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Absolute Stability Conditions for Switched Positive Coupled Systems with Delays

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Abstract: The paper addresses the problem of stability analysis of a switched positive coupled system that describes the interaction of a nonlinear differential subsystem with nonlinearities of a sector type and a linear functional difference one. The case is studied where the considered system contains both point-wise and distributed delays. A special construction of a diagonal Lyapunov–Krasovskii functional is suggested that allows for deriving new absolute stability conditions, i.e., conditions under which the zero solution of the system is asymptotically stable for any admissible nonlinearities and any admissible switching signal. These conditions are formulated in terms of solvability of a system of linear algebraic inequalities. A constructive approach for finding the Lyapunov–Krasovskii functional parameters via the solution of the above inequalities is proposed. In addition, a discrete-time counterpart of a switched positive coupled system is considered. Constraints on the digitization step and the system parameters are obtained under which the discrete-time system is absolutely stable, as well. An illustrative example is given to demonstrate the application of the developed approaches.

Keywords: coupled system, switching, delay, positive system, absolute stability, discretization

1. INTRODUCTION

One of the urgent and challenging problems in modern control theory is the stability investigation of positive systems [10, 15, 18]. Positivity of a dynamical system means that its trajectories starting from non-negative initial conditions remain non-negative as time increases. Interest in such systems is driven by their wide range of applications (see [4, 7, 10, 15, 18, 21] and the references cited therein).

Most known stability conditions have been established for linear time-invariant positive systems [10, 12, 15, 18]. However, when solving numerous applied problems, the mathematical models employed should account for factors such as nonlinearity, time-varying behavior, uncertainties in system parameters, as well as time-delay effects.

Among non-stationary systems, switched systems are of particular interest. These systems consist of families of subsystems and rules governing the switching between them (switching laws) [20]. It is well known [20] that the properties of individual subsystems cannot be directly extended to the corresponding switched system. Moreover, in practice, the operating sequence of subsystems is often unknown; therefore, it is necessary to ensure stability of the resulting modes under any admissible switching signal.

The main approach to addressing this problem involves finding a common Lyapunov function or a Lyapunov–Krasovskii functional for the studied family of subsystems. With the aid of this approach, a set of results on the stability of both linear and nonlinear positive switched systems has been obtained (see, e.g., [2, 5, 6, 20, 25] and the bibliography therein). The present paper continues research in this direction.

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We consider a nonlinear switched positive coupled system with point-wise and distributed delay that describes the interaction between a Persidskii-type differential system [18] and a linear functional difference system. Coupled systems are widely used for modeling processes in chemical engineering and hydrodynamics, lossless transmission lines in electrical engineering, epidemiological processes, as well as in various other applications [1, 24, 27]. Furthermore, neutral-type delay differential equations can be reduced to this form [13].

Some methods for the stability analysis of linear coupled systems were proposed in [12, 16, 19, 23, 26, 28]. For nonlinear systems, results based on the Lyapunov direct method have been obtained in [14]. Stability conditions were formulated in terms of the existence of Lyapunov–Krasovskii functionals possessing specific properties. However, it should be noted that general constructive approaches for finding such functionals have not yet been developed.

Stability of positive coupled systems has been investigated in [3, 5, 8, 22, 23, 28]. With the aid of the comparison method, conditions for exponential stability of nonlinear systems and linear systems with time-varying delays were established [22, 23]. A special approach based on input–output mathematical models was proposed in [8] to derive robust stability conditions for linear systems. Linear and diagonal Lyapunov–Krasovskii functionals were constructed for a class of nonlinear positive switched coupled system with constant delay in [5] and [3], respectively; the existence of such functionals guaranteed the asymptotic stability of the considered system under any admissible switching law.

At the same time, it is worth mentioning that in wide range of practical problems models in the form of coupled systems with distributed delays are used (see, e.g., [1, 11, 12, 19]). The aim of the present work is to extend the results of [3] to the case of coupled systems containing both point-wise and distributed delays.

Furthermore, a discrete-time counterpart of a switched positive coupled system will be studied. In [9], it was noted that the stability problem for discrete-time coupled systems is insufficiently investigated. In particular, an exponential stability condition for linear time-invariant discrete-time coupled systems with delays was derived in [9], whereas a sufficient condition for ensuring the stability of the discrete-time coupled homogeneous delay systems was presented in [17]. However, in the above papers, non-switched systems were considered. In this contribution, with the aid of a special construction of Lyapunov–Krasovskii functional, a new result on the asymptotic stability under arbitrary switching will be obtained for the investigated discrete-time coupled system.

