

# On the Best Constant in the Jackson–Stechkin Inequality

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**Abstract:** In this article, we obtain an upper bound for the exact constant in the Jackson–Stechkin inequality in the space  $L^p(\mathbb{T})$ ,  $1 \leq p \leq \infty$ . This inequality relates the best approximation of a function by trigonometric polynomials to its generalized modulus of smoothness of order  $r$  with non-constant steps depending on the parameters  $h$  and  $a$  in the space  $L^p(\mathbb{T})$ .

We consider the modulus of smoothness with a non-constant step depending on both the classical step  $h$  and the auxiliary parameter  $a$ . We obtain an upper bound for the exact constant in the Jackson–Stechkin inequality that depends only on the smoothness parameter  $r$  and is independent of the other parameters  $a$  and  $p$ . For  $r = 2$  and  $r = 3$ , we derive more precise bounds than in the general case.

**Keywords:** Jackson–Stechkin inequality, approximation by trigonometric polynomials

## 1. INTRODUCTION

This paper examines the Jackson–Stechkin inequality on the set of trigonometric polynomials. The primary objective is to obtain an upper estimate for the best constant in the Jackson–Stechkin inequality. Prior to reviewing classical results, we present the fundamental definitions.

Let  $\mathbb{T}$  be the one-dimensional torus realized as the interval  $[-\pi, \pi]$  with its endpoints identified. By  $L_p(\mathbb{T})$ ,  $1 \leq p < \infty$ , we denote the space of real-valued functions on  $\mathbb{T}$  whose  $p$ -th power is integrable, equipped with the norm

$$\|f\|_p = \left( \frac{1}{2\pi} \int_{\mathbb{T}} |f(x)|^p dx \right)^{1/p}.$$

For  $p = \infty$ , it is assumed that  $f \in C(\mathbb{T})$  and  $\|f\|_\infty = \max_{x \in \mathbb{T}} |f(x)|$ .

Let  $r \in \mathbb{N}$ ,  $a > 0$ , and  $h \in \mathbb{R}$ . We define the generalized difference operator of order  $r$  by the formula

$$\Delta_h^{a,r}(f, x) = \Delta_h \Delta_{ah} \cdots \Delta_{a^{r-1}h}(f, x),$$

where

$$\Delta_h f(x) = f(x) - f(x + h).$$

The modulus of smoothness is defined by the expression

$$\omega_{a,r}(f, \delta)_p = \sup_{|h| \leq \delta} \|\Delta_h^{a,r} f\|_p.$$

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Let  $T_n$  denote the set of all trigonometric polynomials of degree at most  $n$ .

We denote by

$$E_n(f)_p = \inf_{t \in T_n} \|f - t\|_p$$

the best approximation of the function  $f$  by a trigonometric polynomial of degree not exceeding  $n$ .

Let a natural number  $r$  and an increasing sequence of positive numbers  $\{\lambda_n\}_{n \in \mathbb{N}}$  be given, with  $\lim_{n \rightarrow +\infty} \lambda_n = +\infty$ . Problems concerning the determination of the quantity  $C = C(\lambda_n, p, r, a)$  in the inequality

$$E_{n-1}(f)_p \leq C \cdot \omega_{a,r} \left( f; \frac{1}{\lambda_n} \right)_p. \quad (1.1)$$

have been studied.

The main idea of methods for obtaining estimates of this type consists in constructing a “convenient” trigonometric polynomial  $T_{n-1}(f)$  from the function  $f$ , by means of which one obtains an estimate of the form

$$\|f - T_{n-1}(f)\|_p \leq C \cdot \omega_{a,r} \left( f; \frac{1}{\lambda_n} \right)_p. \quad (1.2)$$

Since  $E_{n-1}(f)_p \leq \|f - T_{n-1}(f)\|_p$ , this also yields an estimate of the form (1.1). As a rule, one takes  $\lambda_n^{-1} = \pi/\frac{\tau}{n}/\frac{\tau}{n+\gamma}$ , where  $\tau > 0$  and  $\gamma > 0$ . In [20], the case  $\lambda_n = n$  was considered, whereas in a number of other works different approximation methods were studied in which the arguments of the modulus of continuity were taken in the form  $\tau/(n + \gamma)$ . For example, in [7] the value  $\lambda_n^{-1} = \frac{2\pi}{3(n+1/2)}$  was used.

