

# Application of the Homotopy Perturbation Sumudu Transformation Method Approach to the Time-Fractional Cahn-Hilliard Equation with a Double-Obstacle Potential

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**Abstract:** The model for the Cahn-Hilliard equation that we present in this study incorporates a time-fractional derivative, as defined by Caputo. The Sumudu transform, Homotopy perturbation, and He's polynomials are integrated in a novel way to discover the solutions for this fractional nonlinear problem in both one and two dimensions. We show the findings from numerical simulations for each dimension in order to examine the patterns and behaviors of the system in different scenarios.

**Keywords:** Cahn-Hilliard, Sumudu Transform, He polynomials, Homotopy perturbation method, Time-Fractional Derivatives.

## 1. INTRODUCTION

In the mid-1950s, physicists Cahn and Hilliard undertook an extensive review of the previous century's research on the interfacial energies of non-uniform systems [6]. During this time, there was a notable surge in the number of experiments conducted to record surface tension [7].

Their research has significant implications for both fluid mechanics and industrial applications. The Cahn-Hilliard theory is a crucial tool for modeling phase separation in multiphase fluid flows, providing valuable insights into the behavior of immiscible fluids. Furthermore, in industrial settings, the Cahn-Hilliard theory is applied to the custom design of materials with specific properties via controlled phase separation processes. This is evident in fields like materials science [8], cell biology [9], and image processing [10, 11], as well as in more recent initiatives [12].

Fractional calculus, a branch of applied mathematics, studies derivatives and integrals of non-integer orders. Its applications encompass a wide range of scientific disciplines including fluid mechanics, acoustics, chemistry, electromagnetism, control systems, and viscoelasticity. The mathematical representation of many physical systems often includes fractional differential equations, which are common in various fields of physics and engineering. Fractional-time partial differential equations (PDEs) play a significant role in viscoelasticity [13] and plasticity [14], as well as in applications like image processing [15]. Moreover, recent research has emphasized their importance in studying contact problems

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that involve thermo-viscoelastic friction [16, 17].

The homotopy perturbation method (HPM) was pioneered by Ji-Huan He [18]. Since its introduction, numerous authors have applied this method to address a wide range of both linear and nonlinear problems. Momani and Odibat utilized the homotopy perturbation method to solve fractional-order differential equations, showcasing its effectiveness as an alternative analytical technique. Furthermore, the homotopy perturbation method (HPM) has been integrated with transformations such as Laplace, Sumudu [19, 22, 23], and Elzaki [20] to solve ordinary differential equations and fractional-order partial differential equations. Subsequently, Asiru [21] developed solutions using the Sumudu transform for convolution-type integral equations and discrete dynamic systems.

In this study, we investigate a Cahn-Hilliard model incorporating a double-obstacle potential. This article introduces two innovations: firstly, the incorporation of fractional-order derivatives for the time evolution. Fractional derivatives extend beyond integer-order derivatives, offering a global correlation that better captures the historical dependence of system function evolution. In contrast, integer-order derivatives are more localized and less adept at describing historical dependencies.

The second innovation involves applying a hybrid method that combines homotopy perturbation and the Sumudu transform in both one and two dimensions. This method produces approximate series solutions that closely approximate the exact solutions. Additionally, we analyze the Ginzburg-Landau energy (total energy) to examine the system's equilibrium behavior.

The paper is structured as follows: Section 2 outlines the derivation of the fractional time Cahn-Hilliard equation from the theory of continuous mixing using a double-obstacle potential. Here, we introduce the Caputo fractional time derivative operator. In Section 3, we introduce the HPSTM (Homotopy Perturbation Sumudu Transformation Method), which combines Homotopy and Sumudu methods, to address a nonlinear and non-homogeneous fractional-order partial differential equation involving temporal derivatives. Sections 4 and 5 focus on applying the HPSTM method to the Cahn-Hilliard model, emphasizing its computational efficiency compared to traditional methods. We present numerical results in one and two dimensions to demonstrate performance improvements. Additionally, we analyze the Ginzburg-Landau energy to illustrate the temporal decay of the system's total energy.

## 2. TIME-FRACTIONAL CAHN-HILLIARD MODEL

Consider an alloy consisting of a binary mixture of two constituents, denoted as  $\mathcal{A}$  and  $\mathcal{B}$ , with concentrations  $c_{\mathcal{A}}$  and  $c_{\mathcal{B}}$ , respectively. This alloy occupies a spatial domain  $\Omega$  in  $\mathbb{R}^d$ , where  $d$  can be 1, 2, or 3, and is bounded by  $\partial\Omega$ . The distribution of the mixture's composition at any time  $t$  and position  $x$  in space is represented by a continuous function known as the order parameter  $\Phi$ , defined as:

$$\begin{aligned}\Phi : \Omega \times [0, T] &\rightarrow [-1, 1] \\ (x, t) &\mapsto \Phi(x, t) = \frac{\varrho_{\mathcal{A}} - \varrho_{\mathcal{B}}}{\varrho_{\mathcal{A}} + \varrho_{\mathcal{B}}},\end{aligned}\tag{2.1}$$

where  $\varrho = \varrho(x, t)$  denotes the density, which depends on the spatial variable  $x$  and the time variable  $t$ .

Here,  $\varrho_{\mathcal{A}}$  and  $\varrho_{\mathcal{B}}$  represent the densities of the two phases  $\mathcal{A}$  and  $\mathcal{B}$ , respectively.

The numerator  $\varrho_{\mathcal{A}} - \varrho_{\mathcal{B}}$  represents the difference between the densities of the two phases, which characterizes the local dominance of one phase over the other.

The denominator  $\varrho_{\mathcal{A}} + \varrho_{\mathcal{B}}$  corresponds to the sum of the densities of the two phases and is used to normalize the expression.

This normalization ensures that the phase-field variable satisfies

$$-1 \leq \Phi \leq 1.$$

In particular,  $\Phi = 1$  corresponds to the pure phase  $\mathcal{A}$ ,  $\Phi = -1$  corresponds to the pure phase  $\mathcal{B}$ , and intermediate values  $-1 < \Phi < 1$  describe the transition region between the two phases.

The order parameter  $\Phi$  transitions between 1 and  $-1$ , representing distinct phases, and changes smoothly but rapidly across the interface between them.