2. STATEMENT OF THE PROBLEM

We introduce some notations that will be used in this paper.

Let $\mathbb{R}$ be the field of real numbers, whereas $\mathbb{R}^l$ and $\mathbb{R}^{l\times p}$ denote the l-dimensional Euclidean space and the vector space of $l\times p$ matrices, respectively. Inequalities between vectors are understood component-wise. For a set of vectors $z_1,\dots,z_N\in\mathbb{R}^l$, $\max_{s=1,\dots,N} z_s$ is also understood component-wise. A matrix $A\in\mathbb{R}^{l\times p}$ is nonnegative if all its entries are greater than or equal to zero. A matrix is called Metzler if all its off-diagonal entries are nonnegative. A matrix $B\in\mathbb{R}^{l\times l}$ whose eigenvalues are all less than one in absolute value is called a Schur matrix. A diagonal matrix $\Omega\in\mathbb{R}^{l\times l}$ with the elements $\omega_1,\dots,\omega_l$ on the main diagonal will be denoted by $\mathrm{diag}\{\omega_1,\dots,\omega_l\}$. Let $1_l\in\mathbb{R}^l$ be the vector whose components are all equal to 1 and I_l be the identity matrix in $\mathbb{R}^{l\times l}$. If $A_s=\left\{a_{ij}^{(s)}\right\}_{i,j=1}^{l}$, $s=1,\dots,N$, then $\widehat{A}=\{\hat{a}_{ij}\}_{i,j=1}^{l}$, where $\hat{a}_{ij}=\max_{s=1,\dots,N} a_{ij}^{(s)}$, $i,j=1,\dots,l$. For a given number $d>0$, $PC([-d,0),\mathbb{R}^m)$ denotes the space of bounded right piece-wise continuous functions $\theta(\xi):[-d,0)\mapsto\mathbb{R}^m$ with the norm $\|\theta\|_d=\sup_{\xi\in[-d,0)}\|\theta(\xi)\|$, where $\|\cdot\|$ is the Euclidean norm of a vector.

Let the switched coupled differential-difference system

$$\begin{cases} \dot{x}(t) = A_\sigma \Psi(x(t)) + B_\sigma y(t-\tau) + C_\sigma \int_{t-\beta}^{t} y(\xi)d\xi, \\ y(t) = F_\sigma \Psi(x(t)) + G_\sigma y(t-\tau) + Q_\sigma \int_{t-\beta}^{t} y(\xi)d\xi \end{cases} \tag{1}$$

be given. We assume that $t \geq 0$, $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^m$, the switching signal $\sigma = \sigma(t)$ is a piece-wise constant function, $\sigma : [0, +\infty) \mapsto \{1, \dots, N\}$, β and τ are constant positive delays, $A_s \in \mathbb{R}^{n\times n}$, $B_s, C_s \in \mathbb{R}^{n\times m}$, $F_s \in \mathbb{R}^{m\times n}$, $G_s, Q_s \in \mathbb{R}^{m\times m}$ are constant matrices, $s = 1, \dots, N$, $\Psi(x)$ is continuous for $x \in \mathbb{R}^n$ vector function of the separable type, i.e., $\Psi(x) = (\psi_1(x_1), \dots, \psi_n(x_n))^\top$ and the scalar functions $\psi_i(x_i)$ satisfy the sector conditions $x_i\psi_i(x_i) > 0$ for $x_i \neq 0$, $i = 1, \dots, n$. Let the function $\sigma(t)$ admit only finitely many discontinuities on every bounded time interval.

Thus, (1) is a complex system describing the interaction of the Persidskii type system (see [18])

$$\dot{x}(t) = A_\sigma \Psi(x(t))$$

and the functional difference system

$$y(t) = G_\sigma y(t-\tau) + Q_\sigma \int_{t-\beta}^{t} y(\xi)d\xi.$$

Solutions $(x^\top(t), y^\top(t))^\top = (x^\top(t, t_0, x_0, \theta), y^\top(t, t_0, x_0, \theta))^\top$ of (1) are defined via initial conditions $t_0 \geq 0$, $x(t_0) = x_0 \in \mathbb{R}^n$, $y(t_0 + \xi) = \theta(\xi) \in PC([-d, 0), \mathbb{R}^m)$, where $d = \max\{\tau; \beta\}$. Denote by y_t the restriction of the corresponding component of a solution to the segment $[t-d, t)$, i.e., $y_t : \xi \mapsto y(t+\xi)$ for $\xi \in [-d, 0)$.

