Adv Syst Sci Appl 25; **3**:96–108
Published online at https://ijassa.ipu.ru.

The Sturm–Liouville Operator with Rapidly Growing Potential and the Asymptotics of its Spectrum

A. V. Kachkina[1,2*]

[1] *Lomonosov Moscow State University, Moscow, Russia*
[2] *Moscow Center for Fundamental and Applied Mathematics, Moscow, Russia*

Abstract: In this paper, we study the asymptotic behavior of the discrete spectrum of the Sturm–Liouville operator given on $\mathbb{R}_+$ by the expression $-y'' + q(x)y$ and the zero boundary condition $y(0)\cos\alpha + y'(0)\sin\alpha = 0$, for rapidly growing potentials $q(x)$. For this class of operators, asymptotic formulas for the eigenvalues are derived, which describe the rate of their growth at infinity.

Keywords: Differential operator, Sturm-Liouville operator, operator spectrum, asymptotics.

INTRODUCTION

In the Hilbert space $L_2[0, +\infty)$, we consider the Sturm–Liouville operator $\mathbb{L}_q$ generated by the differential expression:

$$l_q(y) = -y''(x) + q(x)y(x),$$

and the boundary condition at zero:

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0,$$

where $q(x)$ is a continuous real-valued function on $[0, +\infty)$. The domain of the operator $\mathbb{L}_q$: $D(\mathbb{L}_q) = \{y \in L_2[0, +\infty) : y, y'$ are absolutely continuous on any $[a, b] \subset [0, +\infty),\ -y'' + q(x)y \in L_2[0, +\infty)$ and $y(0)\cos\alpha + y'(0)\sin\alpha = 0\}$.

If the function (potential) $q(x) \to +\infty,\ x \to +\infty$, then the operator $\mathbb{L}_q$ is semi-bounded from below and has a purely discrete spectrum $\{\lambda_n\}_{n\in\mathbb{N}},\ \lambda_n \to +\infty,\ n \to +\infty$ (E. C. Titchmarsh [1], A. M. Molchanov [4]). Let us numerate the eigenvalues of the operator $\mathbb{L}_q$ in ascending order: $\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$.

The distribution of the spectrum (E. C. Titchmarsh [1]) in the case of power-law growth of potential q has been well studied. For example, if $q(x) = x^k$, $k > 0$, then the eigenvalues λ_n of the operator $\mathbb{L}_q$ have the asymptotics:

$$\lambda_n \sim \left\{ \frac{\pi k \Gamma(\frac{3}{2} + \frac{1}{k})}{\Gamma(\frac{3}{2})\Gamma(\frac{1}{k})} n \right\}^{\frac{2k}{k+2}}, \quad n \to +\infty, \tag{0.1}$$

where $\Gamma(z)$ is the Euler's Gamma function.

The asymptotics of the eigenvalues of the operator $\mathbb{L}_q$ in the case $\alpha = 0$ for potentials of the form $q(x) = x^k + V(x),\ k > 0$ was obtained in the works of H. H. Murtazin and

*Corresponding author: alisa-kachkina@mail.ru

T. G. Amangil'din [5] for $V(x) \in C_0^2[0,+\infty)$ and H. K.Ishkin [6] for $V(x) \in C_0^1[0,+\infty)$, where functions from the class $C_0^m[0,+\infty)$ are compactly supported functions of the class $C^m[0,+\infty)$.

The distribution of the spectrum of the Airy and Weber operators perturbed by the delta interaction (Dirac delta function) was found by A.S.Pechentsov [19], [20], [21].

If the potential q increases at infinity faster than any power function, then the eigenvalues of the operator $\mathbb{L}_q$ do not have a power asymptotics (0.1). A. I. Kozko [3] established that for the potential $q(x) = e^x$ the relation $\lambda_n \sim \left(\dfrac{\pi n}{2\ln(\pi n)}\right)^2$, $n \to +\infty$ holds.

In this paper, we obtain asymptotics of the eigenvalues of the operator $\mathbb{L}_q$ for classes of potentials that increase rapidly at infinity.

1. CLASSES OF RAPIDLY GROWING POTENTIALS. AUXILIARY STATEMENTS

Let $\mathfrak{Q}$ denote the class of functions $q \in C[0,+\infty) \cap C^2(0,+\infty)$ satisfying the conditions:

$$q''(x) \geq 0, \quad x \geq x_0, \tag{1.2}$$

$$\lim_{x\to+\infty} \frac{xq'(x)}{q(x)} = +\infty. \tag{1.3}$$

In particular, from the last equality it follows that there exists a number $\widetilde{x}$ such that for all the values of the argument $x > \widetilde{x}$ the values $q(x)$ are not equal to zero, and the inequality $\dfrac{q'(x)}{q(x)} > 0$ is also satisfied. Without loss of generality, we will assume that these relations are satisfied for all the $x > 0$ (i.e. $\widetilde{x} = 0$).

Lemma 1.1:
Let q be an arbitrary function in the class $\mathfrak{Q}$. The following statements are true.

