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Dynamic Linearizing Feedback Design for a Single-Link Manipulator under Unmatched Disturbances

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Abstract: Using the output feedback linearization method, we consider the design of a servo system for a single-link manipulator, which is elastically coupled to a gearbox shaft and controlled by a sensorless DC motor. Unlike typical problem formulations, we assume significant parametric uncertainty in the controlled plant model and the presence of external, unmatched disturbances. The output (measured and controlled) variable is the angular position of the manipulator, which, during control, tracks the reference trajectory. There is no analytical description of the reference signal; only its current value is known. The design of a tracking system is based on the equivalent canonical form of the "input – output" relationship with respect to mixed variables, which are functions of state variables, external influences, and their derivatives. A combined control law has been formalized that linearizes a closed-loop virtual "input – output" system under conditions of an uncertain control multiplier. A reduced observer with piecewise linear corrective actions in the form of nested saturators has been developed. The observer is constructed on the basis of a canonical system with an uncertain input and, based on the tracking error measurements, estimates mixed variables and generalized disturbances with a given accuracy without dynamic generators of external influences and additional expansion of the state space. It is shown that observer gain coefficients tuning can be performed using a reference Hurwitz polynomial with a large stability margin. The actual gains will be many times smaller than those of a typical observer with linear correction without saturation, which generates large spikes in estimated signals and control action at the beginning of the transient process. Numerical simulation results are presented, demonstrating the performance of the tracking system and the advantages of the developed observer compared to a typical observer with linear correction without saturation.

Keywords: tracking, uncertain parameters, unmatched disturbances, reduced-order state observer, piecewise linear function with saturation, single-link manipulator.

1. INTRODUCTION

The tracking process is one of the most common technological modes of operation of automatic control systems and consists of the synthesis of static or dynamic feedback, ensuring the tracking of reference trajectories by output (regulated) variables.

Almost always, this automatic control problem must be solved under conditions of uncertain factors, including: the presence of parametric and functional mathematical models of the controlled plant; the action of external, uncontrolled disturbances related to various classes of functions; an incomplete set of sensors and measurement noise; and incomplete information about the reference signals. These factors reduce the performance of the closed-loop system and degrade its target indicators. Improving the quality of tracking systems operating under uncertainty is largely associated with the improvement of information-control systems and the development and use of effective control and evaluation algorithms, which demonstrates the relevance of the research topic.

A new paradigm for the synthesis of robust systems operating under conditions of uncertain factors of various natures consists of the development of universal and easy-to-

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implement methods for synthesizing dynamic feedback that do not require readjustment of gain factors when uncertain factors change during operation. These approaches include the output feedback linearization method, which allows for the design of universal control laws for output variables from a unified perspective for a large class of nonlinear systems whose mathematical models allow multiple differentiation and are differentially flat [1]. When a number of conditions are met, mathematical models of such systems are reduced to a block-canonical "input – output" form with matched generalized nonlinearities operating through the same control channels. In the stabilization problem, the output is considered to be the controlled variables, in the tracking problem – the discrepancies between the controlled variables and the reference trajectories (tracking errors), while the generalized nonlinearities include the derivatives of the reference trajectories. Under conditions of parametric certainty of the plant model and complete measurements of internal and external signals, generalized nonlinearities are compensated for using combined control with a stabilizing linear component. As a result, the closed-loop virtual "input – output" system becomes linear, allowing the application of known modal synthesis methods to it, ensuring the specified behavior of the output variables.

Within the framework of this approach, nonlinear systems with matched external disturbances and uncertain parameters were also considered, but with significant restrictions on the type of permissible nonlinearities and uncertainties, to the assessment of which adaptive control methods are applicable [2–8]. Under conditions of uncertainty and incomplete measurements, linearizing control is implemented using dynamic feedback. To obtain real-time estimates of uncertain signals and parameters, dynamic state and disturbance observers, uncertain parameter identifiers, and dynamic generators of external influences are introduced into the feedback loop.

In [9–10] it is shown that the output feedback linearization method is applicable to nonlinear systems with unmatched external disturbances if they are assumed to be smooth functions of time and, upon repeated differentiation of the output variables, do not change the relative order inherent in the corresponding undisturbed system. Then, using diffeomorphic transformations of variables, the model of the control plant is reduced to an equivalent block-canonical form "input – output" in the coordinate basis of mixed variables, which are functions of the state variables of the plant, external influences and their derivatives. This canonical form, where only the generalized matched disturbance is explicitly present, is a unified basis for solving tracking and observation problems, which greatly simplifies the microprocessor implementation of the tracking system. It is shown that when measuring output variables and parametric certainty of input control channels, all the necessary information for the synthesis of dynamic feedback can be obtained using a reduced observer of mixed variables and generalized disturbances with piecewise linear corrective actions with saturation (sat-correction). This approach does not require individual evaluation of uncertain parameters, external disturbances and unmeasured variables of the state vector using appropriate dynamic algorithms and models, which increase the dynamic order of the closed system several times and complicate the structure of the controller.

In this paper, this approach is applied to the design of a single-link manipulator angular position tracking system operating under conditions of incomplete information about external and internal signals, as well as the parameters of the controlled plant. It is shown that, with parametric uncertainty in the input channel, the control portion with an uncertain multiplier can be attributed to a generalized disturbance, enabling the implementation of both an observer and a combined control that linearizes the closed-loop system. The scientific novelty lies in the proposed simplified algorithm for setting up sat-correction of a reduced mixed variable observer and a generalized disturbance using a reference polynomial.