The study of various  $\lambda_n$  is of particular interest in connection with the determination of the so-called Chernykh point, beginning from which the best constant in inequality (1.1) attains its global minimum; see [2]. Let us illustrate this with an example. For the best constant in inequality (1.1), in the case  $a = 1$ ,  $p = 2$ , and  $r = 1$ , it follows from [6], [1] that  $C(\lambda_n, 2, 1, 1) = 1/\sqrt{2}$ ,  $0 < \lambda_n \leq n/\pi$  and  $C(\lambda_n, 2, 1, 1) > 1/\sqrt{2}$ ,  $\lambda_n > n/\pi$ . That is, for  $a = 1$ ,  $p = 2$ , and  $r = 1$ , the Chernykh point is  $\lambda_n = n/\pi$ . Determining the Chernykh point presents a significant challenge; nevertheless, results of this type have been obtained for certain parameter values (the resulting  $\lambda_n$  depend on  $r$ ,  $p$ , and  $n$  when  $a = 1$ ).

The case  $a = 1$  and  $r = 1$  in the spaces  $C(\mathbb{T}) = L_\infty(\mathbb{T})$  with  $\lambda_n = 2nk$ ,  $k \in \mathbb{N}$ , was studied in the works of N. P. Korneichuk [9], [10], where he proved that the exact constant  $C_{1,k}$  in the inequality  $E_n(f)_{L_\infty(\mathbb{T})} \leq C_{1,k} \omega_{1,1}(f, \frac{1}{2kn})_\infty$  satisfies  $(1 - \frac{1}{2n})^{\frac{k+1}{2}} \leq C_{1,k} \leq \frac{k+1}{2}$ . The upper estimate for this result can be obtained both by comparison theorems and by approximation using an auxiliary class of functions. The lower estimate was obtained by N. P. Korneichuk on a special sequence of functions.

The case  $a = 1$  and  $r = 2$  in the spaces  $C(\mathbb{T}) = L_\infty(\mathbb{T})$  with  $\lambda_n = 4n$  was obtained in the work of V. V. Zhuk and V. A. Shalaev (see [24], Ch. 8, §3, Theorem 3 and the commentary to Ch. 8, §3):  $E_n(f)_{L_\infty(\mathbb{T})} \leq C_1 \omega_{1,2}(f, \frac{1}{4n})_\infty$ , and they proved that the exact constant  $C_1$  satisfies the inequality  $1 - 1/2n \leq C_1 \leq 1$ .

For the case where  $a \in \mathbb{N}$  is odd,  $a \geq 3$ , and  $p = 2$ , an estimate for the best constant in inequality (1.1), namely  $C = C(\lambda_n, 2, r, a)$ , for  $\lambda_n = \frac{n}{\gamma\pi}$ ,  $\gamma \geq 1$ , was obtained in [13]:

$$\frac{1}{\sqrt{2^r}} \leq C(\lambda_n, 2, r, a) \leq \sqrt{1 + \frac{1}{[\gamma]} \frac{1}{\sqrt{2^r}}}.$$

In the present paper, we study the quantity (for  $\lambda_n = n$ ):

$$\varkappa(r) = \sup_{a,p,n} \sup_{\substack{f \in L_p \\ f \notin T_{n-1}}} \frac{E_n(f)_p}{\omega_{a,r}\left(f, \frac{1}{n}\right)_p}. \tag{1.3}$$

We shall obtain an upper estimate for this quantity. To this end, we construct the approximating trigonometric polynomial  $T_{n-1}(f)$  in (1.2), obtained by means of the Jackson–Stechkin construction. For functions  $f \in L_p(\mathbb{T})$ , we consider the approximating operator of the form

$$S_{a,r,n}(f, x) = \int_{\mathbb{T}} (f(x) - \Delta_t^{a,r}(f, x)) K_{n,r}(t) dt.$$

As the kernel  $K_{n,r}(t)$ , one often chooses an even trigonometric polynomial of degree not exceeding  $n - 1$ . In that case, the expression  $S_{a,r,n}(f, x)$  itself is a trigonometric polynomial of degree at most  $n - 1$ . The kernels  $K_{n,r}(t)$  that we shall use in this paper are the Jackson–Stechkin kernels:

$$J_{r,N}(t) = \frac{1}{\Delta_{r,N}} \left[ \frac{\sin\left(\frac{Nt}{2}\right)}{\sin\left(\frac{t}{2}\right)} \right]^{2r}, \quad \Delta_{r,N} = \int_{-\pi}^{\pi} \left[ \frac{\sin\left(\frac{Nt}{2}\right)}{\sin\left(\frac{t}{2}\right)} \right]^{2r} dt.$$

The parameter  $N$  is chosen depending on  $n$  and  $r$  so that

$$(N - 1)r \leq n < Nr, \quad \frac{n}{N} < r \leq \frac{n}{N - 1}, \quad \text{that is,} \quad N = \left[ \frac{n}{r} \right] + 1.$$