In this contribution, we consider the case where the system (1) is positive. From the results of [23] it follows that this is fulfilled if and only if the matrices A_s are Metzler and the matrices B_s, C_s, F_s, G_s, Q_s are nonnegative, $s = 1, \dots, N$.

The system (1) admits the zero solution. Our goal is to determine the conditions of stability for this solution.

Definition 2.1:
The system (1) is absolutely stable if its zero solution is asymptotically stable for any admissible function $\Psi(x)$ and any admissible switching signal.

Our analysis will be based on the Lyapunov direct method. A special construction of the Lyapunov–Krasovskii functional candidate will be proposed, and constraints on the system parameters will be derived under which this functional solves the absolute stability problem for (1).

In addition, we will study the stability of the zero solution of a discrete-time counterpart of (1).

3. ABSOLUTE STABILITY CONDITIONS

We impose the following restrictions on the investigated system.

Assumption 3.1:
Let $\widehat{G} + \beta\widehat{Q}$ be a Schur matrix.

It is known (see [18]) that Assumption 3.1 implies that the matrix $\widehat{G} + \beta\widehat{Q} - I_m$ is Metzler and Hurwitz, whereas $L = (I_m - \widehat{G} - \beta\widehat{Q})^{-1}$ is nonnegative matrix.

Assumption 3.2:
There exist a positive vector $\zeta \in \mathbb{R}^n$ and a number α_1 such that

$$(A_s + B_s L\widehat{F} + \beta C_s L\widehat{F})\zeta \le \alpha_1 \zeta, \quad s = 1, \ldots, N.$$

Assumption 3.3:
There exist a positive vector $\eta \in \mathbb{R}^n$ and a number α_2 such that

$$(A_s + B_r L\widehat{F} + \beta C_p L\widehat{F})^\top \eta \le \alpha_2 \eta, \quad s, r, p = 1, \ldots, N.$$

Theorem 3.1:
Let Assumptions 3.1–3.3 be satisfied and $\alpha_1 + \alpha_2 < 0$. Then the system (1) is absolutely stable for any $\tau > 0$.

Proof
Construct a diagonal Lyapunov–Krasovskii functional candidate for (1) in the form

$$V(x(t), y_t) = 2\sum_{i=1}^{n} \omega_i \int_0^{x_i(t)} \psi_i(u)du + \int_{t-\tau}^{t} y^\top(\xi)\Lambda y(\xi)d\xi$$
$$+\frac{1}{\beta}\int_{t-\beta}^{t} (\xi - t + \beta) y^\top(\xi)\Gamma y(\xi)d\xi. \tag{2}$$

Here $\Lambda = \operatorname{diag}\{\lambda_1, \ldots, \lambda_m\}$, $\Gamma = \operatorname{diag}\{\gamma_1, \ldots, \gamma_m\}$ and $\omega_i, \lambda_l, \gamma_l$ are positive tuning parameters, $i = 1, \ldots, n$, $l = 1, \ldots, m$. Consider the derivative of the functional (2) along the solutions of the system (1). One obtains

$$\dot{V} = 2\Psi^\top(x(t))\Omega\left(A_\sigma \Psi(x(t)) + B_\sigma y(t-\tau) + C_\sigma \int_{t-\beta}^{t} y(\xi)d\xi\right)$$
$$+\left(F_\sigma \Psi(x(t)) + G_\sigma y(t-\tau) + Q_\sigma \int_{t-\beta}^{t} y(\xi)d\xi\right)^\top \Xi \left(F_\sigma \Psi(x(t)) + G_\sigma y(t-\tau)\right.$$
$$\left.+Q_\sigma \int_{t-\beta}^{t} y(\xi)d\xi\right) - y^\top(t-\tau)\Lambda y(t-\tau) - \frac{1}{\beta}\int_{t-\beta}^{t} y^\top(\xi)\Gamma y(\xi)d\xi,$$

where $\Omega = \operatorname{diag}\{\omega_1, \ldots, \omega_n\}$, $\Xi = \Lambda + \Gamma$.

