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Derivations for a Special Case of Inverse Semigroup Algebras

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Abstract: This paper is aimed at studying the derivations of inverse semigroups algebras. We adress a special case of semigroups which are constructed as a disjoint union of two groups. We use a categorical method, constructing a category and expressing the derivations in terms of characters on said category. The resulting derivation algebra is proven to be divisible into direct sum of derivation algebras for two groups.

Keywords: derivations, inverse semigroups, semigroup algebra, groupoid

1. INTRODUCTION

The objects of this paper's research are derivartions of inverse semigroup algebras.

The derivation problem for associative algebras was studied in the works of Johnson [7] and Losert [8]. The papers by D.E.Bagha, M.Amini [2] and N.D.Gilbert [6] approached the question of module derivations of inverse semigroups.

The derivations of group algebras were largely explored in the works of A.A.Arutyunov, A.S.Mishchenko and A. I. Shtern [3–5]. These papers propose a categorical approach for the derivations study, which consists in constructing a suitable category for given associative algebra. For groups, such category is a groupoid of adjoint action. The derivations can be presented via characters on said groupoid, which are functions from morphisms of the groupoid to complex numbers that preserve composition. In the paper [1] such approach is used for studying the derivations of algebras of Malstev semigroups.

Inverse semigroups can be considered as an object generalising the concept of groups. In this paper we consider a special case of inverse semigroups constructed with two groups that are connected by a homomorphism. We use the categorical approach to study the derivations on their assotiative algebras. The resulting category turns out to be the disjoint union of groupoids for the two groups, and the character space divides into the direct sum of the respective spaces for groups, which implies that the same holds for the Lie algebras of derivations. This result is stated in Theorem 4.1.

2. PRELIMINARIES

Definition 2.1:

A semigroup S is called an inverse semigroup *if for each element $x \in S$ there is a unique inverse $y \in S$ in the sense that $xyx = x$ and $yxy = y$.*

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Definition 2.2:
Let G, H be groups and $\varphi : G \to H$ a homomorphism.

Define a semigroup $G \overset{\varphi}{\sqcup} H$ on the set $G \sqcup H$ with an operation $\circ$:

$$u \circ v = \begin{cases} u \cdot_G v, & u, v \in G \\ u \cdot_H v, & u, v \in H \\ \varphi(u) \cdot_H v, & u \in G, v \in H \\ u \cdot_H \varphi(v), & u \in H, v \in G \end{cases} \tag{2.1}$$

This semigroup is inverse, since for every element $g \in G \overset{\varphi}{\sqcup} H$ its inverse g^{-1} in the respective group is an inverse in the semigroup sense.

We consider the semigroup algebra $\mathbb{C}\left[G \overset{\varphi}{\sqcup} H\right]$. Notice that e_G, which is the identity element of the group G, is neutral for the operation $\circ$, therefore $G \overset{\varphi}{\sqcup} H$ is a monoid, and the algebra $\mathbb{C}\left[G \overset{\varphi}{\sqcup} H\right]$ is a unital algebra.

Notations 2.1:
We will use letters from the beginning of the alphabet (a, b, c) for elements from G and the letters from the middle (p, q, r) for elements from H. We will also assume for $\cdot$ to denote $\cdot_G, \cdot_H$ or multiplication by number if it is clear from context. The use of the operation $\circ$ will always be written explicitly.

Notation 2.1:
We will use the following notation for coefficients. For a given semigroup algebra I, its element $u \in I$ is presented as

$$u = \sum_{h \in I} \lambda_I^u h.$$

Definition 2.3:
Denote as $\widetilde{\varphi}$ the following linear transformation on $\mathbb{C}\left[G \overset{\varphi}{\sqcup} H\right]$:

$$\widetilde{\varphi}(g) = \begin{cases} \varphi(g), & g \in G \\ g, & g \in H \end{cases}$$

Notice that for an arbitrary $u \in \mathbb{C}\left[G \overset{\varphi}{\sqcup} H\right]$ the following holds: $e_H \circ u = u \circ e_H = \widetilde{\varphi}(u)$.

Proposition 2.1:
If $u = v \circ p$ or $u = p \circ v$, where $v \in \mathbb{C}\left[G \overset{\varphi}{\sqcup} H\right], p \in H$, then for each $h \in G$ holds $\lambda_h^u = 0$.