The paper is structured as follows. Section 2 describes the mathematical model of the control plant and formulates the problem. The procedure for obtaining the canonical "input – output" form and the basic law of combined control that linearizes the canonical system are presented. In Section 3, a mixed variable observer with piecewise linear corrective actions with saturation is designed for a canonical system with uncertain input. The main result consists of a simplified algorithm for selecting sat correction gain coefficients that ensure a given accuracy of solving the observation problem. Section 4 presents the results of numerical simulation of the developed algorithms, demonstrating the robust properties of the tracking system and the advantages of the developed observer compared to an observer with linear correction without saturation.

2. OUTPUT FEEDBACK LINEARIZATION METHOD

The control plant is a single-link manipulator, which is elastically connected to the gearbox shaft and controlled by a sensorless DC motor. Without taking into account the rapid dynamics of the current loop, the mathematical model has the following form [11]:

$$\begin{aligned} &\dot{x}_1 = x_2,\ \dot{x}_2 = a_{21}(x_3 - x_1) - a_{22}\sin(x_1) + f_1(t), \\ &\dot{x}_3 = x_4,\ \dot{x}_4 = -a_{43}(x_3 - x_1) - a_{44}x_4 + b_4 u + f_2(t), \end{aligned} \tag{2.1}$$

where $x_{1,2} \in R$ are angular position and angular speed of the manipulator, $x_{3,4} \in R$ are angular position and angular velocity of the gearbox shaft; $a_{ij} > 0$, $b_4 > 0$ are design parameters [12] that are not precisely known, but their values are within known ranges; u is the armature current of a DC motor, which is considered as control, $b_4 u$ is torque applied to the gearbox shaft; $f_{1,2}(t)$ are external disturbances that are assumed to be unknown functions of time, limited together with their derivatives by known constants.

The problem is to synthesize dynamic feedback that ensures tracking of the angular position of the manipulator $x_1(t)$ of the reference trajectory $g(t)$ with a given accuracy in a steady state.

$$|x_1(t) - g(t)| \le \Delta, t \to +\infty. \tag{2.2}$$

It is assumed that there is only an angular position sensor $x_1(t)$. The manipulator with a complete actuator is capable of working out a reference trajectory $g(t)$, which is specified in the form of a deterministic signal, its derivatives are unknown, but limited. The problem is solved using the output feedback linearization method with minimal dynamic expansion; therefore, individual estimation of uncertain parameters and external influences is not used. The auxiliary problem consists of synthesizing a mixed-variable observer of canonical form, based on which the linearizing control will be generated.

In system (2.1), the disturbance $f_2(t)$ acts through the same channel as the control, i.e., it is matched. The disturbance is $f_1(t)$ unmatched, but its input channel is independent of the state variables, so its presence does not change the relative order of control inherent in the unperturbed system at $f_1(t) \equiv 0$ [9], which is obviously controllable and observable with respect to $x_1(t)$. Based on the mathematical model (2.1), we will construct a canonical "input – output" form, considering the tracking error $e_1(t) = x_1(t) - g(t)$ as the output variables and differentiating it until control appears:

$$\dot{e}_i = e_{i+1},\ i = \overline{1,3};\quad \dot{e}_4 = \psi(t) + bu, \tag{2.3}$$

where e_i, $i = \overline{1,4}$ are mixed variables of the new coordinate basis $\psi(t)$, is a generalized disturbance that depends on uncertain parameters, external influences and their derivatives:

$$\begin{aligned}
&e_2 = x_2 - \dot{g}(t), e_3 = a_{21}(x_3 - x_1) - a_{22}\sin x_1 + f_1(t) - \ddot{g}(t),\\
&e_4 = a_{21}(x_4 - x_2) - a_{22}x_2\cos x_1 + \dot{f}(t) - \dddot{g}(t),\ b = a_{21}b_4,\\
&\psi(t) = a_{21}[(a_{43} + a_{21} + a_{22}\cos x_1)(x_1 - x_3) - a_{44}x_4 + f_2(t)] +\\
&+ (a_{21} + a_{22}\cos x_1 + x_2^2)a_{22}\sin x_1 - (a_{21} + a_{22}\cos x_1)f_1(t) + \ddot{f}_1(t) - g^{(4)}(t).
\end{aligned} \tag{2.4}$$

It is assumed that in system (2.3) the control multiplier $b = a_{21}b_4 > 0$, which depends on the uncertain parameters of the plant (2.1), can be represented as the sum of the nominal $b_0 = \text{const} > 0$ (known) and uncertain terms in the following form:

$$b = b_0 + \Delta b,\ \text{sign}(b_0) = \text{sign}(b_0 + \Delta b) = +1 \Rightarrow |\Delta b|/b_0 < 1, t \ge 0. \tag{2.5}$$

Then the control part with the undefined multiplier can be attributed to the generalized disturbance, which is assumed to be an unknown function of time with a bounded derivative. In this case, system (2.3) takes the following form:

$$\dot{e}_i = e_{i+1},\ i = \overline{1,3};\ \ \dot{e}_4 = e_5(t) + b_0 u,\ e_5 = \psi(t) + \Delta b u. \tag{2.6}$$

Thus, a canonical "input – output" form (2.6) with a consistent disturbance and a known control multiplier is obtained, on the basis of which a linearizing combined control is synthesized

$$u = -(c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4 + e_5(t))/b_0, \tag{2.7}$$

where $c_i = \text{const} > 0$ are the coefficients of the reference Hurwitz polynomial

$$\prod_{i=1}^{4}(\lambda - \lambda_i) = \lambda^4 + c_4\lambda^3 + c_3\lambda^2 + c_2\lambda + c_1,\ \text{Re}\,\lambda_i < 0,\ i = \overline{1,4}. \tag{2.8}$$

Its roots are assigned based on the requirements for the transient tracking error process. The closed virtual system (2.6)–(2.7) is linear and stable:

$$\dot{e}_i = e_{i+1},\ i = \overline{1,3};\ \ \dot{e}_4 = -(c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4). \tag{2.9}$$

For the "worst" (as a rule, maximum in absolute value) values of uncertain parameters, external and reference influences, and also taking into account the range of permissible initial values of the state variables of the system (2.1), from expressions (2.4), (2.6), (2.7) one can obtain conservative estimates for the ranges of variation of the generalized disturbance, its derivative, and also the mixed variables of the system (2.9):

$$|e_i(t)| \le E_i,\ i = \overline{1,5};\ |\dot{e}_i(t)| \le E_{51},\ t \ge 0. \tag{2.10}$$

It is obvious that the mixed variables in the closed system (2.9) are maximum in magnitude at the beginning of the transient process.

