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# On Sufficient Conditions for Optimality in the Minimum Frequency Maximization Problem for a Shell of Revolution at a Given Weight

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**Abstract:** We consider shallow elastic shells with a given circular boundary and seek an axisymmetric shell shape maximizing the fundamental shell vibration frequency at a given weight. The choice of functionals considered during optimal design is part of the formulation of optimization problems. The most typical problems in the theory of optimal design of compressed structures are the problems of maximizing the critical $\omega_0$ ($\omega_0$ is the minimum eigenvalue) for a given weight of the structure and the problems of minimizing the weight under the constraint $\omega_0 \geq \mu$, where $\mu$ is a given number. We study an optimal control problem described by an eigenvalue problem for a system of differential equations with variable coefficients some of which are nonintegrable near zero. To solve this problem, in this paper we derive sufficient optimality conditions under the assumption that the frequency domain functional is Fréchet differentiable and the necessary conditions in the optimization problem are satisfied. It is proved that if the necessary conditions are satisfied, then the sufficient conditions are realized as well. Using the sufficient conditions, as an application, we also determine the optimal shape $f = f(r)$ for the case in which $h(r) = h_0$.

***Keywords:*** second Fréchet differentiability, frequency functional, sufficient condition, optimality

## 1. INTRODUCTION

The interest in research in the field of optimal design has grown significantly owing to the rapid development of aviation and space technology, shipbuilding, and precision engineering. Optimal design has led to significant weight reductions in aircraft and improved mechanical properties of structures. Optimization problems also arise in civil engineering. Therefore, research in this area has clear practical applications.

Optimal design problems are also of theoretical interest. The optimal design theory has received considerable development in connection with research into the problem of finding the shape of a compressed rod (column) that has minimal weight and can withstand a given load without loss of stability. This problem was posed by J.L. Lagrange [1].

The present paper contents the study of the minimum frequency maximization problem for a shell of revolution under a given shell weight. It is proved that if the necessary conditions are satisfied, then the sufficient conditions are realized as well.

Consider shallow elastic shells with a given circular boundary. Our aim is to determine the shape $f(r)$ of an axisymmetric shell that has the maximum fundamentalal frequency for a given weight of the shallow shell. We study an optimal control problem described by an eigenvalue problem for a system of differential equations with variable coefficients some of which are nonintegrable near zero.

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To solve this problem, in this paper we derive sufficient optimality conditions under the assumption that the frequency domain functional is Fréchet differentiable and the necessary conditions in the optimization problem are satisfied. Sufficiency conditions implementing the solution of the stated problem are obtained as sufficient conditions for the conditional maximization problem, namely, the minimum vibration frequency maximization problem for a shell with a given weight. Using the sufficient conditions we also determine the optimal shape $f = f(r)$ for the case in which $h(r) = h_0$.

## 2. STATEMENT OF THE PROBLEM

Let a function $W(r)$ define the displacement amplitude of the midsurface points in the axial direction, let $\varphi = \varphi(r)$ be the stress characterizing the tangential displacement, and let the variable $r \in [0, b]$ (where $b > 0$ is a given constant) indicate the current radius.

Under certain assumptions, the transverse vibrations of a shell of revolution are described by a system of two differential equations [2-4].

Let us give a rigorous statement of the optimization problem from the theory of thin shells: find all pairs $(\lambda, u)$, where $\lambda \in \mathbb{R}$ and $u = (\varphi, W)$, such that

$$(Lu)_1 \equiv (rDW'')'' + \left(\left(\nu D' - \frac{D}{r}\right) W'\right)' + (f'\varphi)' = \lambda r h \rho W, \tag{1}$$

$$(Lu)_2 \equiv (ar\varphi')' - \left(\frac{a}{r} + \nu a'\right) \varphi - f'W' = 0, \tag{2}$$

where $f(r)$ determines the shape of the midsurface of the shell of revolution, $\rho(r) > 0$ is the specific gravity of the shell material, $h(r)$ is the a given shell thickness satisfying the condition

$$0 < h_0 \leq h(r) \leq h_1, \quad h_0, h_1 = \text{const}, \tag{3}$$

$D(r)$ is the bending rigidity, $D(r) \geq D_0 > 0$, defined as

$$D(r) = \frac{Eh^3(r)}{12(1-\nu^2)}, \quad a(r) = \frac{1}{Eh(r)}, \tag{4}$$

$E > 0$ is the Young modulus, and $\nu$ $(-1 < \nu < 0.5)$ is the Poisson ratio.