In equilibrium, the system's state minimizes the Ginzburg-Landau energy, as discussed in reference [1].

$$E_{\lambda}(\Phi(x, t)) := \frac{\lambda}{2} \int_{\Omega} |\nabla \Phi|^2 dx + \int_{\Omega} \varphi(\Phi(x, t)) dx. \quad (2.2)$$

In this context,  $\lambda > 0$  and  $\sqrt{\lambda}$  represent the parameters controlling the width of the interface region. The free energy is represented by  $\varphi(\Phi)$ , and the chemical potential  $w$  is defined as the derivative of the free energy with respect to the order parameter  $\Phi$ .

$$w(x, t) := \frac{\partial E_{\lambda}}{\partial \Phi}. \quad (2.3)$$

We have

$$\begin{aligned} \frac{\partial E_{\lambda}}{\partial \Phi}(\Phi)(v) &= \lim_{\delta \rightarrow 0} \frac{E_{\lambda}(\Phi + \delta v) - E_{\lambda}(\Phi)}{\delta} \\ &= \lambda \int_{\Omega} \nabla \Phi \cdot \nabla v dx + \int_{\Omega} \dot{\varphi}(\Phi)(v) dx. \end{aligned} \quad (2.4)$$

Hence, we arrive that

$$w(\Phi) = -\lambda \Delta \Phi(x, t) + \dot{\varphi}(\Phi(x, t)), \quad (2.5)$$

where  $\dot{\varphi}(\Phi(x, t)) = \frac{\partial \varphi}{\partial \Phi}$ .

Under homogeneous Neumann boundary conditions, and following Green's theorem, we obtain:

$$\frac{\partial E_{\lambda}}{\partial \Phi}(\Phi)(v) = \int_{\Omega} (-\lambda \Delta \Phi + \dot{\varphi}(\Phi)) v dx. \quad (2.6)$$

The free energy  $\varphi$  can be modeled using various potentials. While the double-well and logarithmic potentials have been extensively explored in earlier studies [2, 3], the double-obstacle potential emerges as an alternative and effective approximation. Its definition is given by:

$$\varphi(\Phi(x, t)) = \frac{1}{4} (\Phi(x, t)^2 - 1)^2. \quad (2.7)$$

Furthermore, adhering to the law of conservation of mass [4] and accounting for the time derivative as a fractional-order derivative, we obtain:

$${}_0^C D_t^{\alpha} \Phi(t) = -\operatorname{div}(\mathcal{J}), \quad (2.8)$$

where  $\mathcal{J}$  represents the mass flux, which is connected to the chemical potential  $w$  through Fick's law.

$$\mathcal{J} = -\mathcal{M}(\Phi)\nabla w(x, t). \quad (2.9)$$

and  ${}_0^C D_t^\alpha \Phi(t)$  is the Caputo derivative given by

$${}_0^C D_t^\alpha \Phi(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \dot{\Phi}(s) ds, \forall t \in [0, T], \quad (2.10)$$

where  $0 < \alpha \leq 1$ .

The mobility of the mixture, denoted as  $\mathcal{M}(\Phi)$ , typically varies with  $\Phi$ . Nevertheless, for simplicity, we assume it to be constant and fix  $\mathcal{M}(\Phi) = 1$ . Therefore, the law of conservation of mass can be expressed as:

$${}_0^C D_t^\alpha \Phi(t) - \Delta w = 0. \quad (2.11)$$

The equations governing our fractional Cahn-Hilliard model are as follows:

$$\begin{cases} {}_0^C D_t^\alpha \Phi(x, t) - \Delta w(x, t) = 0, \\ \Phi(x, 0) = \Phi_0. \end{cases} \quad (2.12)$$

### 3. HOMOTOPY PERTURBATION SUMUDU TRANSFORM METHOD

Let  $L$  represent a linear operator,  $N$  a nonlinear operator, and  $f(x, t)$  the source term. To illustrate the main concept of the HPSTM method (as discussed in [5]), we consider the following example involving a nonlinear, non-homogeneous fractional-order partial differential equation:

$${}_0^C D_t^\alpha \Phi(x, t) + L\Phi(x, t) + N\Phi(x, t) = f(x, t), \quad 0 < \alpha < 1, \quad (3.13)$$

with the following initial condition:

$$\Phi(x, t) = g(x). \quad (3.14)$$

Next, we apply the Sumudu transform to both sides of equation (3.13) and obtain

$$\mathcal{S} \{ {}_0^C D_t^\alpha \Phi(x, t) + L\Phi(x, t) + N\Phi(x, t) \} = \mathcal{S} \{ f(x, t) \}, \quad (3.15)$$

the property of linearity for the Sumudu transform implies

$$\mathcal{S} \{ {}_0^C D_t^\alpha \Phi(x, t) \} = \mathcal{S} \{ f(x, t) \} - \mathcal{S} \{ L\Phi(x, t) \} - \mathcal{S} \{ N\Phi(x, t) \}. \quad (3.16)$$

The property associated with the Sumudu transform implies

$$\mathcal{S} \{ \Phi(x, t) \} - \Phi(x, 0) = v^\alpha \mathcal{S} \{ f(x, t) \} - v^\alpha \mathcal{S} \{ L\Phi(x, t) \} - v^\alpha \mathcal{S} \{ N\Phi(x, t) \}. \quad (3.17)$$

Applying the inverse Sumudu transform, we obtain:

$$\Phi(x, t) = F(x, t) - \mathcal{S}^{-1} \{ v^\alpha \mathcal{S} \{ L\Phi(x, t) + N\Phi(x, t) \} \}, \quad (3.18)$$

The term  $F(x, t)$  corresponds to the contribution arising from the source term and the initial conditions provided.