The derivation of an upper estimate for the best approximation via the trigonometric polynomial  $T_{n-1}(f)$  with the aid of the kernel  $K_{n,r}(t)$  was first used by D. Jackson and later, in a more general setting, by S. B. Stechkin. These ideas were developed in the works of S. B. Stechkin [19, 20], and were subsequently extended and refined in a large number of papers [ [16], [7] – [14]. For the Jackson–Stechkin kernels, the following estimates are well known (see, for example, [14]):

$$\begin{aligned} E_n(f)_p &\leq \|f(\cdot) - S_{a,r,n}(f, \cdot)\|_p = \left\| \int_{\mathbb{T}} \Delta_t^{a,r}(f, x) K_{n,r}(t) dt \right\|_p \leq \\ &\leq \int_{\mathbb{T}} \|\Delta_t^{a,r}(f, \cdot)\|_p K_{n,r}(t) dt \leq \int_{\mathbb{T}} \omega_{a,r}(f, |t|)_p K_{n,r}(t) dt \leq \\ &\leq \omega_{a,r}\left(f, \frac{1}{n}\right)_p \int_{\mathbb{T}} (n|t| + 1)^r K_{n,r}(t) dt. \end{aligned}$$

Thus, the upper estimate for quantity (1.3) takes the form

$$\varkappa(r) \leq \int_{\mathbb{T}} (n|t| + 1)^r J_{r,N}(t) dt.$$

Below, we estimate the resulting integral, which will yield an upper bound for the quantity  $\varkappa(r)$ .

In this paper, we first examine the special cases  $r = 2$  and  $r = 3$ . The case  $r = 1$  was previously studied in detail by D. Jackson and is therefore not examined here; for reference, a citation to the corresponding publication is provided (see [16]).

After that, we analyze the case of arbitrary  $r$  and present two upper estimates for the quantity  $\varkappa(r)$ . The special cases  $r = 2$  and  $r = 3$  are treated separately because stronger estimates are obtained there than in the case of general  $r$ .

## 2. AUXILIARY LEMMAS

### Lemma 2.1:

Let  $r \in \mathbb{N}$ ;  $r \geq 2$ ;  $r, k \in \mathbb{N}$ ,  $0 \leq k \leq r$ . Then the following inequality holds:

$$\Delta_{r,N} \geq \frac{2^{1+2r} N^{2r-1}}{\pi^{2r-1}}. \quad (2.4)$$

**Proof.** We shall use the inequality  $\sin\left(\frac{t}{2}\right) \leq \frac{t}{2}$ , which is valid for every  $t \geq 0$ .

$$\begin{aligned} \Delta_{r,N} &= \int_{-\pi}^{\pi} \left[ \frac{\sin\left(\frac{Nt}{2}\right)}{\sin\left(\frac{t}{2}\right)} \right]^{2r} dt \geq 2 \int_0^{\pi} \left[ \frac{\sin\left(\frac{Nt}{2}\right)}{\sin\left(\frac{t}{2}\right)} \right]^{2r} dt \geq \\ &\geq 2 \int_0^{\pi} \left( \sin\left(\frac{Nt}{2}\right) \right)^{2r} \left( \frac{2}{t} \right)^{2r} dt = 2^{1+2r} \int_0^{\pi} \left( \frac{\sin\left(\frac{Nt}{2}\right)}{t} \right)^{2r} dt = \\ &= \left| y = \frac{Nt}{2}, \quad dy = \frac{N}{2} dt \right| = 2^{1+2r} \int_0^{\frac{\pi N}{2}} \left( \frac{\sin y}{\frac{2y}{N}} \right)^{2r} \frac{2}{N} dy = \\ &= 2^{1+2r} \int_0^{\frac{\pi N}{2}} (\sin y)^{2r} \left( \frac{N}{2y} \right)^{2r} \frac{2}{N} dy = 4 N^{2r-1} \int_0^{\frac{\pi N}{2}} \left( \frac{\sin y}{y} \right)^{2r} dy. \end{aligned}$$

From the convexity of the function on the interval  $\left[0, \frac{\pi}{2}\right]$ , we obtain  $\sin y \geq \frac{2y}{\pi}$ , for all  $y \in \left[0, \frac{\pi}{2}\right]$ .

$$\begin{aligned} \Delta_{r,N} &\geq 4 N^{2r-1} \int_0^{\frac{\pi}{2}} \left( \frac{2y}{\pi} \cdot \frac{1}{y} \right)^{2r} dy = \frac{2^{2+2r} N^{2r-1}}{\pi^{2r}} \int_0^{\frac{\pi}{2}} dy = \\ &= \frac{2^{2+2r} N^{2r-1}}{\pi^{2r}} \cdot \frac{\pi}{2} = \frac{2^{1+2r} N^{2r-1}}{\pi^{2r-1}}. \end{aligned}$$

□

### Lemma 2.2:

Let  $r \in \mathbb{N}$ ,  $r \geq 2$ ,  $k \in \mathbb{N}$ ,  $0 \leq k \leq r$ . Then the following estimate holds:

$$I_{k,r} := \int_{\mathbb{T}} |t|^k J_{r,N}(t) dt \leq \frac{\pi^{4r-1}}{N^k \cdot 2^{4r-k-1}} \left( \frac{1}{k+1} + \frac{1}{2r-k-1} \right). \tag{2.5}$$