Proof
Let $I = \mathbb{C}\left[G \overset{\varphi}{\sqcup} H\right]$.

$u = v \circ p = \sum\limits_{h \in I} \lambda_h^v (h \circ p)$. By definition (2.1), with $p \in H$ for all $h \in I$ the product $h \circ p$ lies in the group H. The case $u = p \circ v$ is considered similarly. □

Proposition 2.2:
Both group identity elements commute with all elements of $\mathbb{C}\left[G \overset{\varphi}{\sqcup} H\right]$:

$$\forall u \in \mathbb{C}\left[G \overset{\varphi}{\sqcup} H\right] : u \circ e = e \circ u, \quad \textit{where } e \in \{e_G, e_H\}.$$

Proof
Let $e = e_G$. As mentioned earlier, e_G is neutral for the operation $\circ$: for $a \in G$ we have $a \circ e_G = a \cdot e_G = a$, and for $p \in H : p \circ e_G = p \cdot e_H = p$. Hence, e_G commutes with all elements of $G \overset{\varphi}{\sqcup} H$, and therefore with all elements of $\mathbb{C}\left[G \overset{\varphi}{\sqcup} H\right]$.

For $e = e_H$ we have $u \circ e_H = \widetilde{\varphi}(u) \cdot e_H = \widetilde{\varphi}(u) = e_H \circ u$. □

3. DERIVATIONS

Definition 3.1:
Given a semigroup S and its semigroup algebra $\mathbb{C}[S]$, a linear map

$$d : \mathbb{C}[S] \to \mathbb{C}[S],$$

satisfying the Leibniz rule

$$d(u \circ v) = d(u) \circ v + u \circ d(v), \quad u, v \in \mathbb{C}[S]$$

is called a derivation of the algebra $\mathbb{C}[S]$.

Notation 3.1:
We will denote the derivation coeffitients as such: $d(u) = \sum\limits_{h \in S} d^u_h h$, i.e. $d^u_h = \lambda^{d(u)}_h$.

Proposition 3.1:
Consider a semigroup $G \overset{\varphi}{\sqcup} H$ and d — a derivation of the algebra $\mathbb{C}\left[G \overset{\varphi}{\sqcup} H\right]$. Then, for an idempotent $e \in \{e_G, e_H\}$ holds $d(e) = 0$.

Proof
$d(e) = d(e \circ e) = d(e) \circ e + e \circ d(e) = 2d(e) \circ e$. Last equality follows from Proposition 2.2.

1) Let $e = e_G$. Then $d(e) = 2d(e) \circ e = 2d(e)$, hence $d(e) = 0$.
2) $e = e_H$. Then $d(e) = 2d(e) \circ e = 2\widetilde{\varphi}(d(e))$. This implies that $\lambda^{d(e)}_a = 0$ for $a \in G$, therefore $\widetilde{\varphi}(d(e)) = d(e)$. Then $d(e) = 2d(e)$, hence $d(e) = 0$.

□

Proposition 3.2:
Let d be a derivation of $\mathbb{C}\left[G \overset{\varphi}{\sqcup} H\right]$. Then

$$\forall p \in H, a \in G \quad d^p_a = 0. \tag{3.2}$$

Proof
$d(p) = d(p \circ e_H) = d(p) \circ e_H + \underbrace{p \circ d(e_H)}_{=0} = d(p) \circ e_H$.

Proposition 2.1 implies $\lambda^{d(p)}_a = 0$, i.e. $d^p_a = 0$. □

Definition 3.2:
For a homomorphism $\varphi : G \to H$ the preimage *of an element $p \in H$ is*

$$\varphi^{-1}(p) = \{a \in G \mid \varphi(a) = p\}.$$

This set can be empty (if p is not in $\operatorname{Im} \varphi$).

Lemma 3.1:
Let d *be a derivation of* $\mathbb{C}\left[G \overset{\varphi}{\sqcup} H\right]$. *Then*

$$\forall a \in G, p \in H \quad d_p^a = d_p^{\varphi(a)} - \sum_{b \in \varphi^{-1}(p)} d_b^a. \tag{3.3}$$

Proof
Notice that $d(\varphi(a)) = d(a \circ e_H) = d(a) \circ e_H = \widetilde{\varphi}(d(a))$.