Using Jensen's integral inequality (see [13]), we arrive at the estimate

$$\dot{V} \le \frac{1}{\beta}\int_{t-\beta}^{t} \begin{pmatrix} \Psi(x(t)) \\ y(t-\tau) \\ y(\xi) \end{pmatrix}^\top W_\sigma \begin{pmatrix} \Psi(x(t)) \\ y(t-\tau) \\ y(\xi) \end{pmatrix} d\xi,$$

where

$$W_\sigma = \begin{pmatrix} \Omega A_\sigma + A_\sigma^\top \Omega + F_\sigma^\top \Xi F_\sigma & \Omega B_\sigma + F_\sigma^\top \Xi G_\sigma & \beta(\Omega C_\sigma + F_\sigma^\top \Xi Q_\sigma) \\ B_\sigma^\top \Omega + G_\sigma^\top \Xi F_\sigma & G_\sigma^\top \Xi G_\sigma - \Lambda & \beta G_\sigma^\top \Xi Q_\sigma \\ \beta(C_\sigma^\top \Omega + Q_\sigma^\top \Xi F_\sigma) & \beta Q_\sigma^\top \Xi G_\sigma & \beta^2 Q_\sigma^\top \Xi Q_\sigma - \Gamma \end{pmatrix}.$$

We will choose the values of the parameters $\omega_i, \lambda_l, \gamma_l$ so that the matrices $W_1, \ldots, W_N$ are negative definite. It is worth noting that W_s are symmetric and Metzler matrices. Therefore (see [18]), to prove their negative definiteness, it is sufficient to find a positive vector

$\mu \in \mathbb{R}^{n+2m}$ such that

$$W_s\mu < 0, \quad s = 1, \dots, N. \tag{3}$$

Let η_i and ζ_i be components of the vectors η and ζ, respectively, $\omega_i = \eta_i/\zeta_i$, $i = 1, \dots, n$,

$$\mu = \begin{pmatrix} \zeta \\ L(\widehat{F}\zeta + \delta 1_m) \\ L(\widehat{F}\zeta + \delta 1_m) \end{pmatrix}$$

and $\delta = \mathrm{const} > 0$. Then $\mu > 0$,

$$W_s\mu = \begin{pmatrix} \Omega H_s + A_s^\top \eta + F_s^\top \Xi F_s \zeta + F_s^\top \Xi (G_s + \beta Q_s) L(\widehat{F}\zeta + \delta 1_m) \\ B_s^\top \eta + G_s^\top \Xi F_s \zeta - \Lambda L(\widehat{F}\zeta + \delta 1_m) + G_s^\top \Xi (G_s + \beta Q_s) L(\widehat{F}\zeta + \delta 1_m) \\ \beta C_s^\top \eta + \beta Q_s^\top \Xi F_s \zeta + (\beta Q_s^\top \Xi (G_s + \beta Q_s) - \Gamma) L(\widehat{F}\zeta + \delta 1_m) \end{pmatrix}$$

$$\leq \begin{pmatrix} \Omega H_s + A_s^\top \eta + \widehat{F}^\top \Xi \big((I_m + (\widehat{G} + \beta\widehat{Q})L)\widehat{F}\zeta + \delta(\widehat{G} + \beta\widehat{Q})L 1_m\big) \\ B_s^\top \eta + \widehat{G}^\top \Xi \big((I_m + (\widehat{G} + \beta\widehat{Q})L)\widehat{F}\zeta + \delta(\widehat{G} + \beta\widehat{Q})L 1_m\big) - \Lambda L(\widehat{F}\zeta + \delta 1_m) \\ \beta C_s^\top \eta + \beta \widehat{Q}^\top \Xi \big((I_m + (\widehat{G} + \beta\widehat{Q})L)\widehat{F}\zeta + \delta(\widehat{G} + \beta\widehat{Q})L 1_m\big) - \Gamma L(\widehat{F}\zeta + \delta 1_m) \end{pmatrix}$$

$$= \begin{pmatrix} \Omega H_s + A_s^\top \eta + \widehat{F}^\top \Xi L \widehat{F}\zeta + \delta \widehat{F}^\top \Xi (G_s + \beta Q_s) L 1_m \\ B_s^\top \eta + \widehat{G}^\top \Xi \big(L\widehat{F}\zeta + \delta(\widehat{G} + \beta\widehat{Q})L 1_m\big) - \Lambda L(\widehat{F}\zeta + \delta 1_m) \\ \beta C_s^\top \eta + \beta \widehat{Q}^\top \Xi \big(L\widehat{F}\zeta + \delta(\widehat{G} + \beta\widehat{Q})L 1_m\big) - \Gamma L(\widehat{F}\zeta + \delta 1_m) \end{pmatrix},$$

where $H_s = (A_s + B_s L\widehat{F} + \beta C_s L\widehat{F})\zeta + \delta(B_s + \beta C_s)L 1_m$, $s = 1, \dots, N$.