**Proof.** Since  $J_{r,N}$  and the function  $|t|^k$  are even, we have

$$I_{k,r} = \int_{\mathbb{T}} |t|^k J_{r,N}(t) dt = 2 \int_0^\pi t^k J_{r,N}(t) dt.$$

We use the estimate

$$\frac{1}{\sin\left(\frac{t}{2}\right)} \leq \frac{\pi}{t}, \quad t \in (0, \pi]:$$

$$\begin{aligned} I_{k,r} &= \frac{2}{\Delta_{r,N}} \int_0^\pi t^k \left( \frac{\sin\left(\frac{Nt}{2}\right)}{\sin\left(\frac{t}{2}\right)} \right)^{2r} dt \leq \\ &\leq \frac{2}{\Delta_{r,N}} \int_0^\pi t^k \sin^{2r}\left(\frac{Nt}{2}\right) \left(\frac{\pi}{t}\right)^{2r} dt = \frac{2\pi^{2r}}{\Delta_{r,N}} \int_0^\pi \sin^{2r}\left(\frac{Nt}{2}\right) t^{k-2r} dt. \end{aligned}$$

Next, we apply Lemma 2.1. Let us make the substitution  $y = \frac{Nt}{2}$ ,  $dt = \frac{2}{N} dy$ . Then we obtain

$$\begin{aligned} I_{k,r} &\leq \frac{\pi^{4r-1}}{2^{2r} N^{2r-1}} \int_0^{\frac{\pi N}{2}} \sin^{2r}(y) \left(\frac{2y}{N}\right)^{k-2r} \frac{2}{N} dy = \\ &= \frac{\pi^{4r-1}}{N^k 2^{4r-k-1}} \int_0^{\frac{\pi N}{2}} \frac{\sin^{2r}(y)}{y^{2r-k}} dy \leq \frac{\pi^{4r-1}}{N^k \cdot 2^{4r-k-1}} \int_0^{+\infty} \frac{\sin^{2r}(y)}{y^{2r-k}} dy \leq \\ &\leq \frac{\pi^{4r-1}}{N^k \cdot 2^{4r-k-1}} \left( \int_0^1 \frac{\sin^{2r}(y)}{y^{2r-k}} dy + \int_1^{+\infty} \frac{\sin^{2r}(y)}{y^{2r-k}} dy \right). \end{aligned}$$

Let us estimate the latter integrals separately.

1. From the inequality  $|\sin y| \leq |y|$ ,  $y \in \mathbb{R}$ , it follows that

$$\int_0^1 \frac{(\sin y)^{2r}}{y^{2r-k}} dy \leq \int_0^1 \frac{y^{2r}}{y^{2r-k}} dy = \int_0^1 y^k dy = \frac{1}{k+1}.$$

2. From the inequality  $|\sin y| \leq 1$ ,  $y \in \mathbb{R}$ , it follows that

$$\begin{aligned} \int_1^{+\infty} \frac{\sin^{2r}(y)}{y^{2r-k}} dy &\leq \int_1^{+\infty} \frac{1}{y^{2r-k}} dy = \int_1^{+\infty} y^{-(2r-k)} dy = \\ &= \frac{y^{1-(2r-k)}}{1-(2r-k)} \Big|_1^{+\infty} = \frac{1}{2r-k-1}, \end{aligned}$$

provided that  $2r - k > 1$ .

Thus, we obtain the estimate

$$I_{k,r} \leq \frac{\pi^{4r-1}}{N^k \cdot 2^{4r-k-1}} \left( \frac{1}{k+1} + \frac{1}{2r-k-1} \right).$$

□

### 2.1. The case of $r = 2$ and $r = 3$

#### Theorem 2.1:

Let  $f \in L_p(\mathbb{T})$ ,  $1 \leq p \leq \infty$ . Then for the following estimate holds:

$$E_n(f)_p \leq \varkappa(2) \omega_{a,2} \left( f, \frac{1}{n} \right)_p,$$

where the constant  $\varkappa(2)$  satisfies the inequality

$$\varkappa(2) \leq \frac{23}{96} \pi^7 \approx 723,612.$$

#### Proof.

Let us estimate the corresponding integrals:

$$\xi_k = \int_{\mathbb{T}} (n|t|)^k J_{2,N}(t) dt = \int_{\mathbb{T}} (n|t|)^k \frac{1}{\Delta_{2,N}} \left[ \frac{\sin\left(\frac{Nt}{2}\right)}{\sin\left(\frac{t}{2}\right)} \right]^4 dt, \quad k = 0, 1, 2.$$