By grouping the terms from G in $d(a)$, we get

$$d(a) = \sum_{q \in \operatorname{Im} \varphi} \left(\sum_{b \in \varphi^{-1}(q)} d_b^a b + d_q^a q \right) + \sum_{r \notin \operatorname{Im} \varphi} d_r^a r.$$

Then

$$d(\varphi(a)) = \widetilde{\varphi}(d(a)) = \sum_{q \in \operatorname{Im} \varphi} \left(\sum_{b \in \varphi^{-1}(q)} d_b^a + d_q^a \right) q + \sum_{r \notin \operatorname{Im} \varphi} d_r^a r,$$

which implies

$$d_p^{\varphi(a)} = \begin{cases} \sum\limits_{b \in \varphi^{-1}(p)} d_b^a + d_p^a, & p \in \operatorname{Im} \varphi \\ d_p^a, & p \notin \operatorname{Im} \varphi. \end{cases} \tag{3.4}$$

Notice that for $p \notin \operatorname{Im} \varphi$ its preimage is empty, hence $\sum\limits_{b \in \varphi^{-1}(p)} d_b^a = 0$. Therefore, the formula (3.4) can be rewritten as $d_p^{\varphi(a)} = \sum\limits_{b \in \varphi^{-1}(p)} d_b^a + d_p^a$ for all $p \in H$, which is equivalent to the formula (3.3). □

3.1. General Formula for Derivations

Notation 3.2:
Further we will use the notation $[\cdot]$ *for indicator, i.e.*

$$[\textit{condition}] = \begin{cases} 1, & \textit{condition holds} \\ 0, & \textit{condition does not hold.} \end{cases}$$

Lemma 3.2:
Let d *be a linear operator on* $\mathbb{C}[I]$, *where* $I = G \overset{\varphi}{\sqcup} H$. *Then* d *is a derivation if and only if the following conditions are met simultaneously: the relations* (3.2), (3.3) *hold, and also for each pair* $u, v \in I$ *the following formulas hold true (everywhere we assume that* $a, b \in G$; $p, q \in H$*).*

$$u, v \in G \qquad d_{ua=bv}^{uv} = d_a^v + d_b^u \tag{3.5}$$

$$u, v \in G \qquad d_{\varphi(u)p=q\varphi(v)}^{uv} = d_p^v + d_q^u \tag{3.6}$$

$$u \in G, v \in H \qquad d_{\varphi(u)p=qv}^{\varphi(u)v} = d_p^v + d_q^{\varphi(u)} \tag{3.7}$$

$$u \in H, v \in G \qquad d_{up=q\varphi(v)}^{u\varphi(v)} = d_p^{\varphi(v)} + d_q^u \tag{3.8}$$

$$u, v \in H \qquad d_{up=qv}^{uv} = d_p^v + d_q^u \tag{3.9}$$

Proof

For arbitrary $u, v \in I$ write out the Leibniz rule:

$$d(u \circ v) = d(u) \circ v + u \circ d(v),$$

which is equivalent to the following:

$$\sum_{h \in I} d_h^{u \circ v} h = \sum_{h' \in I} d_{h'}^u (h' \circ v) + \sum_{h'' \in I} d_{h''}^u (v \circ h''). \tag{3.10}$$

The equality holds if and only if for each element of I the coefficients on both sides of the equation coincide. We equate the elements on the right to h and consider all possible cases of u, v, h being in groups G and H.

- $h = h' \circ v$.
  1) $h \in G$. This is equivalent to $h', v \in G$ and $h' = hv^{-1}$.
  2) $h \in H$. Three cases are possible:
     (i) $h', v \in H$. Then, $h' = hv^{-1}$.
     (ii) $h' \in G, v \in H$. Then, $h' \circ v = \varphi(h')v = h \Leftrightarrow \varphi(h') = hv^{-1} \Leftrightarrow h' \in \varphi^{-1}(hv^{-1})$.
     (iii) $h' \in H, v \in G$. Then, $h = h' \circ v = h'\varphi(v) \Leftrightarrow h' = h(\varphi(v))^{-1} = h\varphi(v^{-1})$.
- $h = u \circ h''$. Similarly, we get:
  1) $h \in G \Longleftrightarrow u \in G,\ h'' = u^{-1}h$
  2) $h \in H$ is equvalent to one of these cases:
     (i) $u \in H \Longleftrightarrow h'' = u^{-1}h$ or $h'' \in \varphi^{-1}(u^{-1}h)$;
     (ii) $u \in G \Longleftrightarrow h'' = \varphi(u^{-1})h$.

