequence $\{u^{(l_m)}\}$ with the property $\text{dist}(u^{(l_m)}, \mathcal{U}_N^*) > \delta > 0$ for some $\delta > 0$. Passing to a subsequence again if needed, we have $\{u^{(l_m)}\}$ converges to some $\bar{u} \in \mathcal{U}_N$. Then, by Theorem 5.1, we obtain $\bar{u} \in \mathcal{U}_N^*$. But since $\text{dist}(\cdot, \mathcal{U}_N^*)$ is continuous on $\mathcal{U}_N$, we have $\text{dist}(\bar{u}, \mathcal{U}_N^*) \geq \delta > 0$ – a contradiction.

Now, the function in the left hand side of (6.10) is continuous in $\hat{u}$, and it is equal to $\bar{J}_N > 1$ for each $\hat{u} \in \mathcal{U}_N^*$ by Lemma 6.1. Therefore, the condition (6.12) holds for the given sequence $\{u^{(l)}\}$ and some $l^* \geq 0$. $\square$

It may happen, that the condition (6.12) holds for $l^* = 0$, means that no elements of the sequence $\{u^{(l)}\}$ should be constructed by the rule (4.6) to ensure the inequality $\bar{J}_N > 1$ and the minimal value $N_{\min}$ in the problem of getting into the area T to be greater than N . In the general case, inequality (6.12) can be used as a natural internal stopping rule in an algorithm of finding such $N_{\min}$ via checking different values of N in the context of problem (3.5).

Next, we propose the simplest version of the algorithm utilizing the results of the paper.

Algorithm 6.1: 0. Put $N = 1$.

1. Put $l = 0$, define/supplement the initial approximation with the equality $u^{(0)}(N-1) = 0 \in \mathbb{R}^n$ and calculate the matrices $\mathcal{A}_N(k) = A(N-1) \dots A(k)$, $k = 0, \dots, N-1$, put the matrix $\mathcal{A}_N(N)$ to be the identity matrix of size $n \times n$.

2. Calculate the values $\xi^{(l)}(k)$, $k = 1, \dots, N$, by the formulas

$$\xi^{(l)}(k) = \xi^{(l)}(k+1) - \mathcal{A}_N(k+1)B(k)u^{(l)}(k), \quad \xi^{(l)}(N) = 0.$$

3. For each $k = 0, \dots, N-1$, sequentially find and fix some solution $u^{(l+1)}(k)$ to the extremum problem

$$\|\mathcal{A}_N(k+1)B(k)\mu + \mathcal{A}_N(k)x^{(l+1)}(k) - \xi^{(l)}(k+1)\|_Q \rightarrow \min_{\mu \in U(k)},$$

where the values $x^{(l+1)}(k)$ are calculated step by step using the formulas

$$x^{(l+1)}(k+1) = A(k)x^{(l+1)}(k) + B(k)u^{(l+1)}(k), \quad x^{(l+1)}(0) = x_0.$$

4. Check the external stopping condition

$$\|x^{(l+1)}(N)\|_Q^2 \leq 1,$$

if true, terminate the calculations with the answer $N_{\min} = N$, $u^* = u^{(l+1)}$.

5. Calculate the estimate $E_N^{(l+1)}$ by the formula

$$E_N^{(l+1)} = \|x^{(l+1)}(N)\|_Q^2 - 2 \sum_{j=0}^{N-1} \max_{\mu \in U(j)} \langle x^{(l+1)}(N), Q\mathcal{A}_N(j+1)B(j)(u^{(l+1)}(j) - \mu) \rangle.$$

6. Check the internal stopping condition

$$E_N^{(l+1)} > 1,$$

if true, put the found $u^{(l+1)}$ as the new initial approximation, i.e. put $u^{(0)}(k) = u^{(l+1)}(k)$, $k = 0, \dots, N-1$, increment N by one and go to step 1; otherwise increment l by one and go to step 2.

7. EXAMPLES

Here we provide a number of examples to demonstrate Algorithm 6.1 at work.