The eigenvalue problem is the system (1), (2) supplemented with the boundary conditions

$$W'|_{r=0} = \left[(rDW'')' + \left(\nu D' - \frac{D}{r}\right) W'\right]\Big|_{r=0} = 0, \tag{5}$$

$$W|_{r=b} = W'|_{r=b} = 0, \tag{6}$$

$$a(\nu\varphi - r\varphi')|_{r=0} = a(\nu\varphi - r\varphi')|_{r=b} = 0. \tag{7}$$

Let us denote by $\lambda_1 = w^2(f')$ the minimum eigenvalue of problem (1)-(7), where $w(f')$ is the minimum vibration frequency of the shell.

Note that the boundary conditions (5)-(7) correspond to the case of a clamped edge at $r = b$. In the case of a hinged edge, conditions (6) acquire the form

$$W|_{r=b} = -D\left(W'' + \nu \frac{W'}{r}\right)\Big|_{r=b} = 0. \tag{8}$$

Thus, there are two eigenvalue problems, (1)-(7) and (1)-(5), (7), (8). Let us examine problem (1)-(7) in detail; for problem (1)-(5), (7), (8) the argument is similar.

 

Consider the standard shell weight minimization problem

$$\lambda_1(f') = \omega^2(f') \to \sup \tag{9}$$

under the condition

$$J(f') = \int_0^b 2\pi r h\rho \sqrt{1 + (f'(r))^2}\, dr = J_0. \tag{10}$$

Let $p(r) = f'(r)$ be a fixed control element. Then, when studying problem (1)–(7), (9), (10), the question arises: How to solve problem (1)–(7) for this control $p(r)$? Since some coefficients of system (1), (2) are not integrable on the interval $[0, b]$, weighted spaces are introduced in Sec. 2 (see [5]).

## 3. WEIGHTED SPACES. GENERALIZED SOLUTION

Let us introduce the weighted Hilbert spaces $\widetilde{H}^1[0, b]$ and $\widetilde{H}^2[0, b]$ with inner products

$$\langle u, v\rangle_{\widetilde{H}^1} = \int_0^b \left( r u' v' + \frac{uv}{r} \right) dr,$$
$$\langle u, v\rangle_{\widetilde{H}^2} = \int_0^b \left( r u'' v'' + \frac{u'v'}{r} + uv \right) dr.$$

Here $u, v, u', v', u'', v'' \in L_{1,\mathrm{loc}}(0, b)$. The corresponding norms are determined in a standard manner.

**Theorem 3.1:**
*[5,Theorem 1] Any function $u \in \widetilde{H}^1[0, b]$ can be identified with a continuous function on the interval $[0, b]$, any function $v \in \widetilde{H}^2[0, b]$ can be identified with a continuously differentiable function on $[0, b]$, and moreover,*

$$\max_{[0,b]} |u| \le c_1 \|u\|_{\widetilde{H}^1[0,b]}, \quad u(0) = 0,$$

$$\max_{[0,b]} (|v'| + |v|) \le c_2 \|v\|_{\widetilde{H}^2[0,b]}, \quad v'(0) = 0,$$

*where $c_1$ and $c_2$ are positive constants.*

Let us introduce the following linear subspace of the product space $\widetilde{H}^1 \times \widetilde{H}^2$

$$V = \{v \colon v = (v_1, v_2) \in \widetilde{H}^1 \times \widetilde{H}^2,\ v_2(b) = v_2'(b) = 0\}.$$

The mechanical meaning of the boundary conditions included in the definition of $V$ is that the shell is clamped along the entire boundary.

On the space $V \times V$, consider the bilinear form

$$B(u, v) = \int_0^b \Big[ r D u_2'' v_2'' + (D - \nu D' r)\frac{u_2' v_2'}{r} - p u_1 v_2' + a r u_1' v_1' + (a + \nu a' r)\frac{u_1 v_1}{r} + p u_2' v_1 \Big] dr - a\nu u_1 v_1 \Big|_0^b$$

generated by the differential operator and the boundary conditions of problem (5)-(7).