Next, we apply the Homotopy Perturbation Method (HPM)

$$\Phi = \sum_{n=0}^{\infty} p^n \Phi_n. \quad (3.19)$$

Let's decompose the nonlinear term as follows:

$$N\Phi = \sum_{n=0}^{\infty} p^n H_n(\Phi), \quad (3.20)$$

where the He polynomials  $H_n(\Phi(x, t))$  are defined as follows:

$$H_n(\Phi_0, \Phi_1, \dots, \Phi_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[ N \left( \sum_{i=0}^n p^i \Phi_i \right) \right]_{p=0}, \quad n = 0, 1, 2, \dots \quad (3.21)$$

Substituting equations (3.19) and (3.20) into (3.18), we derive

$$\sum_{n=0}^{\infty} p^n \Phi_n = F(x, t) - p \left( \mathcal{S}^{-1} \left\{ v^\alpha \mathcal{S} \left\{ L \sum_{n=0}^{\infty} p^n \Phi_n + \sum_{n=0}^{\infty} p^n H_n(\Phi) \right\} \right\} \right), \quad (3.22)$$

This represents the combination of the Sumudu transformation and the Homotopy Perturbation Sumudu Transformation Method (HPSTM). By equating coefficients of corresponding powers of the parameter  $p$ , we derive the following approximations:

$$\begin{aligned} p^0 : \Phi_0 &= F, \\ p^1 : \Phi_1 &= -\mathcal{S}^{-1} \{ v^\alpha \mathcal{S} \{ L\Phi_0(x, t) + H_0(\Phi) \} \}, \\ p^2 : \Phi_2 &= -\mathcal{S}^{-1} \{ v^\alpha \mathcal{S} \{ L\Phi_1(x, t) + H_1(\Phi) \} \}, \\ p^3 : \Phi_3 &= -\mathcal{S}^{-1} \{ v^\alpha \mathcal{S} \{ L\Phi_2(x, t) + H_2(\Phi) \} \}, \\ &\vdots \end{aligned} \quad (3.23)$$

Following a similar procedure, the remaining components  $\Phi_n(x, t)$  can be fully determined, thereby completing the series solution.

In conclusion, we approximate the analytical solution  $\Phi(x, t)$  using the following series:

$$\Phi = \lim_{m \rightarrow \infty} \sum_{n=0}^m \Phi_n. \quad (3.24)$$

#### 4. SOLVING THE TIME-FRACTIONAL CAHN-HILLIARD EQUATION WITH DOUBLE-OBSTACLE POTENTIAL IN ONE DIMENSION

Referring to equations (2.5) and (2.7), we obtain:

$$\begin{aligned}
 {}^C_0 D_t^\alpha \Phi(x, t) &= -\operatorname{div}(\mathcal{J}) = \operatorname{div}(\nabla w(\Phi(x, t))) = \Delta w(\Phi(x, t)) \\
 &= \Delta (-\lambda \Delta \Phi(x, t) + \Phi(x, t)^3 - \Phi(x, t)) \\
 &= -\lambda \Delta(\Delta \Phi(x, t)) + \Delta(\Phi(x, t)^3) - \Delta(\Phi(x, t)) \\
 &= -\lambda \frac{\partial^4 \Phi(x, t)}{\partial x^4} + 3\Phi(x, t)^2 \frac{\partial^2 \Phi(x, t)}{\partial x^2} + 6\Phi(x, t) \left( \frac{\partial \Phi(x, t)}{\partial x} \right)^2 \\
 &\quad - \frac{\partial^2 \Phi(x, t)}{\partial x^2} \\
 {}^C_0 D_t^\alpha \Phi(x, t) &= -\lambda \frac{\partial^4 \Phi(x, t)}{\partial x^4} + \frac{\partial^2 \Phi(x, t)}{\partial x^2} (3\Phi(x, t)^2 - 1) + 6\Phi(x, t) \left( \frac{\partial \Phi(x, t)}{\partial x} \right)^2.
 \end{aligned} \tag{4.25}$$

Here is the system describing our Cahn–Hilliard Equation with Double-Obstacle Potential in one dimension:

$$\begin{cases} {}^C_0 D_t^\alpha \Phi(x, t) = -\lambda \frac{\partial^4 \Phi(x, t)}{\partial x^4} + \frac{\partial^2 \Phi(x, t)}{\partial x^2} (3\Phi(x, t)^2 - 1) + 6\Phi(x, t) \left( \frac{\partial \Phi(x, t)}{\partial x} \right)^2, \\ \Phi(x, 0) = x. \end{cases} \tag{4.26}$$

Applying the Sumudu transform to equation (4.26), as described in the preceding section, results in:

$$\begin{aligned}
 \mathcal{S}\{\Phi(x, t)\} &= x + v^\alpha \mathcal{S} \left\{ -\lambda \frac{\partial^4 \Phi(x, t)}{\partial x^4} + \frac{\partial^2 \Phi(x, t)}{\partial x^2} (3\Phi(x, t)^2 - 1) \right. \\
 &\quad \left. + 6\Phi(x, t) \left( \frac{\partial \Phi(x, t)}{\partial x} \right)^2 \right\}.
 \end{aligned} \tag{4.27}$$

The inverse of the Sumudu transform implies that

$$\begin{aligned}
 \Phi(x, t) &= x + \mathcal{S}^{-1} \left\{ v^\alpha \mathcal{S} \left( -\lambda \frac{\partial^4 \Phi(x, t)}{\partial x^4} + \frac{\partial^2 \Phi(x, t)}{\partial x^2} (3\Phi(x, t)^2 - 1) \right. \right. \\
 &\quad \left. \left. + 6\Phi(x, t) \left( \frac{\partial \Phi(x, t)}{\partial x} \right)^2 \right) \right\}.
 \end{aligned} \tag{4.28}$$

Proceeding with the application of the Homotopy Perturbation Method (HPM):

$$\begin{aligned}
 \sum_{n=0}^{\infty} p^n \Phi_n(x, t) &= x + p \times \mathcal{S}^{-1} \left\{ v^\alpha \mathcal{S} \left( -\lambda \sum_{n=0}^{\infty} p^n L(\Phi_n(x, t)) + 3 \sum_{n=0}^{\infty} p^n H_n(\Phi(x, t)) \right. \right. \\
 &\quad \left. \left. - \sum_{n=0}^{\infty} p^n L'(\Phi_n(x, t)) + 6 \sum_{n=0}^{\infty} p^n H'_n(\Phi(x, t)) \right) \right\},
 \end{aligned} \tag{4.29}$$

where

$$L(\Phi(x, t)) = \frac{\partial^4 \Phi(x, t)}{\partial x^4}, \quad (4.30)$$

$$\sum_{n=0}^{\infty} p^n H_n(\Phi(x, t)) = \frac{\partial^2 \Phi(x, t)}{\partial x^2} \Phi^2(x, t), \quad (4.31)$$

$$L'(\Phi(x, t)) = \frac{\partial^2 \Phi(x, t)}{\partial x^2}, \quad (4.32)$$

$$\sum_{n=0}^{\infty} p^n H'_n(\Phi(x, t)) = \Phi(x, t) \left( \frac{\partial \Phi(x, t)}{\partial x} \right)^2. \quad (4.33)$$

By equating coefficients of corresponding powers of  $p$ , we obtain:

Calculation of the coefficient  $p^0$ :

$$p^0 : \Phi_0(x, t) = x. \quad (4.34)$$

Calculation of the coefficient  $p^1$ :

$$\begin{aligned} p^1 : \Phi_1 &= \mathcal{S}^{-1} \left\{ v^\alpha \mathcal{S} \left( -\lambda L(\Phi_0) + 3H_0(\Phi(x, t)) - L'(\Phi_0) + 6H'_0(\Phi) \right) \right\} \\ &= \frac{6xt^\alpha}{\Gamma(\alpha + 1)}. \end{aligned}$$

Calculation of the coefficient  $p^2$ :

$$\begin{aligned} p^2 : \Phi_2 &= \mathcal{S}^{-1} \left\{ v^\alpha \mathcal{S} \left( -\lambda L(\Phi_1) + 3H_1(\Phi(x, t)) - L'(\Phi_1) + 6H'_1(\Phi) \right) \right\} \\ &= \frac{108xt^{2\alpha}}{\Gamma(2\alpha + 1)}. \end{aligned}$$

Calculation of the coefficient  $p^3$ :

$$\begin{aligned} p^3 : \Phi_3 &= \mathcal{S}^{-1} \left\{ v^\alpha \mathcal{S} \left( -\lambda L(\Phi_2) + 3H_2(\Phi) - L'(\Phi_2) + 6H'_2(\Phi) \right) \right\} \\ &= \frac{2592xt^{3\alpha}}{\Gamma(3\alpha + 1)}. \end{aligned}$$

$$\begin{aligned} p^4 : \Phi_4 &= \mathcal{S}^{-1} \left\{ v^\alpha \mathcal{S} \left( -\lambda L(\Phi_3) + 3H_3(\Phi) - L'(\Phi_3) + 6H'_3(\Phi) \right) \right\} \\ &= \frac{71280xt^{4\alpha}}{\Gamma(4\alpha + 1)}. \end{aligned}$$

$$\begin{aligned} p^5 : \Phi_5 &= \mathcal{S}^{-1} \left\{ v^\alpha \mathcal{S} \left( -\lambda L(\Phi_4) + 3H_4(\Phi) - L'(\Phi_4) + 6H'_4(\Phi) \right) \right\} \\ &= \frac{2122848xt^{5\alpha}}{\Gamma(5\alpha + 1)}. \end{aligned}$$

Therefore, the solution to equation (4.26) is expressed as

$$\Phi(x, t) = x + \frac{6xt^\alpha}{\Gamma(\alpha + 1)} + \frac{108xt^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{2592xt^{3\alpha}}{\Gamma(3\alpha + 1)} + \frac{71280xt^{4\alpha}}{\Gamma(4\alpha + 1)} + \frac{2122848xt^{5\alpha}}{\Gamma(5\alpha + 1)} \dots$$

Here, we present graphical representations of the solution for various values of  $\alpha$ .

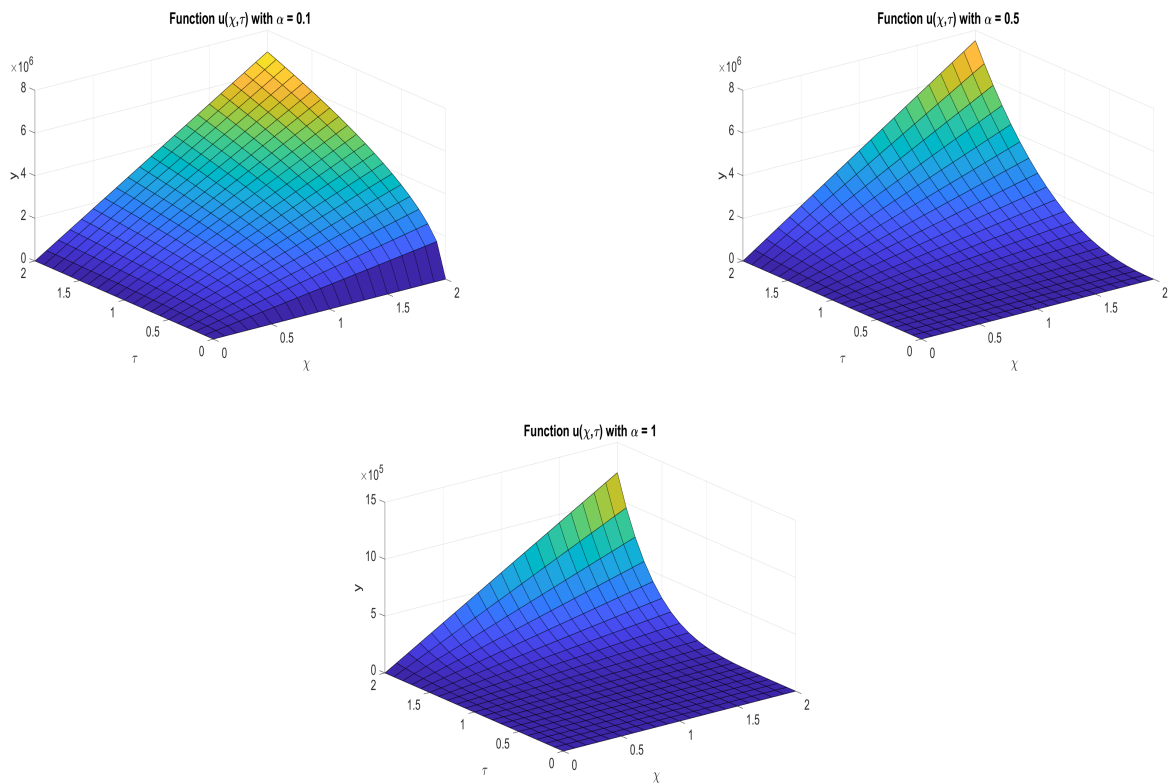
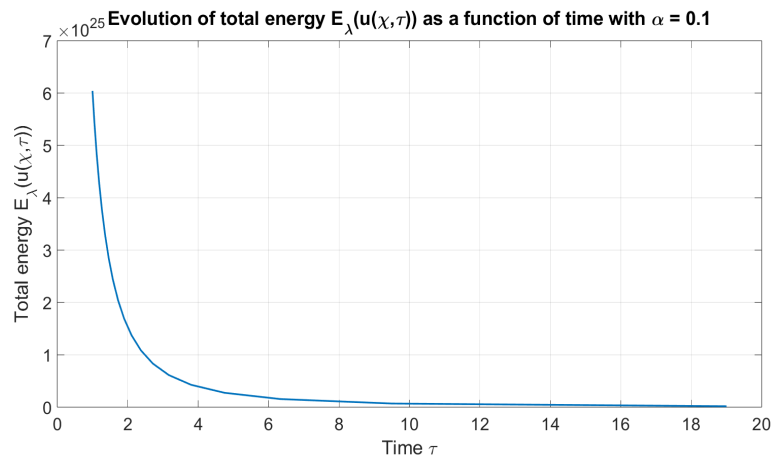
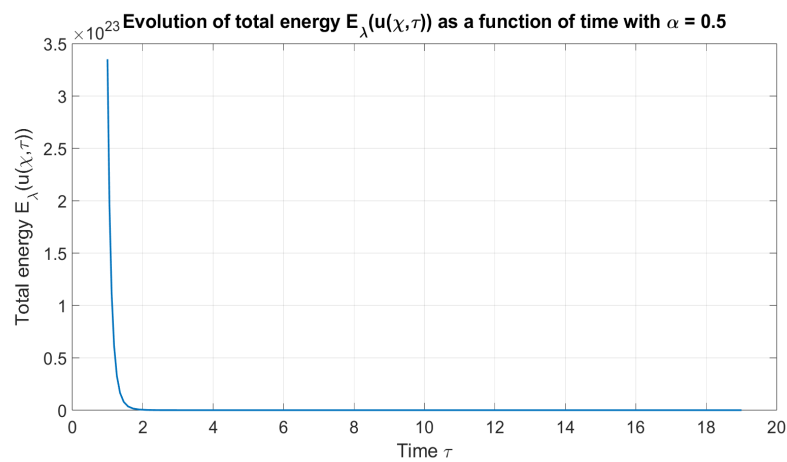
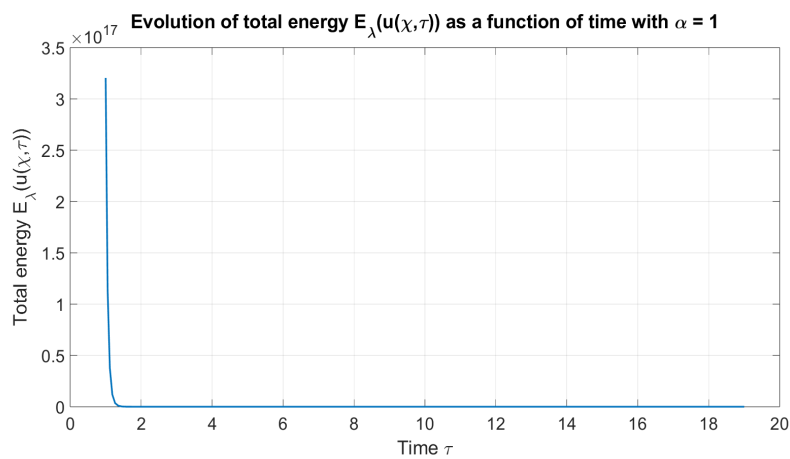