Example 7.1:

Let $n = m = 2$, and the matrices in system (2.1) be given as

$$A(k) = \begin{pmatrix} e^{-1/(k+1)} \cos(1/(k+1)) & -e^{-1/(k+1)} \sin(1/(k+1)) \\ e^{-1/(k+1)} \sin(1/(k+1)) & e^{-1/(k+1)} \cos(1/(k+1)) \end{pmatrix}, \quad B(k) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let the sets $U(k)$ have form

$$U(k) = \{v \in \mathbb{R}^2 \mid 2v_1^2 + 2v_2^2 \leq 1 + e^{-2/(k+1)} - 2e^{-1/(k+1)} \cos(1/(k+1))\}.$$

Consider the initial condition

$$x_0 = (100, 100),$$

and the target area T be determined via (2.3) by the matrix

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 100 \end{pmatrix}.$$

Applying Algorithm 6.1, we obtain the following results, presented in Fig. 7.1 and 7.2.

Example 7.2:

In Example 7.1, change the target area T to be defined by the matrix

$$Q = \begin{pmatrix} 100 & 0 \\ 0 & 10 \end{pmatrix}.$$

Then we have the results in Fig. 7.3 and 7.4.

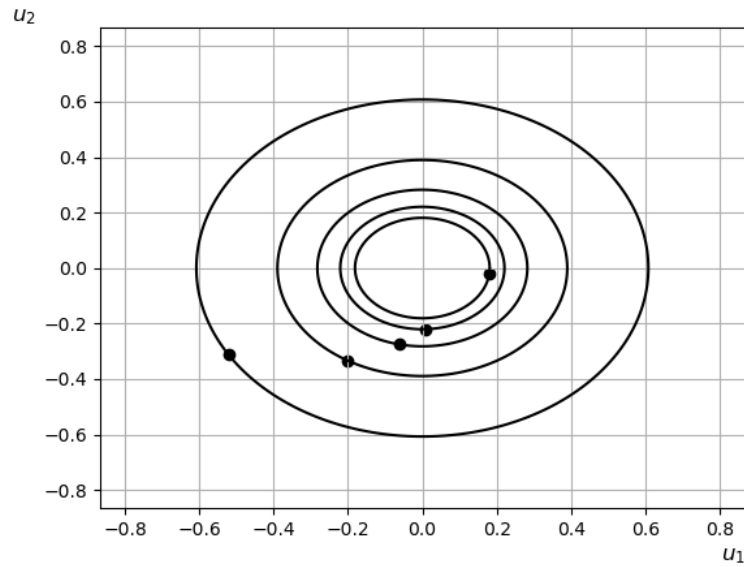


Fig. 7.1. Optimal control in Example 7.1

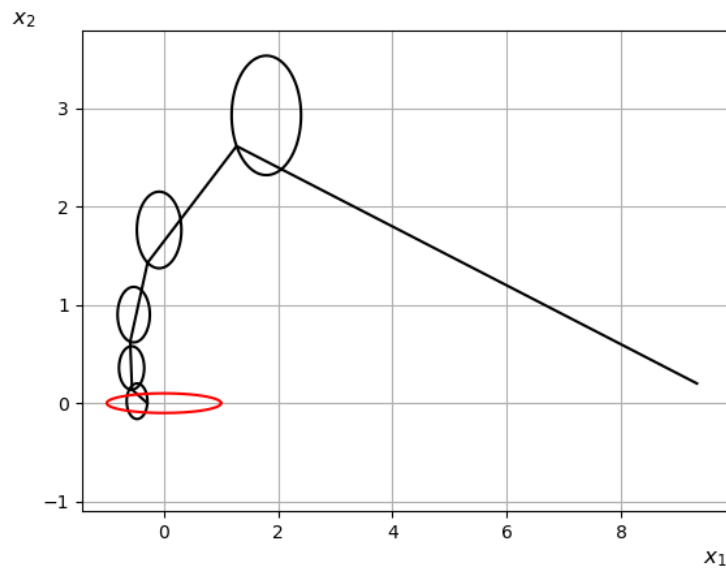


Fig. 7.2. Optimal trajectory in Example 7.1

Example 7.3:

In Example 7.1, change the target area T to be defined by the matrix

$$Q = \begin{pmatrix} 50.6 & -49.5 \\ -49.5 & 50.5 \end{pmatrix}.$$

The results are presented in Fig. 7.5 and 7.6.