To study the eigenvalue problem, we first need to study the boundary value problem for the equations

$$(Lu)_1 = s_1, \tag{11}$$

$$(Lu)_2 = s_2 \tag{12}$$

with the boundary conditions (5)-(7).

We need the following definition.

**Definition 3.1:**
*For any $s = (s_1, s_2)$, $s_1, s_2 \in L_2[0, b]$, a function $u \in V$ is called a generalized solution of problem (5)–(7), (11), (12), if the relation*

$$B(u, v) = \langle s_1, v_2\rangle_{L_2} + \langle s_2, v_1\rangle_{L_2}, \quad v = (v_1, v_2) \in V, \tag{13}$$

*holds true.*

As shown in [5, Theorem 5], the bilinear form $B(u, v)$ is positive definite and bounded if two of the following three conditions are satisfied:

1. $D(r),\ D(r) - \nu D'(r)r,\ a'(r) \in L_\infty[0, b],\ p \in L_1[0, b]$.
2. $D(r) \geq D_0 > 0,\ D(r) - \nu D'(r)r \geq D_{10} > 0,\ a(r) \geq a_0 > 0$.
3. $t(r) = h(r)\rho(r) \in L_\infty[0, b],\ t(r) \geq t_0$.

**Theorem 3.2:**
*[5, Theorem 2] Assume that conditions 1 and 2 are satisfied. Then for any pair of functions $s = (s_1, s_2)^\top$, $s_1, s_2 \in L_2[0, b]$, problem (13) has a unique solution $u \in V$.*

*In addition, the solution $u$ satisfies the estimate*

$$\|u\|_V \leq c\big(\|s_1\|^2_{L_2} + \|s_2\|^2_{L_2}\big)^{1/2}, \quad c = \text{const}, \quad c > 0.$$

Take any function $\psi \in L_2[0, b]$. Then the condition

$$B(u, v) = \left\langle \sqrt{rt}\psi, v_2 \right\rangle_{L_2}, \quad v \in V,$$

obtained from (13) for $s = (\sqrt{rt}\psi, 0)^\top$, defines a function $u \in V$. Thus, the operator $G : \psi \in L_2[0, b] \to u = G\psi \in V$ is defined, and one has the estimate

$$\|u\|_{\widetilde{H}^1\times\widetilde{H}^2} = \|G\psi\|_{\widetilde{H}^1\times\widetilde{H}^2} \leq c\big\|\sqrt{rt}\psi\big\|_{L_2}.$$

The operator $F_1\psi = \sqrt{rt}(G\psi)_2$ as an operator from the space $L_2[0, b]$ into itself is compact and self-adjoint [5].

It is easily seen that if $\mu$ is an eigenvalue of the operator $F_1$ and $\psi$ is the corresponding eigenfunction, i.e., $F_1\psi = \mu\psi$, then $\lambda = 1/\mu$ and $u = G\psi$ are an eigenvalue and an eigenfunction, respectively, of the problem

$$B(u, v) = \lambda\langle rtu_2, v_2\rangle_{L_2}, \quad v \in V. \tag{14}$$

Indeed,

$$B(u, v) = B(G\psi, v) = \left\langle \sqrt{rt}\psi, v_2 \right\rangle_{L_2} = \frac{1}{\mu}\left\langle rt(G\psi)_2, v_2\right\rangle_{L_2} = \lambda\langle rtu_2, v_2\rangle_{L_2}, \quad v \in V.$$

In the same way, one can prove that if $\lambda$ is an eigenvalue and $u$ is the corresponding eigenfunction of problem (14), then $\mu = 1/\lambda$ and $\psi = \sqrt{rt}u_2$ are an eigenvalue and an eigenfunction of the operator $F_1$.

As stated in [5,Theorem 6], if conditions 1-3 are satisfied, then

1. There exists a sequence of positive eigenvalues $\{\lambda_k\}$ of problem (14) such that $\lambda_k \to +\infty$ as $k \to +\infty$.
2. To each eigenvalue $\lambda_k$, there correspond only finitely many linearly independent eigenfunctions in $V$.

## 4. AUXILIARY RESULTS

Let us study some properties of solutions of the problem

$$B(u, v; p) = \lambda \langle rtu_2, v_2 \rangle_{L_2}, \quad v \in V. \tag{15}$$

The following theorems are important for analysis. We need the gradient from[6] and the second Fre'chet derivative of the functional $\lambda_1(p)$ on the tangent subspace from[7].