Fig. 4.1. The graphical representation of the solution with three different values of  $\alpha$ .

These graphical results illustrate that the solution of the Cahn-Hilliard equation exhibits increased stability as  $\alpha$  approaches 1, indicating proximity of the fractional-order derivative to the natural derivative.

In theory, the Ginzburg-Landau energy, referred to as the total energy, plays a crucial role in solving the Cahn-Hilliard problem. Therefore, it is essential to express this energy as a function of time.

Now, we express the Ginzburg-Landau energy as given in equation (2.2) with  $\lambda = 1$ , considering three distinct values of  $\alpha$ .

Fig. 4.2. Energy graph with  $\alpha = 0.1$ .Fig. 4.3. Energy graph with  $\alpha = 0.5$ .Fig. 4.4. Energy graph with  $\alpha = 1$ .

These figures depicting the evolution of energy show that the system progresses towards a lower-energy state over time, indicating a stable equilibrium and potentially leading to phase

separation.

Moreover, the energy increases as  $\alpha$  is incremented by 0.

## 5. SOLVING THE TIME-FRACTIONAL CAHN-HILLIARD EQUATION WITH DOUBLE-OBSTACLE POTENTIAL IN TWO DIMENSIONS

In this section, we will employ the HPSTM (Homotopy Perturbation Sumudu Transformation Method) introduced in Section 3 to solve the fractional-order differential equation (2.12), now in two dimensions.

Referring to equations (2.5) and (2.7), we derive:

$$\begin{aligned}
 {}^C_0 D_t^\alpha \Phi(x, y, t) &= -\operatorname{div}(\mathcal{J}) = \operatorname{div}(\nabla w(\Phi(x, y, t))) = \Delta w(\Phi(x, y, t)) \\
 &= \Delta (-\lambda \Delta \Phi(x, y, t) + \Phi(x, y, t)^3 - \Phi(x, y, t)) \\
 &= -\lambda \Delta(\Delta \Phi(x, y, t)) + \Delta(\Phi(x, y, t)^3) - \Delta(\Phi(x, y, t)) \\
 &= -\lambda \left( \frac{\partial^4 \Phi(x, y, t)}{\partial x^4} + 2 \frac{\partial^4 \Phi(x, y, t)}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi(x, y, t)}{\partial y^4} \right) \\
 &\quad + \frac{\partial^2 \Phi^3(x, y, t)}{\partial x^2} + \frac{\partial^2 \Phi^3(x, y, t)}{\partial y^2} - \frac{\partial^2 \Phi(x, y, t)}{\partial x^2} + \frac{\partial^2 \Phi(x, y, t)}{\partial y^2}.
 \end{aligned} \tag{5.35}$$

Here is the system describing our Cahn–Hilliard equation with a double-obstacle potential in two dimensions:

$$\begin{cases} {}^C_0 D_t^\alpha \Phi(x, t) &= -\lambda \left( \frac{\partial^4 \Phi(x, y, t)}{\partial x^4} + 2 \frac{\partial^4 \Phi(x, y, t)}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi(x, y, t)}{\partial y^4} \right) \\ &\quad + \frac{\partial^2 \Phi^3(x, y, t)}{\partial x^2} + \frac{\partial^2 \Phi^3(x, y, t)}{\partial y^2} - \frac{\partial^2 \Phi(x, y, t)}{\partial x^2} + \frac{\partial^2 \Phi(x, y, t)}{\partial y^2}. \\ \Phi(x, 0) &= x + y. \end{cases} \tag{5.36}$$