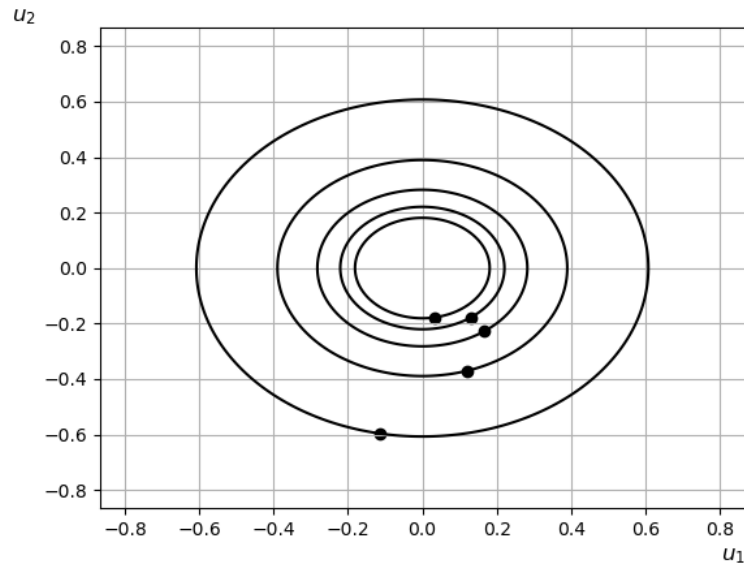


Fig. 7.3. Optimal control in Example 7.2

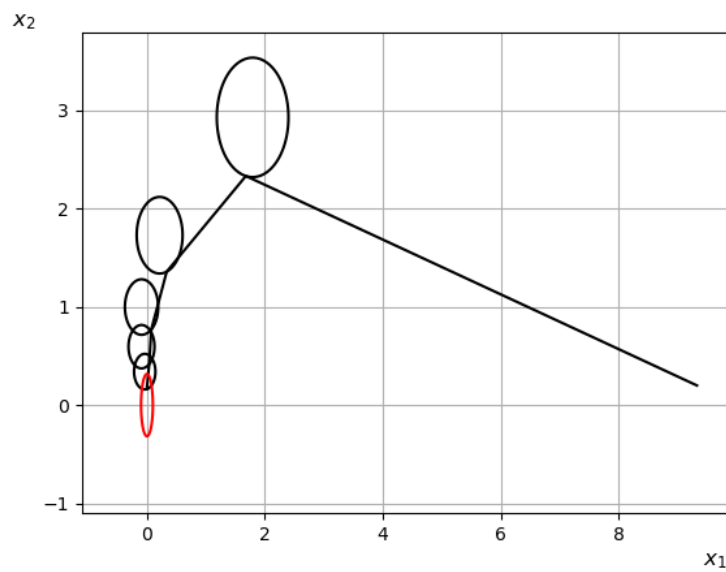


Fig. 7.4. Optimal trajectory in Example 7.2

Example 7.4:

Let $n = m = 2$, and in the system (2.1) – (2.2) be given the following:

$$A(k) = (-1)^k \begin{pmatrix} \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{pmatrix}; \quad B(k) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

$$U(k) = \left\{ v \in \mathbb{R}^2 \mid \max\{|v_1|, |v_2|\} \leq \frac{\sqrt{3}}{2}, \|v\| \leq (0.9)^k \right\};$$