These results can be combined into one formula:

$$\begin{aligned} d_h^{u \circ v} = \quad & [u \in G] \cdot \Big([h \in G] \cdot d^v_{u^{-1}h} + [h \in H] \cdot d^v_{\varphi(u^{-1})h} \Big) \quad + \\ & [v \in G] \cdot \Big([h \in G] \cdot d^u_{hv^{-1}} + [h \in H] \cdot d^v_{h\varphi(v^{-1})} \Big) \quad + \\ & [u \in H] \cdot [h \in H] \cdot \left(d^v_{u^{-1}h} + \sum_{c \in \varphi^{-1}(u^{-1}h)} d^v_c \right) \quad + \\ & [v \in H] \cdot [h \in H] \cdot \left(d^u_{hv^{-1}} + \sum_{c \in \varphi^{-1}(hv^{-1})} d^u_c \right). \end{aligned} \tag{3.11}$$

Consider the term in the third line: $d^v_{u^{-1}h} + \sum\limits_{c \in \varphi^{-1}(u^{-1}h)} d^v_c$. If $v \in H$, then by (3.2) we have $d^v_c = 0$, hence the term equals $d^v_{u^{-1}h}$. If $v \in G$, then by (3.3) the term equals $d^{\varphi(v)}_{u^{-1}h}$. In both cases we can say that it equals $d^{\widetilde{\varphi}(v)}_{u^{-1}h}$. Similar transformation can be carried out for the last string. By performing said transformations, we get the formula

$$\begin{aligned} d_h^{u\circ v} = \quad & [u \in G] \cdot \Big([h \in G] \cdot d_{u^{-1}h}^{v} + [h \in H] \cdot d_{\varphi(u^{-1})h}^{v} \Big) \quad + \\ & [v \in G] \cdot \Big([h \in G] \cdot d_{hv^{-1}}^{u} + [h \in H] \cdot d_{h\varphi(v^{-1})}^{u} \Big) \quad + \\ & [h \in H] \cdot \Big([u \in H] \cdot d_{u^{-1}h}^{\widetilde{\varphi}(v)} + [v \in H] \cdot d_{hv^{-1}}^{\widetilde{\varphi}(u)} \Big) . \end{aligned} \tag{3.12}$$

Let us present the formula in a different way for different cases of u and v being in the groups G and H. By doing so, we will get the relations (3.5)–(3.9).

1) $u, v \in G$. In this case the formula (3.12) is transformed into

$$d_h^{uv} = [h \in G] \cdot (d_{u^{-1}h}^{v} + d_{hv^{-1}}^{u}) + [h \in H] \cdot \Big(d_{\varphi(u^{-1})h}^{v} + d_{h\varphi(v^{-1})}^{u} \Big) .$$

Consider $h \in G$. Define $u^{-1}h = a, hv^{-1} = b$, then $h = ua = bv$. By changing the indices in the formula, we get

$$d_{ua=bv}^{uv} = d_a^v + d_b^u,$$

which is the relation (3.5).

Consider $h \in H$. Define $\varphi(u^{-1})h = a, h\varphi(v^{-1}) = b$, then $h = \varphi(u)a = b\varphi(v)$. So, we get

$$d_{\varphi(u)p=q\varphi(v)}^{uv} = d_p^v + d_q^u,$$

which is the relation (3.6).

2) $u, v \in H$. In this case the formula (3.12) is transformed into

$$d_h^{uv} = d_{u^{-1}h}^{v} + d_{hv^{-1}}^{u}, \quad h \in H.$$

Define $u^{-1}h = p, hv^{-1} = q$, then $h = up = qv$. The formula then transforms into

$$d_{up=qv}^{uv} = d_p^v + d_q^u,$$

which is the relation (3.9).