Exact linearization of the closed-loop canonical system (2.9) and asymptotic convergence to zero of its variables (including the tracking error $e_1(t)$) are ensured under conditions of complete certainty. Under a priori assumptions, only the output $e_1(t)$ of system (2.6) is known. Based on these measurements, in the next section, an observer is designed to estimate the remaining variables used to synthesize feedback (2.7).

Let the $\tilde{e}_i(t),\ i = \overline{2,5}$ are the estimates of mixed variables obtained using a dynamic observer with some errors in the steady state be:

$$|e_i(t) - \tilde{e}_i(t)| \le \delta_i,\ i = \overline{2,5},\ t \ge T. \tag{2.11}$$

Combined control (2.7), implemented in a system with an observer in the form

$$u = -\left(c_1 e_1 + c_2 \tilde{e}_2 + c_3 \tilde{e}_3 + c_4 \tilde{e}_4 + \tilde{e}_5(t)\right)/b_0, \tag{2.12}$$

introduces a generalized estimation error $\phi(t)$ into the last equation of the closed virtual system (2.6), (2.11):

$$\begin{aligned} &\dot{e}_i = e_{i+1},\ i = \overline{1,3};\ \dot{e}_4 = -(c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4) + \phi(t), \\ &|\phi(t)| \le c_2\delta_2 + c_3\delta_3 + c_4\delta_4 + \delta_5 = \Phi,\ \Phi = \mathrm{const} > 0,\ t \ge T, \end{aligned} \tag{2.13}$$

which generates a non-zero tracking error in the steady state (2.2). For the vector of mixed variables $e = (e_1, e_2, e_3, e_4)^{\mathrm{T}} \in R^4$ of the system (2.13), the estimate holds

$$\|e(t)\| \le C\left(\exp(-\upsilon t)\|e(T)\| + \Phi/\upsilon\right) = \hat{E}, t \ge T, \tag{2.14}$$

where $\hat{E} \ge \max\{E_1, ..., E_4\}$, $0 < \upsilon \le \min\{-\mathrm{Re}\,\lambda_i\}_{i=\overline{1,4}}$ is the degree of stability, $C(\upsilon) = \mathrm{const} > 0$, $C = \|D\|\|D^{-1}\| \ge 1$. For example, in the case of different real eigenvalues $\lambda_i < 0,\ i = \overline{1,\ 4}$ (2.8), $D^{4\times 4}$ is the Vandermonde matrix with columns $\mathrm{col}(1,\ \lambda_i,\ \lambda_i^2,\ \lambda_i^3)$, $i = \overline{1,4}$.

In the “worst” case, system (2.13) can be represented as a linear inhomogeneous differential equation (LIDE) with a constant right side

$$e_1^{(4)} + c_4 \ddot{e}_1 + c_3 \ddot{e}_1 + c_2 \dot{e}_1 + c_1 e_1 = \Phi,\ e_1(t) = e_{1\mathrm{c}}(t) + \Phi/c_1. \tag{2.15}$$

Neglecting the damped free motions of mixed variables $\lim_{t\to\infty} e_{1\mathrm{c}}(t) = 0$ in (2.15), for system (2.6) with dynamic feedback (2.12) we have an estimate of the tracking error in steady state

$$|e_1(t)| \le (c_2\delta_2 + c_3\delta_3 + c_4\delta_4 + \delta_5)/c_1,\ t \to +\infty. \tag{2.16}$$

From (2.16) follow the requirements for estimation errors (2.11)

$$\frac{c_2\delta_2 + c_3\delta_3 + c_4\delta_4 + \delta_5}{c_1} \le \Delta \Rightarrow \max(\delta_2, ..., \delta_5) \le \delta \le \frac{c_1\Delta}{c_2 + c_3 + c_4 + 1}, \tag{2.17}$$

which must be ensured when synthesizing an observer to fulfill the target condition (2.2).

3. MAIN RESULT

To realize combined control (2.12), it is necessary to reconstruct the mixed variables $e_2(t), ..., e_5(t)$. This problem is solved using a dynamic observer, which is constructed based on system (2.6) in the following form:

$$\dot{z}_i = z_{i+1} + v_i,\ i = \overline{1,3};\ \ \dot{z}_4 = b_0 u + v_4, \tag{3.1}$$

where $z = (z_1, z_2, z_3, z_4)^{\mathrm{T}} \in R^4$ is the state vector, v_i ($i = \overline{1,4}$) are corrective actions of the observer, which are formed on the basis of measurements $e_1(t)$ in such a way as to ensure stabilization with a given accuracy (2.17) of the system with respect to observation errors $\varepsilon = e - z = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)^{\mathrm{T}} \in R^4$. By virtue of (2.6), (3.1), we have a virtual system

$$\dot{\varepsilon}_i = \varepsilon_{i+1} - v_i,\ i = \overline{1,3};\ \dot{\varepsilon}_4 = e_5 - v_4, \tag{3.2}$$

where $e_5(t)$ is assumed to be a bounded disturbance with bounded derivative (2.10).