We obtain

$$J_{3s} = \beta C_s^\top \eta + \beta \widehat{Q}^\top \Xi \big(L\widehat{F}\zeta + \delta(\widehat{G} + \beta\widehat{Q})L 1_m\big) - \Gamma L(\widehat{F}\zeta + \delta 1_m)$$

$$= \beta C_s^\top \eta + \beta \widehat{Q}^\top \Lambda \big(L\widehat{F}\zeta + \delta(\widehat{G} + \beta\widehat{Q})L 1_m\big) + (\beta\widehat{Q}^\top - I_m)\Gamma L(\widehat{F}\zeta + \delta 1_m) - \delta\beta\widehat{Q}^\top \Gamma 1_m.$$

Let

$$\Gamma L(\widehat{F}\zeta + \delta 1_m) = (I_m - \beta\widehat{Q}^\top)^{-1}\Big(\beta\widehat{Q}^\top \Lambda \big(L\widehat{F}\zeta + \delta(\widehat{G} + \beta\widehat{Q})L 1_m\big) + \beta \max_{r=1,\dots,N}\{C_r^\top \eta\} + \varepsilon 1_m\Big),$$

where $\varepsilon = \mathrm{const} > 0$. Then the diagonal entries of Γ are positive and

$$J_{3s} \leq -\varepsilon 1_m - \delta\beta\widehat{Q}^\top \Gamma 1_m < 0.$$

Next, consider

$$J_{2s} = B_s^\top \eta + \widehat{G}^\top \Xi \big(L\widehat{F}\zeta + \delta(\widehat{G} + \beta\widehat{Q})L 1_m\big) - \Lambda L(\widehat{F}\zeta + \delta 1_m).$$

We obtain

$$J_{2s} = B_s^\top \eta + (\widehat{G}^\top - I_m)\Lambda L(\widehat{F}\zeta + \delta 1_m) + \widehat{G}^\top \Gamma L(\widehat{F}\zeta + \delta 1_m) - \delta\widehat{G}^\top \Xi 1_m$$

$$= B_s^\top \eta - \delta\beta\widehat{G}^\top (I_m - \beta\widehat{Q}^\top)^{-1}\widehat{Q}^\top \Lambda 1_m - (I_m - \widehat{G} - \beta\widehat{Q})(I_m - \beta\widehat{Q}^\top)^{-1}\Lambda L(\widehat{F}\zeta + \delta 1_m)$$

$$+\widehat{G}^\top(I_m-\beta\widehat{Q}^\top)^{-1}\left(\beta\max_{r=1,\dots,N}\{C_r^\top\eta\}+\varepsilon 1_m\right)-\delta\widehat{G}^\top\Xi 1_m.$$

Let

$$(I_m-\beta\widehat{Q}^\top)^{-1}\Lambda L(\widehat{F}\zeta+\delta 1_m)=L\Bigg(\max_{r=1,\dots,N}\{B_r^\top\eta\}$$
$$+\widehat{G}^\top(I_m-\beta\widehat{Q}^\top)^{-1}\left(\beta\max_{r=1,\dots,N}\{C_r^\top\eta\}+\varepsilon 1_m\right)+\varepsilon 1_m\Bigg).$$

Then the diagonal entries of Λ are positive and

$$J_{2s}\le-\delta\beta\widehat{G}^\top(I_m-\beta\widehat{Q}^\top)^{-1}\widehat{Q}^\top\Lambda 1_m-\varepsilon 1_m-\delta\widehat{G}^\top\Xi 1_m<0.$$

Finally, consider

$$J_{1s}=\Omega H_s+A_s^\top\eta+\widehat{F}^\top\Xi L\widehat{F}\zeta+\delta\widehat{F}^\top\Xi(\widehat{G}+\beta\widehat{Q})L1_m.$$