In the case $h \in H$ it must hold that $d_h^{uv} = 0$. This corresponds with (3.2).

3) $u \in G, v \in H$. In this case

$$d_h^{u\circ v} = d_h^{\varphi(u)v} = [h \in G] \cdot d_{u^{-1}h}^{v} + [h \in H] \cdot \Big(d_{\varphi(u^{-1})h}^{v} + d_{hv^{-1}}^{\varphi(u)} \Big) .$$

If $h \in G$, the relation (3.2) implies $d_{u^{-1}h}^{v} = 0$. Therefore, we have

$$d_h^{\varphi(u)v} = d_{\varphi(u^{-1})h}^{v} + d_{hv^{-1}}^{\varphi(u)}.$$

Define $\varphi(u^{-1})h = p, hv^{-1} = q$, then $h = \varphi(u)p = qv$. The formula then transforms into

$$d_{\varphi(u)p=qv}^{\varphi(u)v} = d_p^v + d_q^{\varphi(u)},$$

which is the relation (3.7).

4) $u \in H, v \in H$. This case is similar to the precious one. Namely,

$$d_h^{u \circ v} = d^{u\varphi(v)} = [h \in G] \cdot \underbrace{d^u_{hv^{-1}}}_{=0 \text{ by (3.2)}} + [h \in H] \cdot \left(d^{\varphi(v)}_{u^{-1}h} + d^u_{h\varphi(v^{-1})} \right).$$

Define $u^{-1}h = p, h\varphi(v^{-1}) = q$, then $h = up = q\varphi(v)$, and the formula transforms into

$$d^{u\varphi(v)}_{up=q\varphi(v)} = d^{\varphi(v)}_p + d^u_q,$$

which is the relation (3.8).

□

Proposition 3.3:
The relation (3.6) *is implied by* (3.3), (3.5) *and* (3.9).

Proof
Consider (3.6) and express all coefficients via (3.3). Thus, we get

$$d^{\varphi(u)\varphi(v)}_{\varphi(u)p=q\varphi(v)} - \sum_{a \in u\varphi^{-1}(p)=\varphi^{-1}(q)v} d^{uv}_a = d^{\varphi(u)}_q - \sum_{b \in \varphi^{-1}(q)} d^u_q + d^{\varphi(v)}_p + \sum_{c \in \varphi^{-1}(p)} d^v_p.$$

By moving and grouping the terms, we get

$$d^{\varphi(u)\varphi(v)}_{\varphi(u)p=q\varphi(v)} - d^{\varphi(v)}_p - d^{\varphi(u)}_q = \sum_{\substack{c \in \varphi^{-1}(p) \\ b \in \varphi^{-1}(q)}} \left(d^{uv}_{uc=bv} - d^v_b - d^u_c \right).$$

The left parts equals 0 by (3.9), and the right part equals 0 by (3.5). Thus, the equality is proven.

□

Lemma 3.3:
Let d *be a linear operator on* $\mathbb{C}[I]$*, where* $I = G \overset{\varphi}{\sqcup} H$*. Then,* d *is a derivation if an only if the following relations hold simultaneously:* (3.2), (3.3), (3.5) *and* (3.9).

Proof
Notice that the relations (3.7) and (3.8) are special cases of the relation (3.9). With this and Proposition 3.3, the statement being proven follows from Lemma 3.2. □

4. CATEGORY CONTRUCTION

4.1. Group Case

The work [4] proposes a category suitable for studying the derivations of group algebras.

Definition 4.1:
Let G be a group. Construct a small category Γ_G*, associated with the group G.*

1. *The set of objects:* $Obj(\Gamma_G) = G$.
2. *The set of morphisms:* $Hom(\Gamma_G) = G \times G$*. The morphism* $\psi = (u, v)$ *maps from* $s(\psi) = v^{-1}u$ *into* $t(\psi) = uv^{-1}$.