Now, applying the HPSTM method, we obtain:

$$\begin{aligned}
 \sum_{n=0}^{\infty} p^n \Phi_n(x, y, t) &= x + y + p \left( \mathcal{S}^{-1} \left\{ v^\alpha \mathcal{S} \left\{ -\lambda \frac{\partial^4 \Phi(x, y, t)}{\partial x^4} + \sum_{n=0}^{\infty} p^n H_n(x, y, t) \right\} \right\} \right. \\
 &\quad + \mathcal{S}^{-1} \left\{ v^\alpha \mathcal{S} \left\{ -\frac{\partial^2 \Phi(x, y, t)}{\partial x^2} - \lambda \frac{\partial^4 \Phi(x, y, t)}{\partial y^4} \right\} \right\} \\
 &\quad \left. + \mathcal{S}^{-1} \left\{ v^\alpha \mathcal{S} \left\{ + \sum_{n=0}^{\infty} p^n H'_n(x, y, t) + \frac{\partial^2 \Phi(x, y, t)}{\partial y^2} - 2\lambda \frac{\partial^4 \Phi(x, y, t)}{\partial x^2 \partial y^2} \right\} \right\} \right).
 \end{aligned} \tag{5.37}$$

where

$$\sum_{n=0}^{\infty} p^n H_n(\Phi(x, y, t)) = \frac{\partial^2 \Phi^3(x, y, t)}{\partial x^2} \tag{5.38}$$

$$\sum_{n=0}^{\infty} p^n H'_n(\Phi(x, y, t)) = \frac{\partial^2 \Phi^3(x, y, t)}{\partial y^2}. \tag{5.39}$$

By equating the coefficients of corresponding powers of  $p$ , we find:

Calculation of the coefficient  $p^0$ :

$$p^0 : \Phi_0(x, y, t) = x + y. \quad (5.40)$$

Calculation of the coefficient  $p^1$ :

$$\begin{aligned} p^1 : \Phi_1(x, y, t) &= \left( \mathcal{S}^{-1} \left\{ v^\alpha \mathcal{S} \left\{ -\lambda \frac{\partial^4 \Phi_0(x, y, t)}{\partial x^4} + H_0(\Phi(x, y, t)) - \frac{\partial^2 \Phi_0(x, y, t)}{\partial x^2} \right\} \right\} \right. \\ &\quad + \mathcal{S}^{-1} \left\{ v^\alpha \mathcal{S} \left\{ -\lambda \frac{\partial^4 \Phi_0(x, y, t)}{\partial y^4} + H'_0(\Phi(x, y, t)) + \frac{\partial^2 \Phi_0(x, y, t)}{\partial y^2} \right\} \right\} \\ &\quad \left. + \mathcal{S}^{-1} \left\{ v^\alpha \mathcal{S} \left\{ -2\lambda \frac{\partial^4 \Phi(x, y, t)}{\partial x^2 \partial y^2} \right\} \right\} \right) \\ &= \frac{12(x+y)t^\alpha}{\Gamma(1+\alpha)}. \end{aligned} \quad (5.41)$$

Calculation of other coefficients:

$$p^2 : \Phi_2(x, y, t) = \frac{432(x+y)t^{2\alpha}}{\Gamma(1+2\alpha)}. \quad (5.42)$$

$$p^3 : \Phi_3(x, y, t) = \frac{20736(x+y)t^{3\alpha}}{\Gamma(1+3\alpha)}. \quad (5.43)$$

$$p^4 : \Phi_4(x, y, t) = \frac{1140480(x+y)t^{4\alpha}}{\Gamma(1+4\alpha)}. \quad (5.44)$$

$$p^5 : \Phi_5(x, y, t) = \frac{67931136(x+y)t^{5\alpha}}{\Gamma(1+5\alpha)}. \quad (5.45)$$

Ultimately, the solution to equation (5.36) is presented as follows:

$$\begin{aligned} \Phi(x, y, t) &= x + y + \frac{12(x+y)t^\alpha}{\Gamma(1+\alpha)} + \frac{432(x+y)t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{20736(x+y)t^{3\alpha}}{\Gamma(1+3\alpha)} \\ &\quad + \frac{1140480(x+y)t^{4\alpha}}{\Gamma(1+4\alpha)} + \frac{67931136(x+y)t^{5\alpha}}{\Gamma(1+5\alpha)} \dots \end{aligned} \quad (5.46)$$

The following figures depict the solution for  $\alpha = 0.1$  at different stages.

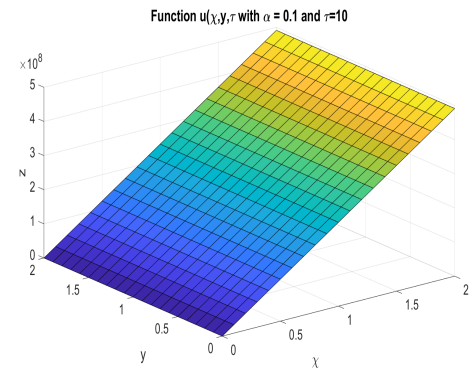
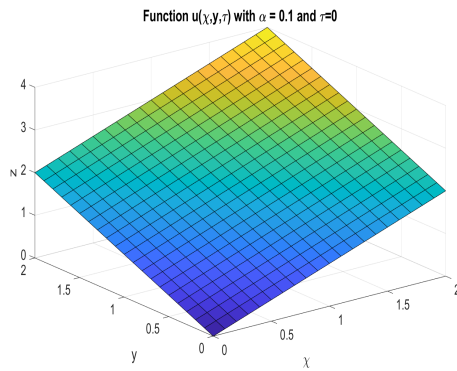


Fig. 5.5. The graphical representation of the solution with  $\alpha = 0.1$  at  $t = 0$  and  $t = 10$ .

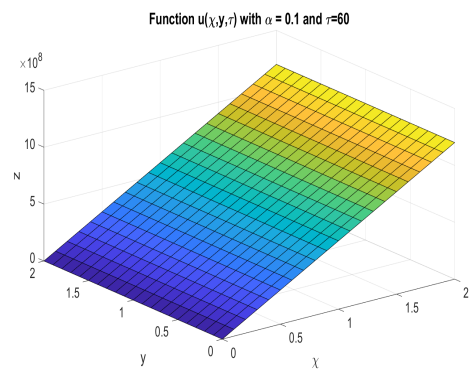
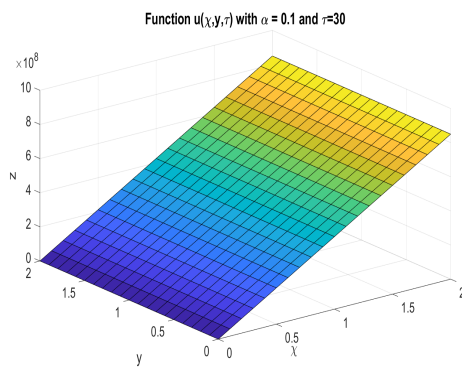


Fig. 5.6. The graphical representation of the solution with  $\alpha = 0.1$  at  $t = 30$  and  $t = 60$ .