We obtain

$$J_{1s}=\Omega H_s+A_s^\top\eta+\widehat{F}^\top\Xi L(\widehat{F}\zeta+\delta 1_m)-\delta\widehat{F}^\top\Xi 1_m$$
$$\le\Omega H_s+A_s^\top\eta+\widehat{F}^\top(I_m-\beta\widehat{Q}^\top)^{-1}\Big(\Lambda L(\widehat{F}\zeta+\delta 1_m)+\beta\max_{r=1,\dots,N}\{C_r^\top\eta\}+\varepsilon 1_m\Big)$$
$$\le\Omega H_s+A_s^\top\eta+\widehat{F}^\top(I_m-\beta\widehat{Q}^\top)^{-1}\Big(\beta\max_{r=1,\dots,N}\{C_r^\top\eta\}+\varepsilon 1_m\Big)+\widehat{F}^\top L\max_{r=1,\dots,N}\{B_r^\top\eta\}$$
$$+\beta\widehat{F}^\top L\widehat{G}^\top(I_m-\beta\widehat{Q}^\top)^{-1}\max_{r=1,\dots,N}\{C_r^\top\eta\}+\varepsilon\widehat{F}^\top L\Big(\widehat{G}^\top(I_m-\beta\widehat{Q}^\top)^{-1}+I_m\Big)1_m$$
$$=\Omega H_s+A_s^\top\eta+\widehat{F}^\top L\max_{r=1,\dots,N}\{B_r^\top\eta\}+\beta\widehat{F}^\top L\max_{r=1,\dots,N}\{C_r^\top\eta\}$$
$$+\varepsilon\widehat{F}^\top(I_m-\beta\widehat{Q}^\top)^{-1}1_m+\varepsilon\widehat{F}^\top L\Big(\widehat{G}^\top(I_m-\beta\widehat{Q}^\top)^{-1}+I_m\Big)1_m$$

From Assumptions 3.2 and 3.3 it follows that

$$J_{1s}\le(\alpha_1+\alpha_2)\eta+\delta\Omega(B_s+\beta C_s)L1_m$$
$$+\varepsilon\widehat{F}^\top(I_m-\beta\widehat{Q}^\top)^{-1}1_m+\varepsilon\widehat{F}^\top L\Big(\widehat{G}^\top(I_m-\beta\widehat{Q}^\top)^{-1}+I_m\Big)1_m$$

Hence, if δ and ε are sufficiently small, then $J_{1s}<0$.

Thus, under the suitable choice of the parameters $\omega_i,\lambda_l,\gamma_l$, $i=1,\dots,n$, $l=1,\dots,m$, δ, ε, the conditions (3) and the inequality

$$\dot V\le-\tilde a\left(\|\Psi(x(t))\|^2+\|y(t-\tau)\|^2+\int_{t-\beta}^t\|y(\xi)\|^2d\xi\right),\qquad \tilde a=\text{const}>0,$$

hold. Applying Theorem 3 from the paper [14], we obtain that the system (1) is absolutely stable for any $\tau>0$. This completes the proof. □

Remark 3.1:
The derived stability conditions depend on β and are independent of τ.

Remark 3.2:
In the case where $N=1$, the result of Theorem 3.1 is equivalent to the known necessary and sufficient absolute stability conditions for the corresponding nonswitched system (see [23]): the matrix $G_1+\beta Q_1$ is Schur and the matrix $A_1+(B_1+\beta C_1)(I_m-G_1-\beta Q_1)^{-1}F_1$ is Hurwitz.

Remark 3.3:
In a similar way, absolute stability conditions for a positive switched coupled system with multiple discrete and distributed delays can be derived.

4. STABILITY ANALYSIS OF A DISCRETE-TIME POSITIVE COUPLED SYSTEM

Next, consider the system

$$\begin{cases} x(k+1) = x(k) + h\left(A_\rho \Psi(x(k)) + B_\rho y(k-\nu)\right), \\ y(k) = F_\rho \Psi(x(k)) + G_\rho y(k-\nu). \end{cases} \tag{4}$$

Here $k = 0, 1, \ldots$, $x(k) \in \mathbb{R}^n$, $y(k) \in \mathbb{R}^m$, the vector function $\Psi(x)$ possesses the same properties as in (1), ν is a positive integer delay, $h > 0$ is a digitization step, $\rho = \rho(k)$ is a switching signal, $\rho: \{0, 1, \ldots\} \mapsto \{1, \ldots, N\}$, A_s, B_s, F_s, G_s are constant matrices of the appropriate dimensions, $s = 1, \ldots, N$. The system (4) can be obtained as a result of the discretization of a continuous coupled system with a constant delay.

Assumption 4.1:
Functions $\psi_i(x_i)$ are globally Lipschitz, i.e., there exists a constant $M > 0$ such that $|\psi_i(x_i') - \psi_i(x_i'')| \leq M|x_i' - x_i''|$ for all $x_i', x_i'' \in (-\infty, +\infty)$, $i = 1, \ldots, n$.

Assumption 4.2:
The matrices A_s are Metzler and the matrices B_s, F_s, G_s are nonnegative, $s = 1, \ldots, N$.

Remark 4.1:
Assumptions 4.1 and 4.2 imply that if $1 + ha_{ii}M \geq 0$, $i = 1, \ldots, n$, then the system (4) is positive.