3. *For any pair of morphisms* $\psi = (u_1, v_1)$ *and* $\xi = (u_2, v_2)$*, such that* $t(\psi) = s(\xi)$*, their composition is defined as follows:*

$$\xi \circ \psi = (v_2 u_1, v_2 v_1).$$

The category Γ_G *is called the groupoid of the adjoint action of the froup* G*.*

Definition 4.2:
A function $\chi : Hom(\Gamma) \to \mathbb{C}$ *is called a* character *on the groupoid* Γ*, if for any two composable morphisms* ψ, ξ *holds*

$$\chi(\xi \circ \psi) = \chi(\xi) + \chi(\psi). \tag{4.13}$$

Remark 4.1:
For convenience, while denoting a character on the morphism (u, v)*, we will write* $\chi(u, v)$ *instead of* $\chi((u, v))$*.*

Definition 4.3:
A character Γ_G *is called* locally finite*, if for any element* $v \in G$ *the inequality* $\chi(u, v) \neq 0$ *holds only for finitely many elements* $u \in G$*.*

Notations 4.1:
The linear space of derivations of an algebra $\mathbb{C}[I]$ *will be denoted as* $Der(\mathbb{C}[I])$*.*

The linear space of locally finite characters on a groupoid Γ *will be denoted as* $X(\Gamma)$*.*

Definition 4.4:
For a group G, we define the following linear mapping:

$$\begin{aligned} \mathfrak{c}_G : \quad & Der(\mathbb{C}[G]) \to X(\Gamma_G) \\ d \mapsto \chi_d \qquad & \qquad \chi_d(v, u) = d_v^u. \end{aligned} \tag{4.14}$$

Proposition 4.1:
For a group G, the linear space $Der(\mathbb{C}[G])$ *is a Lie algebra with Lie bracket defined as commutator:* $\{u, v\} = [u, v] = uv - vu$*.*

The linear space $X(\Gamma_G)$ *is also a Lie algebra with Lie bracket defined as* $\{\chi_{d_1}, \chi_{d_2}\} = \chi_{[d_1, d_2]}$*.*

Moreover, the mapping (4.14) *is an isomorphism of said algebras:* $Der(\mathbb{C}[G]) \overset{\mathfrak{c}_G}{\cong} X(\Gamma_G)$*.*

Proof
The mapping $\mathfrak{c}_G$ is a bijection between derivations and locally finite characters by [4, Propositions 1,2]. The Lie algebra structure in $X(\Gamma_G)$ is inherited from $Der(\mathbb{C}[G])$ because of the way the bracket in $X(\Gamma_G)$ is defined. □

Proposition 4.2:
Given a group G, the following formula holds for characters in $X(\Gamma_G)$*:*

$$\{\chi_{d_1}, \chi_{d_2}\}(a, g) = \sum_{h \in G} \left(\chi_{d_1}(a, h)\chi_{d_2}(h, g) - \chi_{d_2}(a, h)\chi_{d_1}(h, g)\right). \tag{4.15}$$

Proof
This proposition is contained in [1, Proposition 2.4]. □

4.2. Special Semigroup Case

4.2.1. Category We construct a category, following the approach described above.

Definition 4.5:
Given two groupoids Γ_1 *and* Γ_2*, we define their* disjoint union $\Gamma = \Gamma_1 \sqcup \Gamma_2$ *as follows:*

$$Obj(\Gamma_1 \sqcup \Gamma_2) = Obj(\Gamma_1) \sqcup Obj(\Gamma_2) \quad Hom(\Gamma_1 \sqcup \Gamma_2) = Hom(\Gamma_1) \sqcup Hom(\Gamma_2),$$

i.e. as a union of the groupoids without additional morphisms between components.

We introduce a correspondence between derivations and characters similar to (4.14).

Definition 4.6:
For a semigroup $I = G \overset{\varphi}{\sqcup} H$ *and a groupoid* $\Gamma = \Gamma_G \sqcup \Gamma_H$ *we define the following linear mapping:*

$$\begin{array}{ll} \mathfrak{c}_I : & Der(\mathbb{C}[I]) \to X(\Gamma) \\ d \mapsto \chi_d & \quad \chi_d(v,u) = d_v^u, \quad u, v \in G \text{ or } u, v \in H. \end{array} \tag{4.16}$$

Lemma 4.1:
Let $I = G \overset{\varphi}{\sqcup} H$*;* Γ_G, Γ_H *be groupoids of adjoint action for the groups G and H, and a groupoid* $\Gamma = \Gamma_G \sqcup \Gamma_H$*. Then, the mapping* $\mathfrak{c}_I$*, defined by* (4.16)*, is an isomorphism of linear spaces.*

Proof
Notice that χ_d is indeed a locally finite character: the condition (4.13) follows from (3.5) and (3.9), while it is locally finite since d is an operator on $\mathbb{C}[I]$.