**Theorem 4.1:**
*Assume that the functions $D(r)$, $a(r)$, and $t(r)$ satisfy conditions 1-3 and $p \in L_2[0, b]$. Let $\lambda_1(p)$ be the minimum eigenvalue of problem (15). Then its Fréchet derivative exists in Hilbert space and is equal to*

$$\frac{d\lambda_1(p)}{dp} = \frac{-2u_2'(p)u_1(p)}{\langle rtu_2(p), u_2(p)\rangle_{L_2}},$$

*where $u(p) = (u_1(p), u_2(p))$ nonzero function in space $S_1$.*

**Theorem 4.2:**
*Let the functions $D(r)$, $a(r)$, and $t(r)$ satisfy conditions 1-3, let $p \in L_2[0, b]$, and let $\lambda_1(p)$ be the minimum eigenvalue of problem (15). The second derivative Fréchet exists for the Hilbert space $L_2[0, b]$ and is defined as*

$$\left\langle \frac{d^2\lambda_1(p)}{dp^2} h, h \right\rangle_{L_2} = \left\langle \frac{2u_2'(p)u_1(p)}{p\langle rtu_2(p), u_2(p)\rangle_{L_2}} h, h \right\rangle_{L_2} \tag{16}$$

*for any $h$ lying in the subspace*

$$L = \left\{ h : h \in L_2,\ \left\langle \frac{-2u_2'(p)u_1(p)}{\langle rtu_2(p), u_2(p)\rangle_{L_2}}, h \right\rangle_{L_2} = 0 \right\}.$$

We also need these relations from [6],

$$\int_0^{\bar{r}} [ar(u_1'(p))^2 + (a + \nu a' r)\frac{(u_1(p))^2}{r} + pu_2'(p)u_1(p)\,]dr = 0 \quad \forall \bar{r} \in (0, b]. \tag{17}$$

and from [7, relation (24)] for $p + \delta p = \bar{p} \in L_2[0, b]$

$$\left.\frac{\bar{p}u_2'(\bar{p})}{u_1(\bar{p})}\right|_r = \left.\frac{pu_2'(p)}{u_1(p)}\right|_r \quad r \in (0, b]. \tag{18}$$

 

## 5. OPTIMALITY CONDITIONS

Consider the following optimization problem with the variable $p$ in the Banach space $X = L_2[0, b]$:

$$\mathcal{B}_0(p(\cdot)) = \lambda_1(p) \to \sup, \tag{19}$$

$$F(p(\cdot)) = J(p) = \int_0^b 2\pi rt\sqrt{1+p^2}\,dr = J_0, \tag{20}$$

where $J_0$ is a given number and $\lambda_1(p)$ is the minimum eigenvalue of the problem

$$B(u, v; p) = \lambda\langle rtu_2, v_2\rangle_{L_2}, \quad v \in V. \tag{21}$$

Let us construct the Lagrange function for the problem (19)-(21):

$$\mathcal{L}\big(p(\cdot), \alpha_0, \alpha_1\big) = \alpha_0\lambda_1(p) + \alpha_1 J(p).$$

**Theorem 5.1:**
*Let the functions $D(r)$, $a(r)$ and $t(r)$ satisfy conditions 1-3. Assume that $p_*(r)$ is a minimum point in problem (19)-(21). Let $\lambda_1(p_*)$ be the minimum eigenvalue of problem (15). Then there exist a Lagrange multiplier $\bar{\alpha}_1$ and a function $\bar{p_*}(r)$ such that*

$$-\frac{u_2'(p_*(r))u_1(p_*(r))}{\langle rtu_2(p_*(r)), u_{2_*}(p_*(r))\rangle_{L_2}} + \overline{\alpha}_1\frac{\pi rtp_*(r)}{\sqrt{1+p_*(r)^2}} = 0. \tag{22}$$

*Proof*
Previously, we established that the functional $B_0(p)$ has a gradient

$$\mathcal{B}_0'(p) = -\frac{u_2'(p)u_1(p)}{\langle rtu_2(p), u_2(p)\rangle_{L_2}}\,, \quad p \in L_2[0, b]. \tag{23}$$