Assumption 4.3:
Let $\widehat{G}$ be a Schur matrix.

Assumption 4.4:
There exist a positive vector $\zeta \in \mathbb{R}^n$ and a number α_1 such that

$$(A_s + B_s L\widehat{F})\zeta \leq \alpha_1 \zeta, \quad s = 1, \ldots, N,$$

where $L = (I_m - \widehat{G})^{-1}$.

Assumption 4.5:
There exist a positive vector $\eta \in \mathbb{R}^n$ and a number α_2 such that

$$(A_s + B_r L\widehat{F})^\top \eta \leq \alpha_2 \eta, \quad s, r = 1, \ldots, N.$$

Definition 4.1:
The system (4) is absolutely stable if its zero solution is asymptotically stable for any admissible function $\Psi(x)$ and any admissible switching signal.

Theorem 4.1:
Let Assumptions 4.1–4.5 be satisfied and $\alpha_1 + \alpha_2 < 0$. Then one can find a constant $h_0 > 0$ such that the system (4) is absolutely stable for all $h \in (0, h_0)$.

Proof
First, choose h_0 sufficiently small to ensure the positivity of (4).

Next, construct a diagonal Lyapunov–Krasovskii functional in the form

$$V(x(k), y^{(k)}) = 2\sum_{i=1}^{n} \omega_i \int_0^{x_i(k)} \psi_i(u)du + h\sum_{j=1}^{\nu} y^\top(k-j)\Lambda y(k-j)$$
$$+ h\sum_{j=1}^{\nu} \gamma_j \|y(k-j)\|^2. \tag{5}$$

Here $y^{(k)} = (y^\top(k), y^\top(k-1), \ldots, y^\top(k-\nu))^\top$, $\Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_m\}$ and $\omega_i, \lambda_j, \gamma_j$ are positive tuning parameters, $i = 1, \ldots, n$, $j = 1, \ldots, m$.

Consider the difference $\Delta V(k) = V(x(k+1), y^{(k+1)}) - V(x(k), y^{(k)})$ of the functional (5) along the solutions of the system (4). One obtains

$$\Delta V(k) = 2\sum_{i=1}^{n}\omega_i \int_{x_i(k)}^{x_i(k+1)} \psi_i(u)du + hy^\top(k)\Lambda y(k) - hy^\top(k-\nu)\Lambda y(k-\nu)$$

$$+h\gamma_1\|y(k)\|^2 + h\sum_{j=1}^{\nu-1}(\gamma_{j+1}-\gamma_j)\|y(k-j)\|^2 - h\gamma_\nu\|y(k-\nu)\|^2$$

$$= 2h\Psi^\top(x(k))\Omega\Big(A_\rho\Psi(x(k)) + B_\rho y(k-\nu)\Big) - hy^\top(k-\nu)\Lambda y(k-\nu)$$

$$+h\Big(F_\rho\Psi(x(k)) + G_\rho y(k-\nu)\Big)^\top\Lambda\Big(F_\rho\Psi(x(k)) + G_\rho y(k-\nu)\Big) - h\gamma_l\|y(k-\nu)\|^2$$

$$+h\gamma_1\|F_\rho\Psi(x(k)) + G_\rho y(k-\nu)\|^2 + h\sum_{j=1}^{\nu-1}(\gamma_{j+1}-\gamma_j)\|y(k-j)\|^2$$

$$+2\sum_{i=1}^{n}\omega_i\Big(\psi_i(x_i(k)+\pi_{ik}\Delta x_i(k)) - \psi_i(x_i(k))\Big)\Delta x_i(k).$$

Here $0 \le \pi_{ik} \le 1$, $\Delta x_i(k)$ are components of the vector $h(A_\rho\Psi(x(k)) + B_\rho y(k-\nu))$ and $\Omega = \mathrm{diag}\{\omega_1, \ldots, \omega_n\}$.

Similarly to the proof of Theorem 3.1, one can choose the matrices Ω and Λ such that

$$\Big(F_\rho\Psi(x(k)) + G_\rho y(k-\nu)\Big)^\top\Lambda\Big(F_\rho\Psi(x(k)) + G_\rho y(k-\nu)\Big) - y^\top(k-\nu)\Lambda y(k-\nu)$$

$$+2\Psi^\top(x(k))\Omega\Big(A_\rho\Psi(x(k)) + B_\rho y(k-\nu)\Big) \le -\tilde{a}\big(\|\Psi(x(k))\|^2 + \|y(k-\nu)\|^2\big),$$

where $\tilde{a} = \mathrm{const} > 0$. We obtain

$$\Delta V(k) \le -h\tilde{a}\big(\|\Psi(x(k))\|^2 + \|y(k-\nu)\|^2\big) - h\gamma_\nu\|y(k-\nu)\|^2$$

$$+(h\gamma_1 c_1 + h^2 c_2)\big(\|\Psi(x(k))\|^2 + \|y(k-\nu)\|^2\big) + h\sum_{j=1}^{\nu-1}(\gamma_{j+1}-\gamma_j)\|y(k-j)\|^2.$$

Here c_1 and c_2 are positive constants.