Let us estimate the integrals  $\xi_k$ ,  $k = 0, 1, 2$ , using Lemma 2.2:

$$\begin{aligned} \xi_0 &= \int_{\mathbb{T}} J_{2,N}(t) dt \leq \frac{\pi^7}{2^7} \left( 1 + \frac{1}{3} \right) = \frac{\pi^7}{96}, \\ \xi_1 &= \int_{\mathbb{T}} n|t| J_{2,N}(t) dt = n \int_{\mathbb{T}} |t| J_{2,N}(t) dt \leq n \cdot \frac{\pi^7}{N \cdot 2^6} \left( \frac{1}{2} + \frac{1}{2} \right) = \\ &= \frac{n}{N} \cdot \frac{\pi^7}{2^6} < 2 \cdot \frac{\pi^7}{2^6} = \frac{\pi^7}{32}, \\ \xi_2 &= \int_{\mathbb{T}} (n|t|)^2 J_{2,N}(t) dt = n^2 \int_{\mathbb{T}} |t|^2 J_{2,N}(t) dt \leq n^2 \cdot \frac{\pi^7}{N^2 \cdot 2^5} \left( \frac{1}{3} + 1 \right) = \\ &= \left( \frac{n}{N} \right)^2 \cdot \frac{4\pi^7}{3 \cdot 2^5} \leq \frac{\pi^7}{6}. \end{aligned}$$

Therefore, the following estimates hold:

$$\xi_0 \leq \frac{\pi^7}{96}, \quad \xi_1 \leq \frac{\pi^7}{32}, \quad \xi_2 \leq \frac{\pi^7}{6}.$$

Hence it follows that

$$E_n(f)_p \leq (\xi_0 + 2\xi_1 + \xi_2) \omega_{a,2} \left( f, \frac{1}{n} \right)_p.$$

Thus, the upper estimate has been proved:

$$\varkappa(2) \leq \frac{\pi^7}{96} + 2 \cdot \frac{\pi^7}{32} + \frac{\pi^7}{6} = \frac{23}{96} \pi^7 \approx 723,612.$$

□

**Theorem 2.2:**

Let  $f \in L_p(\mathbb{T})$ ,  $1 \leq p \leq \infty$ . Then for  $r = 3$  the following estimate holds:

$$E_n(f)_p \leq \varkappa(3) \omega_{a,3} \left( f, \frac{1}{n} \right)_p,$$

where the constant satisfies the inequality

$$\varkappa(3) \leq \frac{2487\pi^{11}}{20480} \approx 35726,8.$$

**Proof.**

Let us estimate the corresponding integrals:

$$\xi_k = \int_{\mathbb{T}} (n|t|)^k J_{3,N}(t) dt, \quad k = 0, 1, 2, 3.$$

Let us estimate the integrals  $\xi_k$ ,  $k = 0, 1, 2, 3$ , using Lemma 2.2:

$$\xi_0 = \int_{\mathbb{T}} J_{3,N}(t) dt \leq \frac{\pi^{11}}{2^{11}} \left( 1 + \frac{1}{5} \right) = \frac{3\pi^{11}}{5120},$$

$$\begin{aligned} \xi_1 &= \int_{\mathbb{T}} n|t| J_{3,N}(t) dt = n \int_{\mathbb{T}} |t| J_{3,N}(t) dt \leq \\ &\leq n \cdot \frac{\pi^{11}}{2^{10} \cdot N} \left( \frac{1}{2} + \frac{1}{4} \right) = \frac{n}{N} \cdot \frac{3\pi^{11}}{2^{12}} < 3 \cdot \frac{3\pi^{11}}{2^{12}} = \frac{9\pi^{11}}{4096}, \end{aligned}$$

$$\begin{aligned} \xi_2 &= \int_{\mathbb{T}} (n|t|)^2 J_{3,N}(t) dt = n^2 \int_{\mathbb{T}} |t|^2 J_{3,N}(t) dt \leq \\ &\leq n^2 \cdot \frac{\pi^{11}}{N^2 \cdot 2^9} \left( \frac{1}{3} + \frac{1}{3} \right) = \left( \frac{n}{N} \right)^2 \cdot \frac{\pi^{11}}{3 \cdot 2^8} < 9 \cdot \frac{\pi^{11}}{3 \cdot 2^8} = \frac{3\pi^{11}}{256}, \end{aligned}$$

$$\begin{aligned} \xi_3 &= \int_{\mathbb{T}} (n|t|)^3 J_{3,N}(t) dt = n^3 \int_{\mathbb{T}} |t|^3 J_{3,N}(t) dt \leq \\ &\leq n^3 \cdot \frac{\pi^{11}}{N^3 \cdot 2^8} \left( \frac{1}{4} + \frac{1}{2} \right) = \left( \frac{n}{N} \right)^3 \cdot \frac{3\pi^{11}}{2^{10}} < 27 \cdot \frac{3\pi^{11}}{2^{10}} = \frac{81\pi^{11}}{1024}. \end{aligned}$$