It can be easily established that the functional $F(p)$ has a gradient

$$ImF'(p) = \frac{\pi rtp}{\sqrt{1+p^2}}\,, \quad p \in L_2[0, b]. \tag{24}$$

Then, in view of the Language multiplier rule there exist Lagrange multipliers $\alpha_0$, $\alpha_1$ and a function $p_*(r)$ such that

$$\alpha_0\mathcal{B}_0'(p_*) + \alpha_1 F'(p_*) = 0. \tag{25}$$

Let us prove that the range $ImF'(p_*)$ is a closed set, and moreover

$$ImF'(p_*) = \mathcal{R}.$$

Two cases are possible:

i) $u_2'(p_*)u_1(p_*) \equiv 0 \qquad \forall r \in [0, b]$,
ii) There exists $c, \quad c \in (0, b]$, such that

$$\int_0^c u_2'(p_*)u_1(p_*)\,dr \neq 0.$$

In case i) from relation (25) we have that $\alpha_1 F'(p_*) = 0$, so in this case the relation (22) holds Now in the second case two cases are possible:

1) $p_* \equiv 0 \qquad \forall r \in [0, b]$,
2) There exists $c, \quad c \in (0, b]$, such that

$$\int_0^c \frac{2\pi r t p_*}{\sqrt{1 + p_*^2}} \, dr \neq 0.$$

In case 1) we prove that

$$u_1(p_*) \equiv 0. \tag{26}$$

Indeed, from (15) with $p = p_*$, $u = u(p_*), v = (v_1, 0)^T$ we obtain the relation

$$\int_0^b (a r u_1'(p_*) v_1' + (a + \nu a' r) \frac{u_1(p_*) v_1}{r} + p_* u_2'(p_*) v_1) \, dr - a\nu u_1(p_*) v_1 |_0^b = 0 \qquad \forall v \in V. \tag{27}$$

Further, let us substitute $v_1 = u_1(p_*)$ into (27). Using 1) and the positive definiteness of the bilinear form $B(u, v)$ we obtain

$$u_1(p_*) \equiv 0. \tag{28}$$

In this case, the relation (22) holds, so the theorem is proved. Now let us prove that, in the second case, the equation

$$\int_0^b \frac{2\pi r t p_*}{\sqrt{1 + p_*^2}} \cdot x \, dr = q \tag{29}$$

has a solution $x = x(r)$ for each $q \in R$.

Indeed, one can readily verify that

$$x(r) = \begin{cases} \dfrac{q}{\int_0^c \frac{2\pi r t p_*}{\sqrt{1+p_*^2}} \, dr}, & 0 \leq r \leq c \\ 0, & c < r \leq b. \end{cases}$$

$$Im F'(p_*) X = \mathbb{R}.$$

Consequently, according to (29) we have that $Im F'$ coincides with $\mathbb{R}$. Thus, the regularity condition for the mapping $F$ is satisfied. Hence it follows that $\alpha_0 \neq 0$.

In view of the Language multiplier rule the proof the theorem is complete. □

## 6. SUFFICIENT CONDITIONS FOR OPTIMALITY

Now, let us prove that if the necessary optimality conditions are satisfied, then the sufficient conditions are realized as well.

**Theorem 6.1:**
*Let the functions $D(r)$, $a(r)$ and $t(r)$ satisfy conditions 1-3. Assume that there exist a Lagrange multiplier $\bar{\alpha}_1$ and a function $\bar{p}(r)$ such that*

$$-\frac{u_2'(\bar{p}) u_1(\bar{p})}{\langle r t u_2(\bar{p}), u_2(\bar{p}) \rangle_{L_2}} + \bar{\alpha}_1 \frac{\pi r t \bar{p}}{\sqrt{1 + \bar{p}^2}} = 0. \tag{30}$$

*Then $\bar{p}(r)$ is a local maximum in problem (19-21).*

 

*Proof*
Let $\bar{p}+h$ be a feasible element; i.e., $F(\bar{p}+h)=0$.