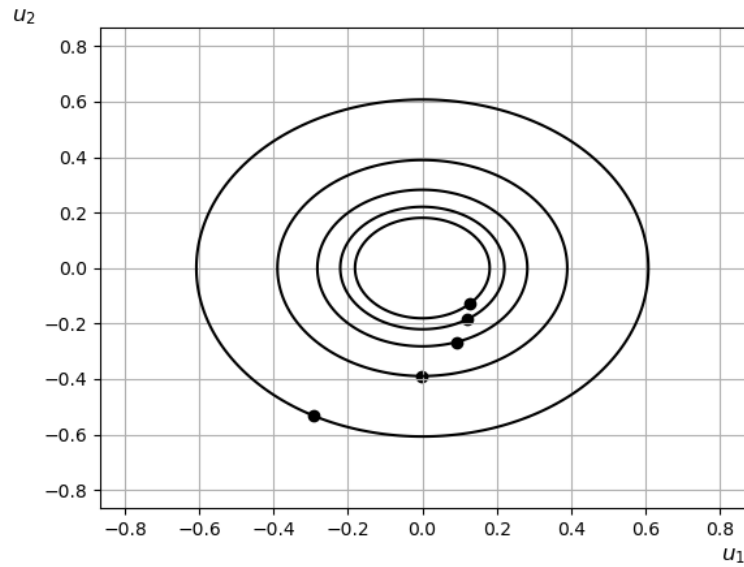


Fig. 7.5. Optimal control in Example 7.3

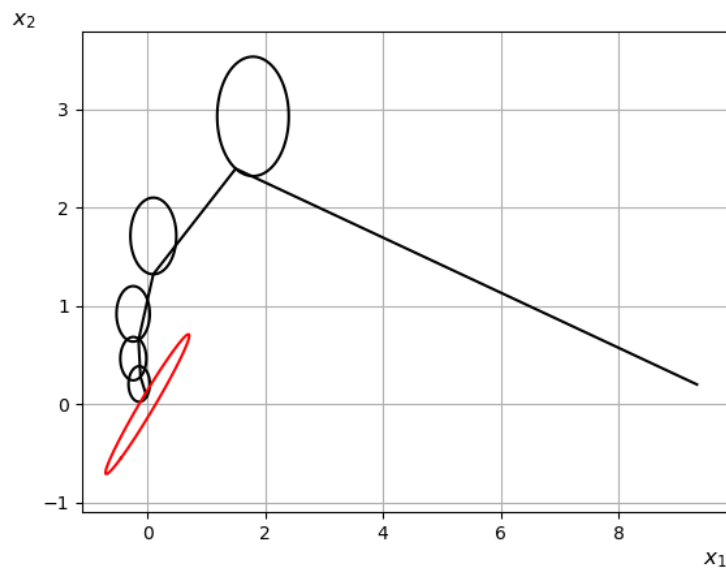


Fig. 7.6. Optimal trajectory in Example 7.3

$$x_0 = (9.33, 0.2).$$

Let the target area T be determined by the matrix

$$Q = \begin{pmatrix} 0.02 & 0 \\ 0 & 0.02 \end{pmatrix}.$$

Applying the proposed algorithm, we obtain the results, presented in Fig. 7.7 and 7.8.