Therefore, the following estimates hold:

$$\xi_0 \leq \frac{3\pi^{11}}{5120}, \quad \xi_1 \leq \frac{9\pi^{11}}{4096}, \quad \xi_2 \leq \frac{3\pi^{11}}{256}, \quad \xi_3 \leq \frac{81\pi^{11}}{1024}.$$

Hence it follows that

$$E_n(f)_p \leq (\xi_0 + 3\xi_1 + 3\xi_2 + \xi_3) \omega_{a,3} \left( f, \frac{1}{n} \right)_p.$$

Thus, for  $\varkappa(3)$ , the following upper estimate has been proved:

$$\varkappa(3) \leq \frac{3\pi^{11}}{5120} + 3 \cdot \frac{9\pi^{11}}{4096} + 3 \cdot \frac{3\pi^{11}}{256} + \frac{81\pi^{11}}{1024} = \frac{2487\pi^{11}}{20480} \approx 35726,8.$$

□

### 3. PROOF OF THE MAIN RESULT

#### Theorem 3.1:

Let  $f \in L_p(\mathbb{T})$ ,  $1 \leq p \leq \infty$ , and  $r \in \mathbb{N}$ . Then for the best approximation by trigonometric polynomials of order not exceeding  $n$ , the following inequality holds:

$$E_n(f)_p \leq \varkappa(r) \omega_{a,r} \left( f, \frac{1}{n} \right)_p,$$

where

$$\varkappa(r) \leq \begin{cases} \frac{\pi^{4r-1}}{2^{4r-1}} \left( \frac{2^{r+1}r}{2r-1} + \frac{(1+2r)^{r+1}}{r^2-1} \right), & r \geq 2, \\ 12, & r = 1. \end{cases}$$

**Proof.** Above, an upper estimate for  $\varkappa(r)$  obtained on the basis of the Jackson–Stechkin kernels  $J_{r,N}$  was established:

$$E_n(f)_p \leq \left( \int_{\mathbb{T}} (n|t|+1)^r J_{r,N}(t) dt \right) \omega_{a,r} \left( f, \frac{1}{n} \right)_p.$$

To prove the theorem, we shall obtain an explicit upper estimate for the integral  $\int_{\mathbb{T}} (n|t|+1)^r J_{r,N}(t) dt$ .

Since  $J_{r,N}(t)$  is an even function, we have

$$\begin{aligned} \varkappa(r) &\leq 2 \int_0^\pi (n|t|+1)^r J_{r,N}(t) dt = \\ &= 2 \left( \underbrace{\int_0^{\frac{1}{n}} 2^r J_{r,N}(t) dt}_{\theta_1} + \underbrace{\sum_{k=0}^r \binom{r}{k} \int_{\frac{1}{n}}^\pi (n|t|)^k J_{r,N}(t) dt}_{\theta_2} \right). \end{aligned}$$

Since  $t^k J_{r,N}(t) \geq 0$ , for any subset  $A \subset \mathbb{T}$  the following inequality holds:

$$\int_A t^k J_{r,N}(t) dt \leq \int_{\mathbb{T}} t^k J_{r,N}(t) dt = I_{k,r}.$$

In particular,

$$\int_0^{\frac{1}{n}} J_{r,N}(t) dt \leq \int_0^\pi J_{r,N}(t) dt = \frac{1}{2} I_{0,r}, \quad \int_{\frac{1}{n}}^\pi t^k J_{r,N}(t) dt \leq \int_0^\pi t^k J_{r,N}(t) dt = \frac{1}{2} I_{k,r}.$$

Then, using Lemma 2.2, we obtain:

$$\begin{aligned} \theta_1 &= \int_0^{\frac{1}{n}} 2^r J_{r,N}(t) dt \leq 2^r \int_0^\pi J_{r,N}(t) dt = 2^r \cdot \frac{1}{2} I_{0,r} \leq \\ &\leq 2^{r-1} \cdot \frac{\pi^{4r-1}}{2^{4r-1}} \left(1 + \frac{1}{2r-1}\right) = \frac{\pi^{4r-1}}{2^{3r}} \cdot \frac{2r}{2r-1} = \frac{r \pi^{4r-1}}{2^{3r-1}(2r-1)}. \end{aligned}$$