Observer (3.1) is reduced: unlike standard approaches [13–14], it is not supplemented by the equation of the dynamics of the generalized disturbance $e_5(t)$. In observer (3.1), we set, for example, zero initial values, then

$$z_i(0) = 0 \Rightarrow \varepsilon_i(0) = e_i(0), \left|\varepsilon_i(0)\right| < E_i, i = \overline{1,4}. \tag{3.3}$$

For system (3.2) the problem is to stabilize not only the observation errors, but also their derivatives:

$$\left|\varepsilon_i(t)\right| = \left|e_i(t) - z_i(t)\right| \le \delta, i = \overline{1,4}; \ \left|\dot{\varepsilon}_4(t)\right| = \left|e_5(t) - v_4(t)\right| \le \delta, t \ge T. \tag{3.4}$$

Then, the corrective action of the last block can be taken as the disturbance estimate. With the estimated signals $\tilde{e}_i(t) = z_i(t), i = \overline{2,4}$, $\tilde{e}_5(t) = v_4(t)$ in a closed-loop system with an observer (3.1), the combined control (2.12) will be implemented as

$$u = -\left(c_1 e_1 + c_2 z_2 + c_3 z_3 + c_4 z_4 + v_4\right)/b_0. \tag{3.5}$$

To solve the stated problem (3.4) in systems (3.1), (3.2), piecewise linear corrective actions with saturation in the form of nested saturators are used:

$$\begin{aligned} v_1 &= m_1 \mathrm{sat}(l_1 \varepsilon_1) = \begin{bmatrix} m_1 \mathrm{sign}(\varepsilon_1), \left|\varepsilon_1\right| > 1/l_1, \\ m_1 l_1 \varepsilon_1, \left|\varepsilon_1\right| \le 1/l_1; \end{bmatrix} \\ v_i &= m_i \mathrm{sat}(l_i v_{i-1}) = \begin{bmatrix} m_i \mathrm{sign}(v_{i-1}), \left|v_{i-1}\right| > 1/l_i, \\ m_i l_i v_{i-1}, \left|v_{i-1}\right| \le 1/l_i, \end{bmatrix} \end{aligned} \tag{3.6}$$

which have two adjustable gain factors each: $m_i, l_i = \mathrm{const} > 0, i = \overline{1,4}$.

In [9, 12, 15], within the framework of the motion separation method, when the total motion of the observation error vector is divided into components of different rates, hierarchical systems of inequalities were obtained for adjusting the observer coefficients (3.1), (3.6), for which the observation problem is solved with a given accuracy in a given time (3.4). In this case, the following hierarchy of estimation errors in the steady state is ensured:

$$0 \le \delta_1 \le \delta_2 \le \delta_3 \le \delta_4 \le \delta_5 \le \delta, \ t \ge T. \tag{3.7}$$

The obtained results are theoretically significant, as they prove the existence of a solution to the problem. However, single-channel systems with high dimension, observer tuning based on the obtained inequalities is quite labor-intensive and leads to conservative estimates. This paper proposes a simplified tuning algorithm using a reference polynomial, recommended for practical use.

The amplitudes $m_i, i = \overline{1,4}$ of the correction actions are chosen to ensure rapid convergence of the observation errors in the closed system (3.2), (3.6) to some neighborhoods of zero, where the correction actions (3.6) become linear. Outside the linear zones, system (3.2), (3.6) can be represented as

$$\dot{\varepsilon}_1 = \varepsilon_2 - m_1 \mathrm{sign}(\varepsilon_1), \ \dot{\varepsilon}_i = \varepsilon_{i+1} - m_i \mathrm{sign}(v_{i-1}) = \varepsilon_{i+1} - m_i \mathrm{sign}(\varepsilon_i - \dot{\varepsilon}_{i-1}), i = \overline{2,4}. \tag{3.8}$$

The simplified setup ignores rapidly decaying transient processes of observation errors, their derivatives, and the short time during which all corrective actions, from top to bottom, converge into their corresponding linear zones. Taking into account (2.10) and (3.3), one can roughly estimate the ranges of change in observation errors during the transient process: $\left|\varepsilon_i(t)\right| \le E_i, i = \overline{1,4}, \ t \ge 0$. Then, from the sufficient conditions $\varepsilon_i \dot{\varepsilon}_i < 0, \ i = \overline{1,4}$ applied to system (3.8), we obtain simplified inequalities for choosing the amplitudes of the corrective actions (3.6), ensuring that they fall within $0 < t < T$, and are located in the linear zones if $t > T$:

$$m_i > \max\{E_{i+1}; m_{i+1}\} \Rightarrow |\varepsilon_i(t)| \le 1/l_i + |\dot{\varepsilon}_{i-1}(t_i)|, t \ge t_i, \ i = \overline{1,4}, \tag{3.9}$$

where $\dot{\varepsilon}_0 \equiv 0,\ m_5 \equiv 0$ and

$$0 < T \le t_1 + t_2 + t_3 + t_4, \ \frac{E_i}{m_i + E_{i+1}} \le t_i \le \frac{E_i}{m_i - E_{i+1}}, i = \overline{1,\ 4}. \tag{3.10}$$

In expression (3.9) a condition $m_i > m_{i+1}, i = \overline{1,3}$ is specially introduced that ensures the required sequence (from top to bottom) of stabilization of observation errors.