Let us estimate $\lambda_1(\bar{p}+h)$ . We have $\lambda_1(\bar{p}+h)=\lambda_1(\bar{p}+h)+\bar{\alpha}_1 F(\bar{p}+h)=\mathcal{L}(\bar{p}+h,1,\bar{\alpha}_1)$. The estimate is based on a straightforward expansion of $\lambda_1(\bar{p})$,

$$\lambda_1(\bar{p}+h)=\mathcal{L}(\bar{p}+h,1,\bar{\alpha}_1)=\mathcal{L}(\bar{p},1,\bar{\alpha}_1)+\frac{d\mathcal{L}(\bar{p},1,\bar{\alpha}_1)}{dp}[h]+\frac{1}{2}\frac{d^2\mathcal{L}(\bar{p},1,\bar{\alpha}_1)}{dp^2}[h,h]+r_1(h),$$

where the remainder $r_1(h)$ is $o(\|h\|^2_{L_2})$.

From this, with regard to (16),(23) and (24), we derive

$$\begin{aligned}\lambda_1(\bar{p}+h)&=\mathcal{L}(\bar{p},1,\bar{\alpha}_1)+\left\langle\bar{\alpha}_1\frac{\pi r t\bar{p}}{\sqrt{1+\bar{p}^2}}-\frac{u_2'(\bar{p})u_1(\bar{p})}{\langle rtu_1(\bar{p}),u_1(\bar{p})\rangle_{L_2}},h\right\rangle_{L_2}\\&\quad+\frac{1}{2}\left\langle\left(\bar{\alpha}_1\frac{\pi rt}{\sqrt{(1+\bar{p}^2)^3}}+\frac{u_2'(\bar{p})u_1(\bar{p})}{\bar{p}\langle rtu_2(\bar{p}),u_2(\bar{p})\rangle_{L_2}}\right)h,h\right\rangle_{L_2}+r_1(h)\qquad(31)\\&=\lambda_1(\bar{p})+\frac{1}{2}\left\langle\bar{\alpha}_1\left(\frac{\pi rt}{\sqrt{(1+\bar{p}^2)^3}}+\frac{\pi rt}{\sqrt{1+\bar{p}^2}}\right)h,h\right\rangle_{L_2}+r_1(h).\end{aligned}$$

In addition, let us substitute $p=\bar{p},v_1=u_1(\bar{p})$ in (27). Now using the positive definiteness of the bilinear form $B(u,v)$ and the optimality condition we obtain $\bar{\alpha}_1<0$. Thus , since

$$\left\langle\left(\frac{\pi rt}{\sqrt{(1+\bar{p}^2)^3}}+\frac{\pi rt}{\sqrt{1+\bar{p}^2}}\right)h,h\right\rangle_{L_2}>0$$

for nontrivial $h$, it follows by (31) that $\lambda_1(\bar{p}+h)<\lambda_1(\bar{p})$. The proof of the theorem is complete. □

## 7. EXAMPLE

Let us determine the optimal shape $f=f(r)$ of the shell for the case in which the function $h(r)=h_0$ is given and one must find $p=\bar{p}(r)$, where $\bar{p}(r)=f'(r)$. Then $D(r)=D_0$, $a(r)=a_0$, and $t=t_0$.

Let us find the solution of problem (21) for $\lambda=\lambda_0$ and $p=p_0(r)$. We seek the solution in the form

$$p_0(r)=cr,$$
$$u_1(p_0)|_r=dr^3+\bar{d}r^5,$$
$$u_2(p_0)|_r=r^4-2b^2r^2+b^4,$$

where $c$, $d$, and $\bar{d}$ are constants.

We have

$$\lambda_0=64D_0/(b^4t_0),\qquad c=-\sqrt{a_0\lambda_0 t_0},$$
$$d=b^2c/(4a_0),\qquad \bar{d}=c/(6a_0).$$

Now, from the optimality condition (22) and the relation (17) for $p=\bar{p}$ we obtain

$$\bar{\alpha}_1\frac{\pi rt_0\bar{p}^2}{\sqrt{1+\bar{p}^2}}=\frac{\bar{p}u_2'(\bar{p})u_1(\bar{p})}{\langle rt_0u_2(\bar{p}),u_2(\bar{p})\rangle_{L_2}}=-\frac{a_0(r(u_1'(\bar{p}))^2+(u_1(\bar{p}))^2/r)}{\langle rt_0u_2(\bar{p}),u_2(\bar{p})\rangle_{L_2}}.\qquad(32)$$