Let $\gamma_1 > \gamma_2 > \ldots > \gamma_\nu$. If γ_1 and h_0 are sufficiently small, then

$$\Delta V(k) \le -\frac{1}{2}h\tilde{a}\big(\|\Psi(x(k))\|^2 + \|y(k-\nu)\|^2\big) - h\bar{a}\sum_{j=1}^{\nu-1}\|y(k-j)\|^2$$

for $h \in (0, h_0)$, where $\bar{a} = \mathrm{const} > 0$. Hence, for these values of h, the system (4) is absolutely stable. This completes the proof. □

Remark 4.2:
From the proof of Theorem 4.1 it follows that the constant h_0 generally depends on the magnitude of the delay ν.

Remark 4.3:
In a similar way, absolute stability conditions for a discrete-time positive switched coupled system with multiple delays can be derived.

5. EXAMPLE

Consider the system (1) in the case where $n = m = N = 2$, $\beta = 1$,

$$A_1 = \begin{pmatrix} -\chi & 1 \\ 0 & -4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -16 & 0 \\ 1 & -8 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix},$$

$$G_1 = \frac{1}{8}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad G_2 = \frac{1}{8}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q_1 = \frac{1}{8}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad Q_2 = \frac{1}{8}\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}.$$

Here χ is a positive parameter. To find the values of this parameter for which the investigated system will be absolutely stable, verify the conditions of Theorem 3.1.

We obtain

$$\widehat{G} + \beta\widehat{Q} = \frac{1}{2}I_2.$$

Hence, Assumption 3.1 is satisfied.

Next, construct the system of inequalities

$$\begin{cases} \big(A_s + 2(B_s + C_s)\widehat{F}\,\big)\zeta \leq \alpha_1\zeta, \quad s = 1, 2, \\ \big(A_s + 2(B_r + C_p)\widehat{F}\,\big)^{\top}\eta \leq \alpha_2\eta, \quad s, r, p = 1, 2, \\ \alpha_1 + \alpha_2 < 0, \\ \zeta > 0, \\ \eta > 0, \end{cases}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$, $\zeta, \eta \in \mathbb{R}^2$.

Eliminating the vectors ζ, η from this system, we obtain

$$\begin{cases} (\alpha_1 + \chi - 12)(\alpha_1 + 6) \geq 1, \\ (\alpha_1 + 16)(\alpha_1 + 6) \geq 4, \\ (\alpha_2 + \chi - 4)(\alpha_2 + 2) \geq 10, \\ (\alpha_2 + \chi - 4)(\alpha_2 + 4) \geq 2, \\ (\alpha_2 + \chi - 4)(\alpha_2 + 6) \geq 8, \\ (\alpha_2 + 8)(\alpha_2 + 4) \geq 1, \\ (\alpha_2 + 8)(\alpha_2 + 2) \geq 5, \\ (\alpha_2 + 8)(\alpha_2 + 6) \geq 4, \\ \alpha_2 + \chi - 12 > 0, \\ \alpha_1 + \alpha_2 < 0, \\ \alpha_1 + 4 > 0, \\ \alpha_2 + 2 > 0. \end{cases} \tag{6}$$

It can be verified that (6) admits solutions if and only if

$$\chi > 9 + \frac{\sqrt{38}}{2}. \tag{7}$$

Thus, under the condition (7), the considered coupled system is absolutely stable for any delay τ.

6. CONCLUSION

In the present paper, new absolute stability conditions are obtained for a nonlinear switched positive coupled system and its discrete-time counterpart. The considered system contains both point-wise and distributed delays, and the derived constraints on the system parameters are independent of the value of the point-wise delay and depend on the value of the distributed delay. Unlike known stability conditions which require the existence of solutions to linear matrix inequalities, in this paper, the conditions are formulated in terms of the solvability of linear algebraic inequalities.

A possible direction for further research is to weaken the obtained restrictions on the parameters of systems that guarantee stability by introducing additional constraints on the switching signals.

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