Let us find $\operatorname{Ker} \mathfrak{c}_I$. Let χ be a zero character, i.e. $\chi(h,u) = 0$ for all h, u. Then, for an arbitrary $d \in \mathfrak{c}_I^{-1}(\chi)$ holds $d_h^u = 0$ for all pairs (h,u), where $h, u \in G$ or $h, u \in H$. Then, (3.3) and (3.2) imply that $d_h^u = 0$ for all u, h, therefore d is a trivial derivation. Hence, $\operatorname{Ker} \mathfrak{c}_I$ is trivial, therefore $\mathfrak{c}_I$ is injective.

Now, let us show that $\mathfrak{c}_I$ is surjective. Indeed, any arbitrary locally finite character $\chi \in X(\Gamma)$ has a preimage. Namely, it is a linear operator d defined as such:

1) for $a, b \in G : d_b^a = \chi(b,a)$;
2) for $p, q \in H : d_q^p = \chi(q,p)$;
3) for $a \in G, p \in H : d_a^p = 0$, and d_p^a is defined by (3.3).

The conditions (3.2) and (3.3) are satisfied from the construction of d. Since χ is a character, the conditions (3.5) and (3.9) hold. Then, by Lemma 3.3, d is indeed a derivation.
□

4.3. Operation on Characters

We define an operation on the character space as such:

$$\{\chi_{d_1}, \chi_{d_2}\} \overset{def}{=} \chi_{[d_1,d_2]}. \tag{4.17}$$

$Der(\mathbb{C}[I])$ with a commutator operation $[\cdot,\cdot]$ is a Lie algebra as a space of derivations of an associative algebra. Then, $X(\Gamma)$ with an operation defined by (4.17) is also a Lie algebra.

Then, we can strengthen the statement of Lemma 4.1.

Proposition 4.3:
Given a semigroup $I = G \overset{\varphi}{\sqcup} H$ *and a groupoid* $\Gamma = \Gamma_G \sqcup \Gamma_H$*, the mapping* $\mathfrak{c}_I$ *defined by* (4.16) *is an isomorphism of Lie algebras:*

$$Der\left(\mathbb{C}\left[G \overset{\varphi}{\sqcup} H\right]\right) \overset{\mathfrak{c}_I}{\cong} X(\Gamma_G \sqcup \Gamma_H).$$

Proof
The mapping $\mathfrak{c}_I$ is a linear space isomorphism by Lemma 4.1. Moreover the mapping preserves the operation:

$$\mathfrak{c}_I\left([d_1,d_2]\right)=\chi_{[d_1,d_2]}=\left\{\chi_{d_1},\chi_{d_2}\right\}=\left\{\mathfrak{c}_I(d_1),\mathfrak{c}_I(d_2)\right\}.$$

□

Let us obtain a formula for characters, similar to (4.15).

Proposition 4.4:

Consider a semigroup $I=G\overset{\varphi}{\sqcup}H$, a groupoid $\Gamma=\Gamma_G\sqcup\Gamma_H$; derivations $\alpha,\beta\in Der(\mathbb{C}[I])$ and characters $\xi,\eta\in X(\Gamma)$, where $\xi=\chi_\alpha,\eta=\chi_\beta$.

Then, the followig formulas hold:

$$\begin{aligned}\left\{\xi,\eta\right\}(c,a)&=\sum_{b\in G}\left(\xi(c,b)\eta(b,a)-\eta(c,b)\xi(b,a)\right),\quad a,c\in G\\ \left\{\xi,\eta\right\}(r,p)&=\sum_{q\in H}\left(\xi(r,q)\eta(q,p)-\eta(r,q)\xi(q,p)\right),\quad p,r\in H\end{aligned}\tag{4.18}$$

Proof
Notice that the morphisms in Γ are of form (h,g), where $h,g\in G$ or $h,g\in H$, therefore the fromulas are given in such form.