At $t > T$ (3.10) the corrective actions become linear

$$v_1 = m_1 l_1 \varepsilon_1,\ v_2 = m_2 l_2 v_1,\ v_3 = m_3 l_3 v_2,\ v_4 = m_4 l_4 v_3, \tag{3.11}$$

then the virtual closed system (3.2), (3.11) can be represented in the following matrix form

$$\dot{\varepsilon} = \overline{A}\varepsilon + \overline{B}e_5,\ \overline{A} = \begin{pmatrix} -m_1 l_1 & 1 & 0 & 0 \\ -m_2 l_2 m_1 l_1 & 0 & 1 & 0 \\ -m_3 l_3 m_2 l_2 m_1 l_1 & 0 & 0 & 1 \\ -m_4 l_4 m_3 l_3 m_2 l_2 m_1 l_1 & 0 & 0 & 0 \end{pmatrix}, \overline{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{3.12}$$

To analyze the convergence of the derivatives of observation errors, we will also compose an auxiliary system, which, when $t > T$, has a form similar to (3.12):

$$\ddot{\varepsilon} = \overline{A}\dot{\varepsilon} + \overline{B}\dot{e}_5. \tag{3.13}$$

The correction coefficients $l_i, i = \overline{1,4}$ are selected to ensure stabilization of systems (3.12), (3.13) with a given accuracy (3.4) in steady-state mode. To simplify their adjustment, a reference Hurwitz polynomial is introduced

$$\prod_{i=1}^{4}(\lambda - \overline{\lambda}_i) = \lambda^4 + p_1\lambda^3 + p_2\lambda^2 + p_3\lambda + p_4,\ \overline{\lambda}_i \in C\colon\ \mathrm{Re}\,\overline{\lambda}_i < 0,\ i = \overline{1,\ 4}, \tag{3.14}$$

and its parameters play the role of large coefficients in a small neighborhood of zero.

The stability margin in closed systems (3.12), (3.13) must be greater than in the closed system (2.6), (2.7), and, in addition,

$$\min\{-\mathrm{Re}\,\overline{\lambda}_i\}_{i=\overline{1,4}} = \overline{\upsilon} > \max\{-\mathrm{Re}\,\lambda_i\}_{i=\overline{1,4}} > 0. \tag{3.15}$$

As is known, the reference polynomial (3.14) is used in classical linear state observers (3.1), constructed on the basis of canonical forms [16], for the modal synthesis of linear corrective actions in the form

$$v_1 = p_1\varepsilon_1,\ v_2 = p_2\varepsilon_1,\ v_3 = p_3\varepsilon_1,\ v_4 = p_4\varepsilon_1, \tag{3.16}$$

which leads to the following closed system (3.2), (3.16):

$$\dot{\varepsilon} = \overline{A}\varepsilon + \overline{B}e_5,\ \overline{A} = \begin{pmatrix} -p_1 & 1 & 0 & 0 \\ -p_2 & 0 & 1 & 0 \\ -p_3 & 0 & 0 & 1 \\ -p_4 & 0 & 0 & 0 \end{pmatrix}, \overline{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{3.17}$$

By equating the components of the first columns of the matrices of systems (3.12) and (3.17) for specific values of the amplitudes $m_i > 0, i = \overline{1,4}$ adopted on the basis of inequalities (3.9), the gain coefficients $l_i > 0, i = \overline{1,4}$ are sequentially determined, which in a simplified setting (in contrast to the choice based on inequalities [9, 12, 15]) take on specific values:

$$l_1 = \frac{p_1}{m_1}, l_2 = \frac{p_2}{m_2 m_1 l_1}, \; l_3 = \frac{p_3}{m_3 m_2 l_2 m_1 l_1}, l_4 = \frac{p_4}{m_4 m_3 l_3 m_2 l_2 m_1 l_1}. \tag{3.18}$$

A suitable value $\bar{\upsilon}$ (3.15) that ensures a given accuracy of estimation (3.4) can be obtained by simulating a closed-loop system with several trial values or by using a one-parameter tuning procedure [13, 16]. In these papers, linear correction actions without saturation are used to stabilize the canonical system with an uncertain input (3.2). To tune them, a reference polynomial is introduced with a hierarchy of real roots $\tilde{\lambda}_i < 0$: $\left|\tilde{\lambda}_1\right| = 1, \left|\tilde{\lambda}_{i+1}\right| > \left|\tilde{\lambda}_i\right|, i = \overline{2,4}$ and parameter $\mu > 1$:

$$\prod_{i=1}^{4}(\lambda - \mu\tilde{\lambda}_i) = \lambda^4 + \mu\tilde{p}_1\lambda^3 + \mu^2\tilde{p}_2\lambda^2 + \mu^3\tilde{p}_3\lambda + \mu^4\tilde{p}_4\,, \tag{3.19}$$

which results in the following linear corrective actions:

$$v_1 = \mu\tilde{p}_1\varepsilon_1, \; v_2 = \mu^2\tilde{p}_2\varepsilon_1, \; v_3 = \mu^3\tilde{p}_3\varepsilon_1, \; v_4 = \mu^4\tilde{p}_4\varepsilon_1\,. \tag{3.20}$$

In the papers [13, 16] it is shown that for the state vector of the system (3.2), (3.3), (3.20) when (2.10) is satisfied, the estimate holds

$$\|\varepsilon(t)\| \le \tilde{C}(\tilde{\lambda}, \mu)\|\varepsilon(0)\|\exp(-\mu t) + K(\tilde{\lambda})E_5 / \mu, \tag{3.21}$$

where the constant $K(\tilde{\lambda}) > 0$ does not depend on μ. In this scheme, the specified accuracy of estimating mixed variables in the steady state (3.4) is ensured by the choice of $\mu > 1$.