$$\begin{aligned} \theta_2 &= \sum_{k=0}^r \binom{r}{k} \int_{\frac{1}{n}}^{\pi} (n|t|)^k J_{r,N}(t) dt \leq \frac{1}{2} \sum_{k=0}^r \binom{r}{k} n^k I_{k,r} \leq \\ &\leq \frac{1}{2} \sum_{k=0}^r \binom{r}{k} n^k \frac{\pi^{4r-1}}{N^k 2^{4r-k-1}} \left(\frac{1}{k+1} + \frac{1}{2r-k-1}\right) = \\ &= \sum_{k=0}^r \binom{r}{k} \left(\frac{n}{N}\right)^k \frac{\pi^{4r-1}}{2^{4r-k}} \left(\frac{1}{k+1} + \frac{1}{2r-k-1}\right) \leq \\ &\leq \sum_{k=0}^r \binom{r}{k} r^k \frac{\pi^{4r-1}}{2^{4r-k}} \left(\frac{1}{k+1} + \frac{1}{2r-k-1}\right) = \\ &= \frac{\pi^{4r-1}}{2^{4r}} \sum_{k=0}^r \binom{r}{k} (2r)^k \left(\frac{1}{k+1} + \frac{1}{2r-k-1}\right) = \\ &= \frac{\pi^{4r-1}}{2^{4r}} \left( \underbrace{\sum_{k=0}^r \binom{r}{k} \frac{(2r)^k}{k+1}}_{S_1} + \underbrace{\sum_{k=0}^r \binom{r}{k} \frac{(2r)^k}{2r-k-1}}_{S_2} \right). \end{aligned}$$

Integrating the identity  $(1+x)^r = \sum_{k=0}^r \binom{r}{k} x^k$  from 0 to  $x$ :

$$\sum_{k=0}^r \binom{r}{k} \frac{x^k}{k+1} = \frac{(1+x)^{r+1} - 1}{(r+1)x}, \quad x \neq 0,$$

we obtain

$$S_1 = \frac{(1+2r)^{r+1} - 1}{(r+1)2r}.$$

Now it remains to estimate  $S_2$ . For  $k \in \{0, \dots, r\}$ , we have

$$2r - k - 1 \geq r - 1 \quad \Rightarrow \quad \frac{1}{2r - k - 1} \leq \frac{1}{r - 1}, \quad r > 1,$$

hence

$$S_2 \leq \frac{1}{r-1} \sum_{k=0}^r \binom{r}{k} (2r)^k = \frac{(1+2r)^r}{r-1}.$$

Therefore, we obtain the following estimate for  $\theta_2$ :

$$\theta_2 \leq \frac{\pi^{4r-1}}{2^{4r}} \left( \frac{(1+2r)^{r+1} - 1}{(r+1)2r} + \frac{(1+2r)^r}{r-1} \right).$$

Thus, we obtain the estimate

$$\begin{aligned} \varkappa(r) = 2(\theta_1 + \theta_2) &\leq 2 \left( \frac{r \pi^{4r-1}}{2^{3r-1}(2r-1)} + \frac{\pi^{4r-1}}{2^{4r}} \left( \frac{(1+2r)^{r+1} - 1}{(r+1)2r} + \frac{(1+2r)^r}{r-1} \right) \right) \leq \\ &\leq \frac{r \pi^{4r-1}}{2^{3r-2}(2r-1)} + \frac{\pi^{4r-1}}{2^{4r-1}} \left( \frac{(1+2r)^{r+1}}{(r+1)2r} + \frac{(1+2r)^{r+1}}{(r-1)(1+2r)} \right) \end{aligned}$$

Using the inequality

$$\frac{1}{2r(r+1)} + \frac{1}{(r-1)(2r+1)} \leq \frac{1}{r^2-1}, \quad r \geq 2,$$

we obtain

$$\frac{\pi^{4r-1}}{2^{4r-1}} \left( \frac{2^{r+1}r}{2r-1} + \frac{(1+2r)^{r+1}}{r^2-1} \right), \quad r \geq 2.$$

The case  $r = 1$  is estimated separately and yields  $\varkappa(1) \leq 12$ . Substituting the obtained estimate into Jackson's inequality, we obtain

$$E_n(f)_p \leq \varkappa(r) \omega_{a,r} \left( f, \frac{1}{n} \right)_p,$$

where

$$\varkappa(r) \leq \begin{cases} \frac{\pi^{4r-1}}{2^{4r-1}} \left( \frac{2^{r+1}r}{2r-1} + \frac{(1+2r)^{r+1}}{r^2-1} \right), & r \geq 2, \\ 12, & r = 1. \end{cases}$$

□

Thus, in the course of the proof, we have obtained the explicit estimate

$$E_n(f)_p \leq C_r \omega_{a,r} \left( f, \frac{1}{n} \right)_p,$$

which holds for every  $f \in L_p(\mathbb{T})$ ,  $1 \leq p \leq \infty$ , and every  $n \in \mathbb{N}$ , where the constant  $C_r$  depends only on the smoothness parameter  $r$  and is given by

$$C_r = \begin{cases} \frac{\pi^{4r-1}}{2^{4r-1}} \left( \frac{2^{r+1}r}{2r-1} + \frac{(1+2r)^{r+1}}{r^2-1} \right), & r \geq 2, \\ 12, & r = 1. \end{cases}$$

We emphasize that the obtained estimate is uniform, since it does not depend on the function  $f$  or on the parameter  $n$ , but only on  $r$ .