The following figures illustrate the solution for  $\alpha = 0.5$  at different stages.

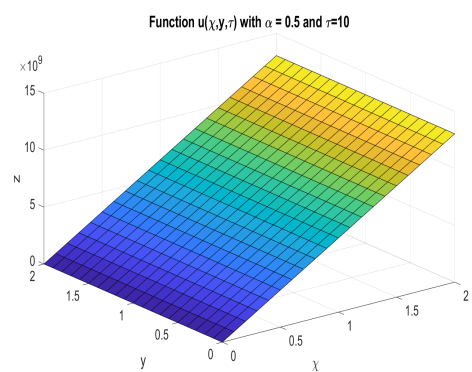
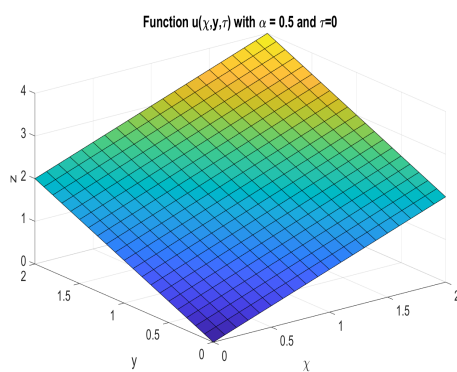


Fig. 5.7. The graphical representation of the solution with  $\alpha = 0.5$  at  $t = 0$  and  $t = 10$ .

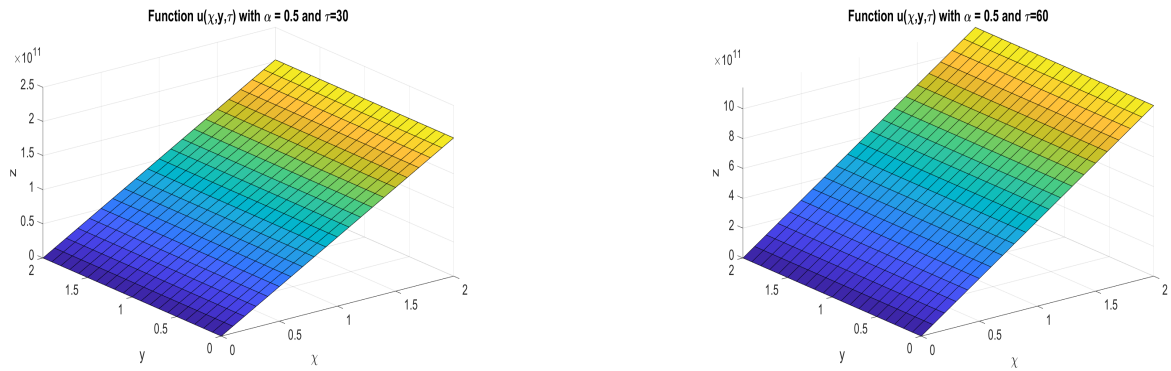


Fig. 5.8. The graphical representation of the solution with  $\alpha = 0.5$  at  $t = 30$  and  $t = 60$ .

The following figures display the solution for  $\alpha = 1$  at different stages.

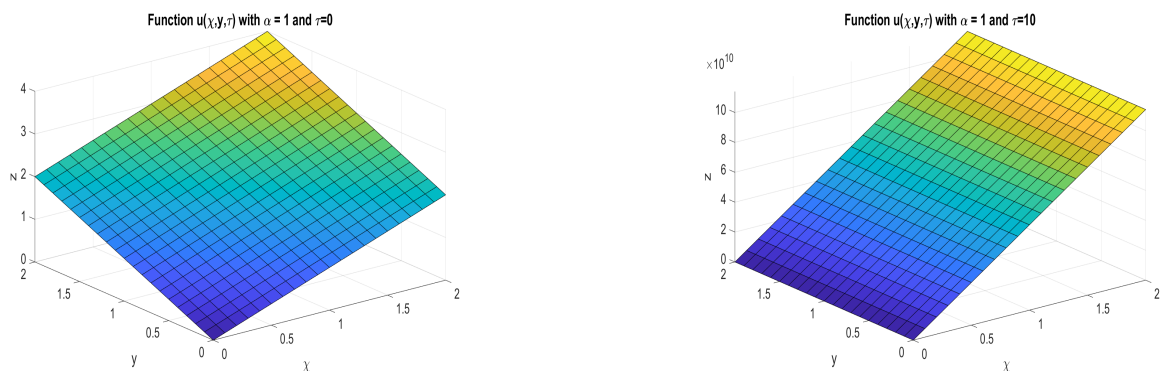


Fig. 5.9. The graphical representation of the solution with  $\alpha = 1$  at  $t = 0$  and  $t = 10$ .