In conclusion, assuming $\bar{\alpha}_1 < 0$, we normalize the function $u = u(p_0)$ as follows:

$$\langle rt_0 u_2(\bar{p}), u_2(\bar{p})\rangle_{L_2} = -\frac{1}{\bar{\alpha}_1}.$$

Then from the relation (18) for $p = p_0$ and from (30) we have the equality

$$\frac{\pi t_0 \bar{p}^2}{\sqrt{1+\bar{p}^2}} = -p_0 u_2'(p_0) u_1(p_0) = -4c(dr^4 + \bar{d}r^6)(r^2 - b^2),$$

whence we find

$$\bar{p}(r) = \bar{q}(r)\sqrt{\frac{1+\sqrt{(\bar{q}(r))^2+4}}{2}},$$

where $\bar{q}(r) = c^2(-2r^8/3 - r^6/3 + r^4)/a_0$.

The figure shows the graph of the optimal shape $f = f(r)$ of the shell.

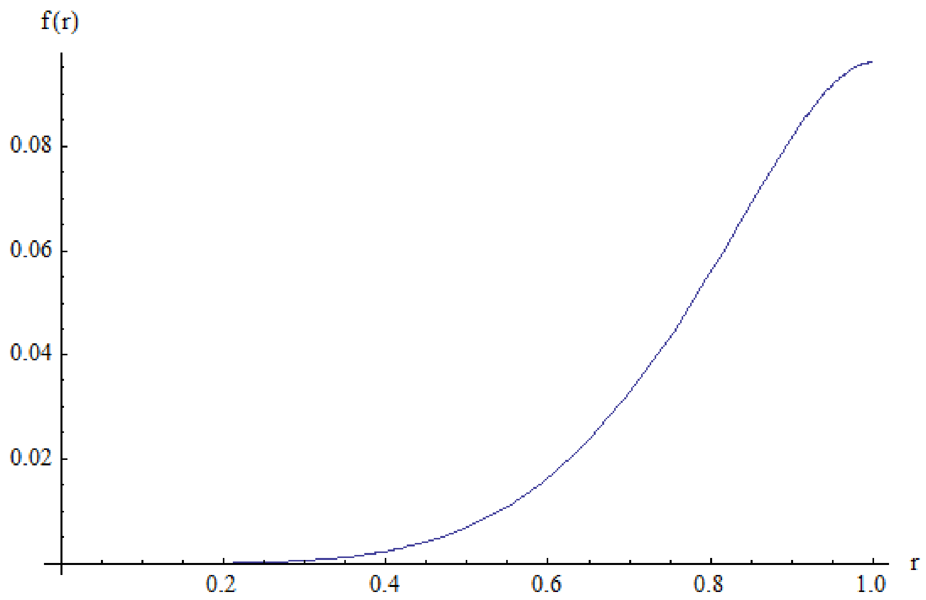


Fig. 7.1. Graph of the optimal shape.

## 8. CONCLUSION

We considered shallow elastic shells with a given circular boundary and seek an axisymmetric shell shape maximizing the fundamental shell vibration frequency at a given weight. It is proved that if the necessary conditions are satisfied, then so are the sufficient conditions. Using the sufficient conditions we also determine the optimal shape $f = f(r)$ for the case in which $h(r) = h_0$.

REFERENCES

 

1. Lagrange, J. L. (1770–1773) *Sur la figure des colonnes*. Paris, France: Miscellanea Taurinensia.
2. Grigolyuk, E.I. (1955) Nonlinear oscillations and stability of shallow rods and shells, *Izv. AN USSR. Otd. Tekh. Nauk*, 3, 33–68.
3. Timoshenko, S. (1959) *Theory of Plates and Shells*. New York, NY: McGraw-Hill.
4. Timoshenko, S., *Strength of Materials. Part 2: Advanced Theory and Problems*. Melbourne, FL: Krieger.
5. Arabyan, M. H. (2019) Boundary-value problems and associated eigenvalue problems for systems describing vibrations of a rotation shell, *New York J. Math.*, **25**, 1350–1367.
6. Arabyan, M.H. (2023) On the optimality conditions in the weight minimization problem for a shell of revolution at a given vibration frequency, *Differ. Equations*, **59**(9), 1262–1279.
7. Arabyan, M. H. (2025) On sufficient conditions in the weight minimization problem for a shell of revolution at a given vibration frequency, *Differ. Equations*, **61**(8), 1071–1081.

 