1. The functions q' and q have only positive values on arguments greater than some x_1. Beyond that, these functions grow at infinity faster than any power function, i.e. for any $k \in \mathbb{N}$ we have $x^k = o(q(x))$, $x \to +\infty$.

2. Let function p be the inverse of the function q, i.e. $q(p(x)) = x$ for $x > x_1$. Then the function p grows slower than any power function, i.e. for any $\delta > 0$ we have $p(x) = o(x^\delta)$, $x \to +\infty$.

1. Let $\varphi(x) = x(\ln|q(x)|)'$ for $x > 0$. Then from the equality (1.3) it follows that $\varphi(x) \to +\infty$ for $x \to +\infty$. For any numbers $x > x_1 > 0$ and any given $k \in \mathbb{N}$ the following is true:

$$\left(\frac{x_1}{x}\right)^k |q(x)| = \left(\frac{x_1}{x}\right)^k |q(x_1)| \exp \int_{x_1}^{x} \frac{\varphi(t)}{t} dt = |q(x_1)| \exp \int_{x_1}^{x} \frac{\varphi(t) - k}{t} dt.$$

Since the function φ is infinitely large, we can choose $x_1 > 0$ so that for all numbers $x > x_1$ the following relations are satisfied:

$$\int_{x_1}^{x} \frac{\varphi(t) - k}{t} dt > \int_{x_1}^{x} \frac{1}{t} dt = \ln\left(\frac{x}{x_1}\right).$$

From this and from the chain of equalities obtained earlier it follows that $|q(x)|x^{-k} \to +\infty$ for $x \to +\infty$ and for any given natural k, that is, the modulus of the function q increases faster than any power function.

Since the inequality $\dfrac{q'(x)}{q(x)} > 0$ holds for $x > 0$, and hence the values $q'(x)$ and $q(x)$ have the same sign, it remains to show that they are positive. Let us assume that this is not the case. Then, by virtue of the relation (1.2), the function $|q|$ is convex upward for values of the argument $x \geq x_0$ and its graph lies below the tangent drawn at some point $x_2 \geq x_0$, and hence $|q|$ cannot grow faster than any power function. The resulting contradiction completes the proof of the first statement of the lemma.

2. As proved earlier, for any $k \in \mathbb{N}$ there exists a $\varkappa_k$ such that for all $x > \varkappa_k$ the inequality $q(x) > x^{k+1}$ holds. Then, due to the strict increase of the function p, we obtain that $p(q(x)) > p(x^{k+1})$, and therefore, $x > p(x^{k+1})$ for $x > \varkappa_k$. Thus, for $t > \varkappa_k^{k+1}$ the inequality $p(t) < t^{\frac{1}{k+1}}$ holds. From here we obtain the following relations:

$$0 < \frac{p(t)}{t^{\frac{1}{k}}} < \frac{t^{\frac{1}{k+1}}}{t^{\frac{1}{k}}} = t^{\frac{1}{k+1}-\frac{1}{k}} \to 0, \quad t \to +\infty.$$

The obtained relation means that $\lim\limits_{t\to+\infty} \dfrac{p(t)}{t^{\frac{1}{k}}} = 0$, $t \to +\infty$, therefore, $p(t) = o(t^{\frac{1}{k}})$ for any $k \in \mathbb{N}$. Since for any number $\delta > 0$ there is a number $k_\delta \in \mathbb{N}$ such that $\delta > \dfrac{1}{k_\delta}$, we obtain that $p(x) = o(x^\delta)$, $x \to +\infty$. □

Further, without loss of generality, we will assume that $q(x) > 0$, $q'(x) > 0$ and $q''(x) \geq 0$ for any $x > 0$.

Example 1.1:
All the entire functions with non-negative Taylor coefficients, other than a polynomial, belong to the class $\mathfrak{Q}$.

Since the logarithm of the maximum absolute value of an entire function $q(z)$ in the disk $|z| \leq x$ is a downward convex function of $\ln x$ (see [2]), then $\varphi(x)$ is non-decreasing. The function $\varphi(x)$ is not bounded above, otherwise the equality $q(x) = O(x^m)$ would hold for some $m \in \mathbb{N}$. As proved earlier, we obtain $\lim\limits_{x\to+\infty} \varphi(x) = +\infty$, that is, relation (1.3) holds, and hence $q(x) \in \mathfrak{Q}$. □

Let $\widetilde{\mathfrak{Q}}$ denote the subclass of functions $q \in \mathfrak{Q}$ satisfying the following condition for at least one value $1 < \gamma < 4/3$

$$q''(x) \leq (q'(x))^\gamma, \quad x \geq x_0. \tag{1.4}$$

Example 1.2:
Entire functions of finite order of the form $q(z) = \sum\limits_{n=0}^{+\infty} a_n z^n$, $a_n \geq 0$, $n \in \mathbb{N}_0$, *other than a polynomial, lie in the class* $\widetilde{\mathfrak{Q}}$.