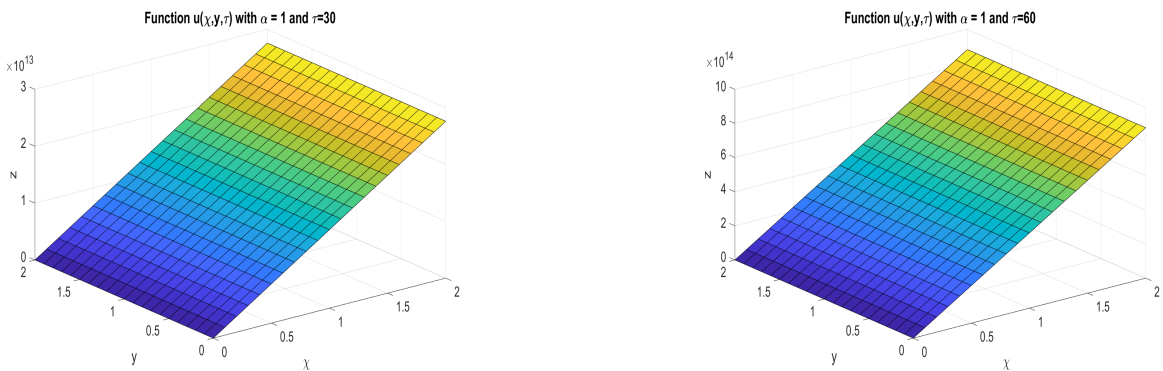
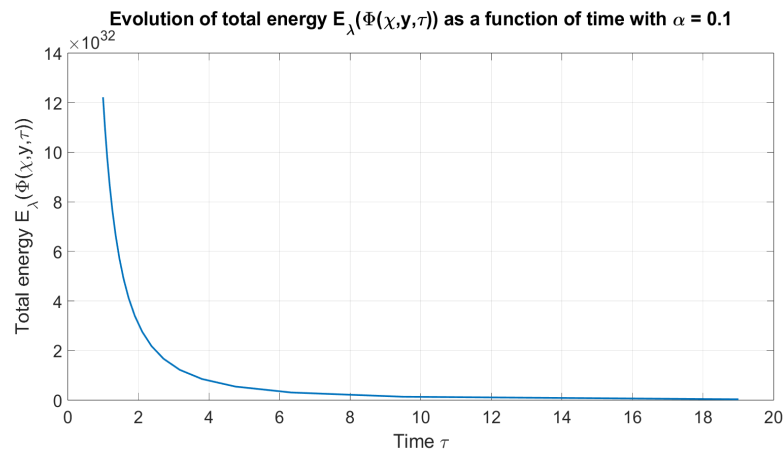
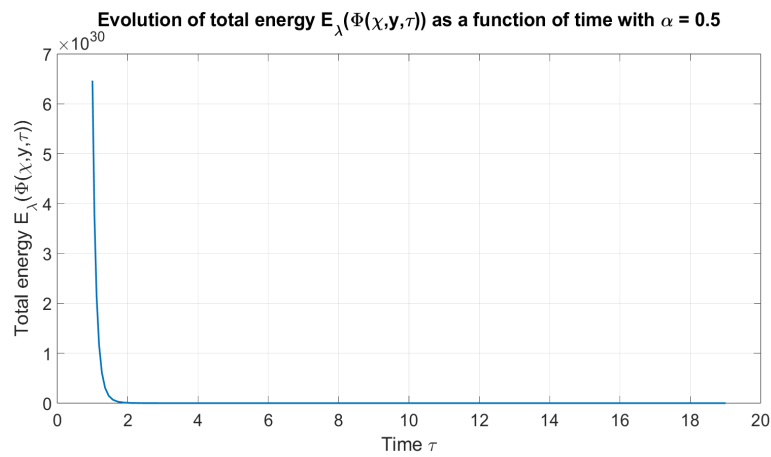
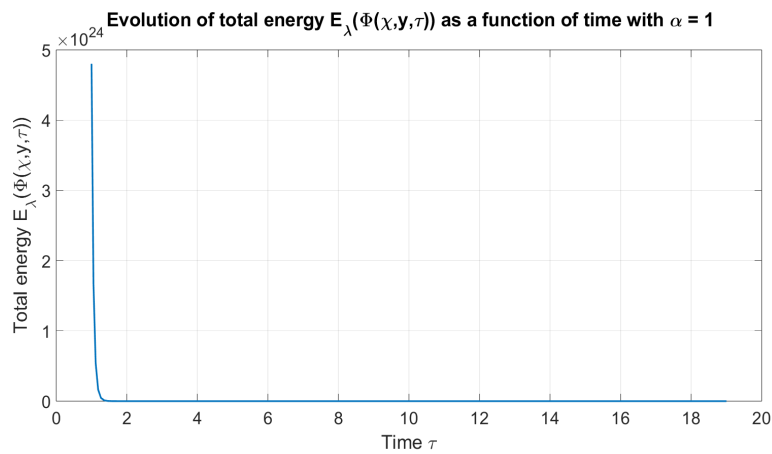


Fig. 5.10. The graphical representation of the solution with  $\alpha = 1$  at  $t = 30$  and  $t = 60$ .

In these 2D numerical simulations and for various values of  $\alpha$ , we observe that the solution undergoes rapid changes starting from time  $t_0$ . However, as time progresses, the solution continues to evolve.

Here, we present the Ginzburg-Landau energy as described in equation (2.2) with  $\lambda = 1$ , considering three distinct values of  $\alpha$  in the context of 2D.

Fig. 5.11. Energy graph with  $\alpha = 0.1$ .Fig. 5.12. Energy graph with  $\alpha = 0.5$ .Fig. 5.13. Energy graph with  $\alpha = 1$ .

From these energy evolution figures, we observe that the system gradually reaches a lower energy state over time, suggesting a stable equilibrium. This trend potentially leads to stable phase separation. These observations align with findings from one-dimensional cases.

**Remark 5.1:**

*The following remarks can be made in this work:*

- 1) *The numerical results presented are derived from the solution obtained and visualized in Matlab in two dimensions. This direct approach is easier to implement and quicker to solve than the variational method involving finite elements. Indeed, in contrast to the finite element method, which requires domain discretization and solving complex linear systems, the direct solution avoids these computationally intensive steps. For detailed applications of the finite element method, refer to [24].*
- 2) *The originality of our work lies in the introduction of a new Cahn-Hilliard model with a double obstacle potential, providing a novel perspective on phase transition phenomena in nonlinear systems.*

## 6. CONCLUSION AND FUTURE DIRECTION

A Caputo fractional nonlinear problem has been solved in one and two dimensions using the Sumudu transform, Homotopy perturbation, and He's polynomials approaches. Additionally, we have used numerical simulations to confirm the theoretical results that were achieved. We can consider this work as a foundation for future studies, addressing other types of Cahn-Hilliard problems with different approximations of free energy, depending on the cooling of the mixture at high temperatures. Boundary conditions, whether mixed, Dirichlet, or Neumann, can also be included. The following points can be emphasized as future perspectives:

- 1) Develop a new Cahn-Hilliard system that accounts for fractional-order derivatives in space.
- 2) Provide numerical simulations in 3D to observe the interplay of temporal and spatial evolution and understand their mutual influence.
- 3) Compare the results with other methods that yield approximate analytical solutions.

### ***Competing interests:***

The authors declare no competing interests.

### ***Data availability statement:***

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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### ***Author contribution:***

All authors have made equal and significant contributions to this article. They have all read and approved the final version of the manuscript.

### ***Ethical approval:***

The research conducted in this study is not associated with either human or animal use.

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