A similar result will be obtained by using the reference polynomial (3.19) to select the gain factors $l_i > 0, i = \overline{1,4}$ of piecewise linear corrective actions with saturation (3.6) in the form (3.18). For $t > T$ (3.10), for the vectors of observation errors and their derivatives, which are the solution of the linear systems (3.12) and (3.13), respectively, taking into account (2.10), (3.17), (3.21), the following estimates are valid:

$$\|\varepsilon(t)\| \le \tilde{C}(\tilde{\lambda}, \mu)\|\varepsilon(T)\|\exp(-\mu t) + K(\tilde{\lambda})E_5 / \mu,$$
$$\|\dot{\varepsilon}(t)\| \le \tilde{C}(\tilde{\lambda}, \mu)\|\dot{\varepsilon}(T)\|\exp(-\mu t) + K(\tilde{\lambda})E_{51} / \mu.$$

From here, neglecting the damped free motions, we have an inequality for choosing the parameter that ensures a given accuracy of estimating (3.4) the mixed variables and the generalized disturbance in the steady state:

$$\mu \ge \frac{K(\tilde{\lambda})}{\delta}\max\{E_5, \, E_{51}\}. \tag{3.22}$$

Let us consider the possibility of obtaining a simpler estimate for selecting a reference polynomial, one that does not require entering a parameter $\mu > 1$ and calculating $K(\tilde{\lambda})$. From the hierarchy of observation errors generated by nested saturators (3.6), it follows that, to satisfy (3.4) in system (3.17) and in the auxiliary system (3.13), it is sufficient to ensure stabilization with a given accuracy of the variables of the last blocks, namely, ε_4 and $\dot{\varepsilon}_4$.

For this purpose, we transform the system (3.17) to the canonical "input – output" form, where the output is the variable of the last block ε_4.

We first perform the appropriate permutation of the state variables $P\varepsilon = \eta$, $\det P \neq 0$, $\eta = (\eta_1 = \varepsilon_4, \eta_2 = \varepsilon_3, \eta_3 = \varepsilon_2, \eta_4 = \varepsilon_1)^{\mathrm{T}}$ and, using a similarity conversion $\hat{A} = P\overline{A}P^{-1}, \hat{B} = P\overline{B}$, obtain the system

$$\dot{\eta} = \hat{A}\eta + \hat{B}e_5,\ \hat{A} = \begin{pmatrix} 0 & 0 & 0 & -p_4 \\ 1 & 0 & 0 & -p_3 \\ 0 & 1 & 0 & -p_2 \\ 0 & 0 & 1 & -p_1 \end{pmatrix}, \hat{B} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, P = P^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (3.23)$$

It is easy to verify that system (3.23) is equivalent to a linear inhomogeneous differential equation similar to (2.15):

$$\eta_1^{(4)} + p_1\dddot{\eta}_1 + p_2\ddot{\eta}_1 + p_3\dot{\eta}_1 + p_4\eta_1 = p_3 e_5 + p_2\dot{e}_5 + p_1\ddot{e}_5 + \dddot{e}_5. \quad (3.24)$$

For the auxiliary system (3.13) we will also compose LIDE, similar to (3.24):

$$\overline{\eta}_1^{(4)} + p_1\dddot{\overline{\eta}}_1 + p_2\ddot{\overline{\eta}}_1 + p_3\dot{\overline{\eta}}_1 + p_4\overline{\eta}_1 = p_3\dot{e}_5 + p_2\ddot{e}_5 + p_1\dddot{e}_5 + e^{(4)}, \quad (3.25)$$

where $\eta_1(t) = \varepsilon_4(t)$, $\overline{\eta}_1(t) = \dot{\eta}_1(t) = \dot{\varepsilon}_4(t)$. Let $\left|e_5^{(i)}(t)\right| \leq E_{5i}, i = \overline{1,4}$ be estimates of the derivatives of the generalized perturbation for $t \geq 0$. In the "worst" cases, when LIDE (3.24), (3.25) have constants on the right-hand sides $p_3E_5 + p_2E_{51} + p_1E_{52} + E_{53} = \Phi_1$ and $p_3E_{51} + p_2E_{52} + p_1E_{53} + E_{54} = \Phi_2$, accordingly, their solutions have the form

$$\eta_1(t) = \varepsilon_4(t) = \varepsilon_{4\mathrm{c}}(t) + \Phi_1 / p_4, \overline{\eta}_1(t) = \dot{\varepsilon}_4(t) = \dot{\varepsilon}_{4\mathrm{c}}(t) + \Phi_2 / p_4, \quad (3.26)$$

where $\varepsilon_{1\mathrm{c}}(t)$, $\dot{\varepsilon}_{1\mathrm{c}}(t)$ are damped proper motions $\lim\limits_{t\to\infty}\varepsilon_{1\mathrm{c}}(t) = 0$, $\lim\limits_{t\to\infty}\dot{\varepsilon}_{1\mathrm{c}}(t) = 0$, that are not taken into account.

For convenience, we assume identical real roots in the reference polynomial (3.14) $\overline{\lambda}_i = -a, i = \overline{1,4}$, ($a >> 1$ is the stability margin of the reference polynomial). Expanding $(\lambda + a)^4$ using Newton's binomial formula, we have $p_1 = 4a,\ p_2 = 6a^2,\ p_3 = 4a^3, p_4 = a^4$. Then, the estimates of the forced motions of the observation errors (3.26) take the form

$$\begin{aligned} \left|\varepsilon_4(t)\right| &\leq \frac{p_3E_5 + p_2E_{51} + p_1E_{52}}{p_4} = \frac{4E_5}{a} + \frac{6E_{51}}{a^2} + \frac{4E_{52}}{a^3} + \frac{E_{53}}{a^4}, \\ \left|\dot{\varepsilon}_4(t)\right| &\leq \frac{p_3E_{51} + p_2E_{52} + p_1E_{53}}{p_4} = \frac{4E_{51}}{a} + \frac{6E_{52}}{a^2} + \frac{4E_{53}}{a^3} + \frac{E_{54}}{a^4}. \end{aligned} \quad (3.27)$$

For a sufficiently large $a >> 4$, where 4 is the dimension of the observer, (for example $a > 400$) the second and third terms in estimates (3.27) can be neglected. Under these assumptions, we have a simpler inequality, compared to (3.22), for choosing the stability margin of the reference polynomial (3.14), ensuring the specified accuracy of the estimate (3.4):

$$a > \kappa \cdot \frac{4}{\delta}\max\{E_5,\, E_{51}\}, \quad (3.28)$$

where is the lower bound multiplied by a coefficient $\kappa > 1$ to compensate for the discarded terms in (3.27).