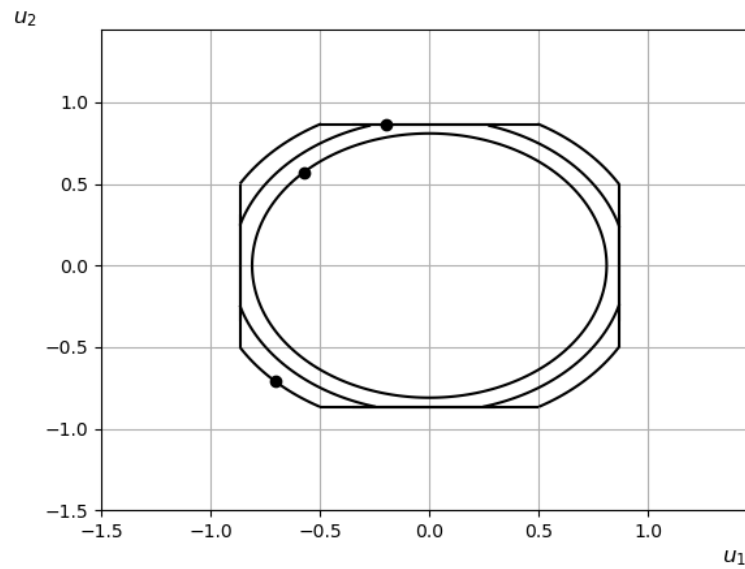


Fig. 7.7. Optimal control in Example 7.4

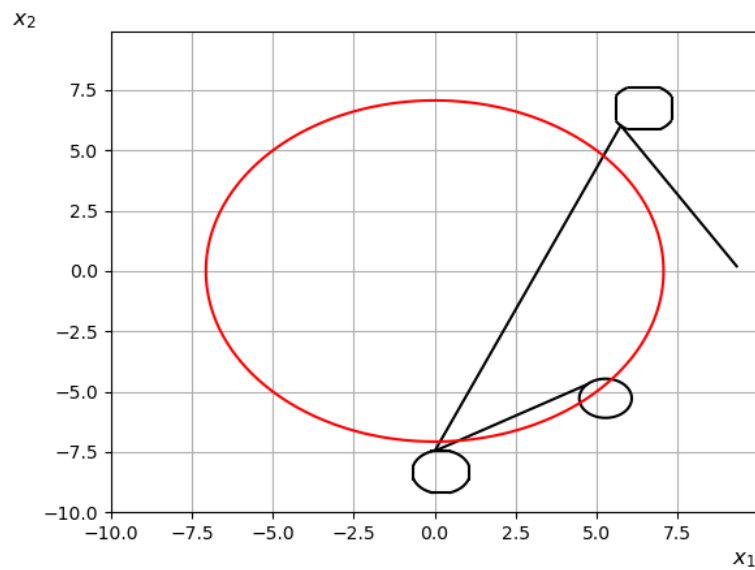


Fig. 7.8. Optimal trajectory in Example 7.4

Example 7.5:

Finally, if we take

$$Q = \begin{pmatrix} 0.04 & 0 \\ 0 & 0.04 \end{pmatrix}$$

in Example 7.4, then we get the results in Fig. 7.9 and 7.10.

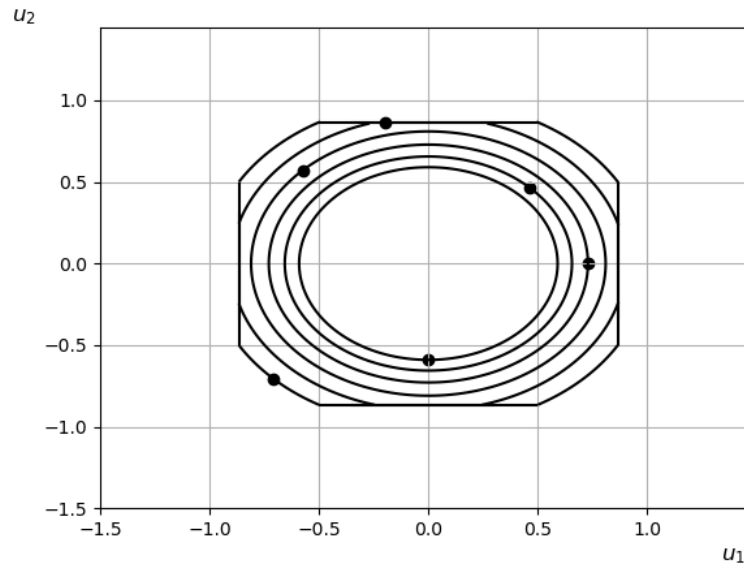


Fig. 7.9. Optimal control in Example 7.5

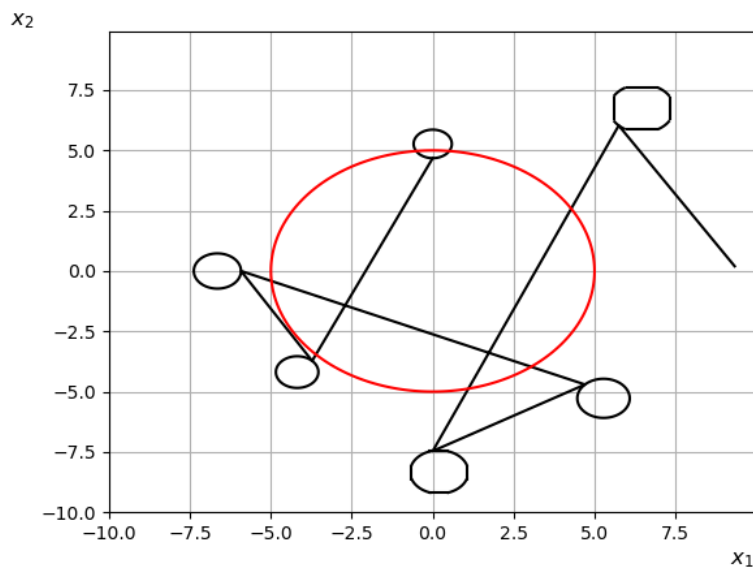


Fig. 7.10. Optimal trajectory in Example 7.5

REFERENCES

1. Tsarkov, K. (2026) Block Coordinate Descent Method Applied to Stabilization and Time-Optimization Problems for Linear Discrete Systems, *Autom. Remote Control*, **9**.
2. Warga, J. (1963) Minimizing Certain Convex Functions, *J. Soc. Indust. Appl. Math.*, **11**(3), 588–593.
3. Bertsekas, D.P. (1999) *Nonlinear Programming*. Nashua, NH: Athena Scientific.