Consider the case $p,r\in H$. Because of (3.2), we have

$$\alpha(p)=\sum_{q\in H}\alpha^p_q q=\sum_{q\in H}\xi(q,p)\cdot q,\qquad \beta(p)=\sum_{q\in H}\beta^p_q q=\sum_{q\in H}\eta(q,p)\cdot q.$$

Similarly, using (3.2) for $\alpha(q)$ and $\beta(q)$, we get

$$(\alpha\beta)(p)=\sum_{q\in H}\eta(q,p)\left(\sum_{r\in H}\xi(r,q)\cdot r\right),\qquad (\beta\alpha)(p)=\sum_{q\in H}\xi(q,p)\left(\sum_{r\in H}\eta(r,q)\cdot r\right).$$

Changing the summation order, we obtain the formula for the commutator:

$$[\alpha,\beta](p)=(\alpha\beta-\beta\alpha)(p)=\sum_{r\in H}\left(\sum_{q\in H}\left(\xi(r,q)\eta(q,p)-\eta(r,q)\xi(q,p)\right)\right)\cdot r.$$

Then, for characters $\{\xi,\eta\}$ we have

$$\{\xi,\eta\}(r,p)=[\alpha,\beta]^p_r=\sum_{q\in H}\left(\xi(r,q)\eta(q,p)-\eta(r,q)\xi(q,p)\right).$$

Now let us consider $a,c\in G$. In this case, we are only interested in the coefficients of elements from G, therefore we will use the notation L_H for an arbitrary linear combination of elements from H. So, we have

$$\alpha(a)=\sum_{b\in G}\alpha^a_b b+\sum_{p\in H}\alpha^a_p=\sum_{b\in G}\xi(b,a)\cdot b+L_H,$$

$$\beta(a)=\sum_{b\in G}\beta^a_b b+\sum_{p\in H}\beta^a_p=\sum_{b\in G}\eta(b,a)+L_H.$$

Notice that by (3.2) for an arbitrary $d \in Der(\mathbb{C}[I])$ holds $d(L_H) = L_H$, therefore

$$(\alpha\beta)(a) = \alpha\left(\sum_{b\in G}\eta(b,a)\cdot b + L_H\right) = \sum_{c\in G}\sum_{b\in G}\xi(c,b)\eta(b,a)\cdot c + L_H,$$

$$(\beta\alpha)(a) = \beta\left(\sum_{b\in G}\xi(b,a)\cdot b + L_H\right) = \sum_{c\in G}\sum_{b\in G}\eta(c,b)\xi(b,a)\cdot c + L_H.$$

Finally, for $a, c \in G$ we obtain the required formula

$$\{\xi,\eta\}(c,a) = \sum_{b\in G}\left(\xi(c,b)\eta(b,a) - \eta(c,b)\xi(b,a)\right).$$

$\square$

4.3.1. Division into direct sum

Notation 4.1:
For a groupoid $\Gamma = \Gamma_G \sqcup \Gamma_H$ we define a mapping representing the character space as a direct sum :

$$\begin{array}{ll} \pi_{G,H}: & X(\Gamma_G \sqcup \Gamma_H) \to X(\Gamma_G) \oplus X(\Gamma_H) \\ \chi \mapsto (\chi^G, \chi^H) & \chi^G(a,b) = \chi(a,b), \quad a,b \in G \\ & \chi^H(p,q) = \chi(p,q), \quad p,q \in H \end{array} \tag{4.19}$$

Theorem 4.1:
For a semigroup $I = G \overset{\varphi}{\sqcup} H$ the following sequence of isomorphisms holds:

$$Der\left(\mathbb{C}\left[I\right]\right) \overset{\mathfrak{c}_I}{\cong} X(\Gamma_G \sqcup \Gamma_H) \overset{\pi_{G,H}}{\cong} X(\Gamma_G) \oplus X(\Gamma_H) \overset{\mathfrak{c}_G^{-1},\mathfrak{c}_H^{-1}}{\cong} Der(\mathbb{C}[G]) \oplus Der(\mathbb{C}[H]).$$

Proof
The first isomorphism holds via Proposition 4.3, while the last isomorphism holds via Proposition 4.1. Now notice that $X(\Gamma_G \sqcup \Gamma_H)$ can obviously be divided into direct sum of character algebras on two groupoids by $\pi_{G,H}$. $\square$

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