Thus, the following statement is true: if in system (3.2), (3.3), (3.6) the generalized disturbance $e_5(t)$ and its derivative are limited by known constants (2.10), then the choice of the amplitudes of the corrective actions (3.9) in a finite time (3.10) ensures the attenuation of rapid movements, and the accuracy of the estimation in the steady-state mode (3.4) is ensured by the choice of the degree of stability (3.28) of the reference polynomial and the gain factors (3.18).

The fundamental difference between observer (3.1) with linear correction with large gains (3.20) and observer (3.1) with sat-correction (3.6) is a large initial burst of observation errors (the first term in (3.21)). Its magnitude rapidly increases in absolute value with increasing parameter and dimension of the observer, and at the same time generates an unacceptable initial burst of the control action. In the observer with sat-correction (3.6) in the transient regime, the correction values, limited by the amplitudes (3.9), restrain the bursts of observation errors, which allows these observers to be easily used in systems with linear feedback (3.5). With the same reference polynomials (3.14) or (3.19) and all other things being equal, both observers demonstrate identical steady-state regimes. In the next section, these conclusions are confirmed by the results of numerical simulation.

4. RESULTS OF NUMERICAL SIMULATION

The simulation was carried out in the MATLAB-Simulink environment, using the Euler method with a constant step of 0.001 for integration. For the control plant (2.1), the initial values $x_{1,2,3,4}(0)=0$, parameters $a_{21}=8{,}8$, $a_{22}=29$, $a_{43}=9{,}9$, $a_{44}=2{,}5$, $b_4=13{,}5$ and external influences

$$f_1(t)=0{,}2\cos(t),\;\; f_2(t)=\sin(0{,}6t),\;\; g(t)=0{,}1\sin(0{,}5t)+0{,}3\cos(0{,}7t)$$

were adopted.

The gain coefficients of the linearizing feedback (2.7) are adopted based on the reference polynomial (2.8) of the form $(\lambda+5)^4=\lambda^4+20\lambda^3+150\lambda^2+500\lambda+625$. It was assumed that the control multiplier $b=a_{21}b_4=8{,}8\cdot 13{,}5=118{,}8$ is not exactly known; the parameter $b_0=100\neq b$ satisfying (2.5) was adopted as the nominal value.

In experiment 1, a closed system (2.1), (3.5) was modeled with an observer (3.1) with piecewise linear corrective actions with saturation (3.6), the amplitudes of which were adopted on the basis of inequalities (3.9) in the form

$$m_1=130,\; m_2=120,\; m_3=110,\; m_4=100. \tag{4.1}$$

Taking into account (3.15), (3.28), the reference polynomial (3.14) is adopted for the observer in the form $(\lambda+900)^4$ with parameters

$$p_1=3\,600\,,\; p_2=4\,860\,000,\; p_3=2{,}916243\text{E+9},\; p_4=6{,}56154675\text{E+11}. \tag{4.2}$$

Next, by substituting the values of (4.1) and (4.2) into formulas (3.18), the following sat correction gain factors (3.6) were obtained:

$$l_1=27{,}692,\; l_2=11{,}25,\; l_3=5{,}455,\; l_4=2{,}25.$$

Figure 4.1 shows the graphs of the controlled variable: the angular position of the manipulator $x_1(t)$ and the reference action $g(t)$. Figure 4.2 shows the tracking error $e_1(t)=x_1(t)-g(t)$ graph. Figure 4.3 shows the graph of the corrective action $v_4(t)$ (3.6), which restores the generalized disturbance $e_5(t)$ (6). Figure 4.4 presents the graph of control $u(t)$ (3.5).