Recall that the constant  $\varkappa(r)$  in Stechkin's inequality is defined as the exact upper bound (i.e., the least possible constant) for which the inequality

$$E_n(f)_p \leq \varkappa(r) \omega_{a,r} \left( f, \frac{1}{n} \right)_p$$

holds for all  $f \in L_p(\mathbb{T})$  and all  $n \in \mathbb{N}$ .

Since we have proved that this inequality holds with the constant  $C_r$ , it follows immediately that the sharp constant  $\varkappa(r)$  does not exceed  $C_r$ , that is,

$$\varkappa(r) \leq C_r.$$

Thus, we obtain the following explicit upper bound for the Stechkin constant:

$$\varkappa(r) \leq \begin{cases} \frac{\pi^{4r-1}}{2^{4r-1}} \left( \frac{2^{r+1}r}{2r-1} + \frac{(1+2r)^{r+1}}{r^2-1} \right), & r \geq 2, \\ 12, & r = 1. \end{cases}$$

We note that this estimate is, in general, not sharp; however, it provides an explicit dependence of  $\varkappa(r)$  on  $r$ , which is of independent interest in the study of inequalities of the Jackson–Stechkin type.

**Corollary 3.1:**

Let  $r \geq 2$ . Then the Stechkin constant  $\varkappa(r)$  admits the estimate under the conditions of the theorem

$$\varkappa(r) \leq \frac{47\pi^{4r-1}r^{r-1}}{2^{3r+2}}.$$

*Proof*

Starting from the previously obtained estimate,

$$\varkappa(r) \leq \frac{\pi^{4r-1}}{2^{4r-1}} \left( \frac{2^{r+1}r}{2r-1} + \frac{(1+2r)^{r+1}}{r^2-1} \right), \quad r \geq 2,$$

we estimate the expression in parentheses.

Factorizing the dominant term  $2^{r+1}r^{r-1}$ , we write

$$\frac{2^{r+1}r}{2r-1} + \frac{(1+2r)^{r+1}}{r^2-1} = 2^{r+1}r^{r-1} \left[ \frac{r^{2-r}}{2r-1} + \frac{r^2}{r^2-1} \left( 1 + \frac{1}{2r} \right)^{r+1} \right].$$

For  $r \geq 2$ , the following estimates hold:

$$r^{2-r} \leq 1, \quad 2r-1 \geq 3, \quad \frac{r^2}{r^2-1} \leq \frac{4}{3}.$$

Moreover, the sequence  $\left(1 + \frac{1}{2r}\right)^{r+1}$  is decreasing for  $r \geq 2$ , hence

$$\left( 1 + \frac{1}{2r} \right)^{r+1} \leq \left( 1 + \frac{1}{4} \right)^3 = \frac{125}{64}.$$

Therefore,

$$\frac{r^{2-r}}{2r-1} + \frac{r^2}{r^2-1} \left( 1 + \frac{1}{2r} \right)^{r+1} \leq \frac{1}{3} + \frac{125}{48} = \frac{47}{16}.$$

Combining the above estimates, we obtain

$$\frac{2^{r+1}r}{2r-1} + \frac{(1+2r)^{r+1}}{r^2-1} \leq \frac{47}{16} 2^{r+1}r^{r-1}.$$

Substituting this into the estimate for  $\varkappa(r)$  yields

$$\varkappa(r) \leq \frac{\pi^{4r-1}}{2^{4r-1}} \cdot \frac{47}{16} 2^{r+1} r^{r-1} = \frac{47\pi^{4r-1} r^{r-1}}{2^{3r+2}}.$$

This completes the proof. □

**Remark 3.1:**

Let us compare the general estimate obtained in Theorem 3.1 with the bounds derived above for the particular cases  $r = 2$  and  $r = 3$ .

Substituting  $r = 2$  and  $r = 3$  into the general bound of Theorem 3.1, we get

$$\varkappa(2) \leq \frac{94\pi^7}{2^8} \approx 1109.01, \quad \varkappa(3) \leq \frac{423\pi^{11}}{2^{11}} \approx 60765.77.$$

These estimates should be understood as consequences of the universal Stechkin-type bound established in Theorem 3.1. They are valid uniformly with respect to  $r$ , but they are not expected to be optimal for fixed small values of  $r$ .

Indeed, in the particular cases  $r = 2$  and  $r = 3$ , more accurate bounds were obtained above by direct computation of the corresponding integrals. The difference between these two approaches is caused by the intermediate estimates used in the proof of the general theorem: they ensure the universality of the bound for all  $r \in \mathbb{N}$ , but inevitably lead to some overestimation of the constant in individual cases.

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