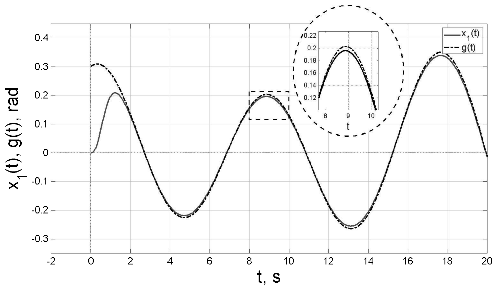

Fig. 4.1. Plots of $x_1(t)$, $g(t)$

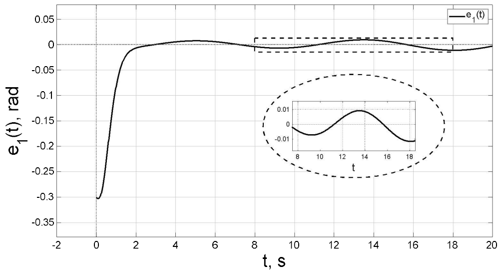

Fig. 4.2. Plot of the tracking error $e_1(t)$

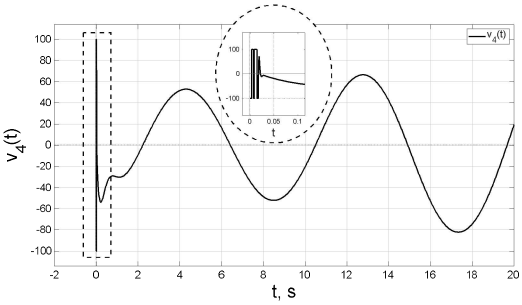

Fig. 4.3. Plot of $v_4(t) \approx e_5(t)$

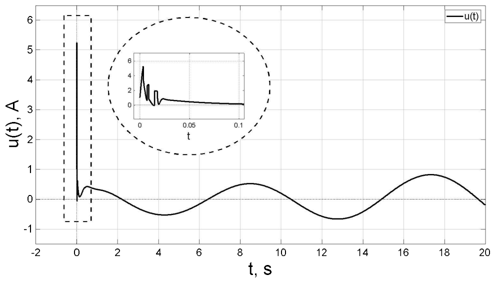

Fig. 4.4. Plot of the control $u(t)$

The graphs show that the transient tracking error lasts approximately 2 seconds, and in the steady state, the tracking error does not exceed 0,012 rad. The control action during the transient period is in the range of $0 \le u(t) \le 5{,}3$ A, $0 \le t \le 1$ sec.

In Experiment 2, a closed-loop system (2.1), (3.5) with an observer (3.1) and linear correction effects (3.16) whose coefficients coincide with the coefficients of the reference polynomial (4.2) was modeled. Figures 4.5–4.8 show graphs of the same variables as in Figures 4.1–4.4.

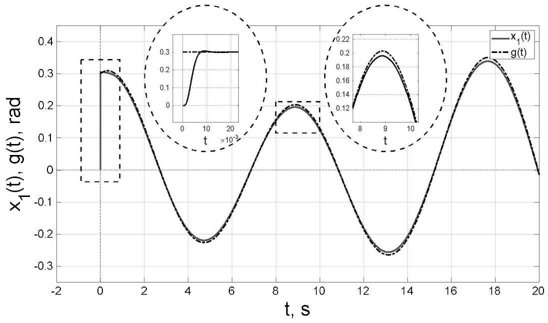

Fig. 4.5. Plots of $x_1(t)$, $g(t)$

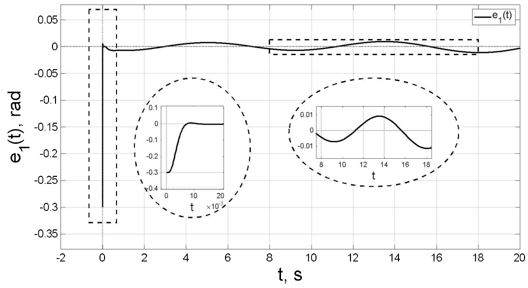

Fig. 4.6. Plot of the tracking error $e_1(t)$

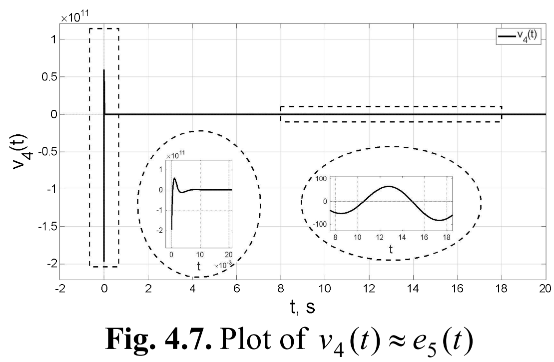

Fig. 4.7. Plot of $v_4(t) \approx e_5(t)$

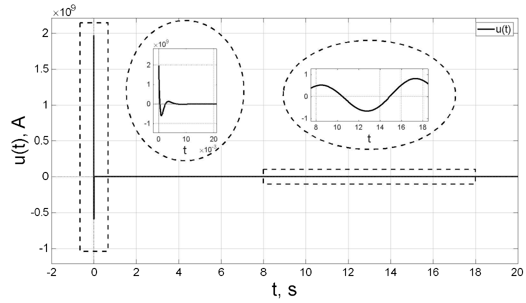

Fig. 4.8. Plot of the control $u(t)$

The graphs show that the transient tracking error is very fast, lasting approximately 0,001 sec. However, in steady-state, the tracking error, as in Experiment 1, does not exceed 0,012 rad. The corrective action $v_4(t)$ initially has spikes in the range $-2\times10^{11} \le v_4(t) \le 0{,}6\times10^{11}$, which generate initial spikes in the control signal $-0{,}5\times10^{9} \le u(t) \le 2\times10^{9}$ A, $0 \le t \le 0{,}001$ sec. In steady-state, in both experiments, the control responds to undamped generalized external signals and is in the range $-0{,}6 \le u(t) \le 0{,}9$ A.

Thus, the simulation results confirm the effectiveness of the developed approach and the advantage of the observer with sat correction compared to a typical linear observer with high gain factors.

5. CONCLUSION

Implementing the approach to synthesizing a tracking system proposed in this article requires only a priori knowledge of the possible ranges of variation of the plant's parameters, external influences, and their derivatives, which depend on operating conditions. If a machine vision system directly measures tracking error, then the electromechanical plant can be freed from all state variable sensors, including the manipulator's angular position sensor. All the necessary information for synthesizing a combined control is provided by a reduced mixed-variable observer during processing of the tracking error signal, which essentially boils down to its dynamic differentiation. Moreover, the closed-loop system order is only doubled, and variable parametric uncertainties and unmatched disturbances are allowed, which distinguishes this approach from the formulation and implementation of tracking systems under uncertainty using adaptive control methods.

Observers with limited corrective actions and, accordingly, limited evaluation signals are in demand in practical applications. The proposed simplified scheme for adjusting the gain factors of piecewise linear corrective actions of a fourth-order observer can be extended, without loss of generality, to single-channel and multi-channel systems of any dimension, reduced to canonical form.

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