Let $f(z) = \sum_{n=0}^{+\infty} b_n z^n$ be an entire function of finite order $\rho > 0$ with non-negative Taylor series coefficients. The derivative $f'(z)$ has the same order ρ as $f(z)$ itself. Therefore, to prove the inequality (1.4) it suffices to show that for any $\varepsilon > 0$:

$$f'(x) = \sum_{n=0}^{+\infty} n b_n x^{n-1} = o(f(x))^{1+\varepsilon}, \ x \to +\infty.$$

Let $\beta > \rho$. Then for some constant $C > 0$ the inequality holds:

$$\max_{|z|\leq x} |f(z)| = f(x) \leq C \exp(x^\beta).$$

Therefore,

$$|b_n| \le \inf_{x>0} x^{-n} f(x) \le C \inf_{x>0} \exp(x^\beta - n\ln x) = C\exp\Big(-\frac{n}{\beta}\ln\frac{n}{e\beta}\Big).$$

From the last inequality for $n > e^{\beta+1}\beta x^\beta$ we find

$$|b_n|x^n \le C\exp\Big(-\frac{n}{\beta}\ln\frac{n}{e\beta} + n\ln x\Big) \le C\exp\Big(-\frac{n}{\beta}\ln(e^\beta x^\beta) + n\ln x\Big) =$$
$$= C\exp(-n\ln(ex) + n\ln x) = C\exp(n(\ln x - \ln(ex))) = Ce^{-n}.$$

It follows that $\sum\limits_{n>e^{\beta+1}\beta x^\beta} nb_n x^n \le C_1,\ C_1 > 0$. Therefore, for $x \ge 1$ we have the inequality

$$f'(x) = \sum_{n\le e^{\beta+1}\beta x^\beta} nb_n x^{n-1} + \sum_{n>e^{\beta+1}\beta x^\beta} nb_n x^{n-1} \le$$
$$\le C_1 + e^{\beta+1}\beta x^\beta \sum_{n\le e^{\beta+1}\beta x^\beta} b_n x^{n-1} \le C_1 + C_2\frac{f(x)}{x}x^\beta.$$

Since f grows faster than any power function, $\forall\varepsilon > 0$ we get $f'(x) = o(f(x))^{1+\varepsilon}$, $x \to +\infty$. □

For $\beta > 1$ and $\mu > 0$ we denote by $\mathfrak{Q}_{\beta,\mu}$ the class of functions $q \in \widetilde{\mathfrak{Q}}$ such that

$$\ln q(x) = \mu\ln^\beta x + o(\ln^{\beta-1} x), \quad x \to +\infty. \tag{1.5}$$

The following statement allows us to rewrite this condition for the potential in terms of the inverse function.

Lemma 1.2:
Let q be an arbitrary function in the class $\mathfrak{Q}_{\beta,\mu}$. Let p be the inverse function to q, $\delta = \mu^{-\frac{1}{\beta}}$. Then the relation is satisfied

$$\ln p(x) = \delta\ln^{\frac{1}{\beta}} x + o(1), \quad x \to +\infty.$$

Let us rewrite the expression (1.5) as follows: $\ln x = \mu\ln^\beta p(x) + o(\ln^{\beta-1} p(x))$, $x \to +\infty$. For some $\varepsilon(x) = o(1)$, $x \to +\infty$, we obtain the expression:

$$\ln x = \mu\ln^\beta p(x)\Big(1 + \frac{\varepsilon(x)}{\ln p(x)}\Big) = \mu\ln^\beta p(x)\cdot\alpha(x),$$

where $\alpha(x) = 1 + \dfrac{\varepsilon(x)}{\ln p(x)} \to 1$, $x \to +\infty$. From here, expressing $\ln p(x)$, we obtain the equality:

$$\ln p(x) = \Big(\frac{1}{\mu}\Big)^{\frac{1}{\beta}}\ln^{\frac{1}{\beta}} x\Big(1 + \frac{\varepsilon(x)}{\ln p(x)}\Big)^{-\frac{1}{\beta}}.$$

Then, taking into account the notation for δ and according to the binomial expansion, we obtain the following relationship:

$$\ln p(x) = \delta\ln^{\frac{1}{\beta}} x - \frac{\delta\ln^{\frac{1}{\beta}} x\cdot\varepsilon(x)}{\beta\ln p(x)}\cdot(1 + o(1)), \quad x \to +\infty.$$

Since $\delta \ln^{\frac{1}{\beta}} x = \ln p(x) \cdot \alpha^{\frac{1}{\beta}}(x)$, the resulting expression can be rewritten as follows:

$$\ln p(x) = \delta \ln^{\frac{1}{\beta}} x - \frac{\ln p(x) \cdot \alpha^{\frac{1}{\beta}}(x) \cdot \varepsilon(x)}{\beta \ln p(x)} \cdot (1 + o(1)), \quad x \to +\infty.$$

This means that $\ln p(x) = \delta \ln^{\frac{1}{\beta}} x + o(1)$, $x \to +\infty$, since $\alpha(x) \to 1$ and $\varepsilon(x) \to 0$ when $x \to +\infty$. □

The expression (1.5) for the parameter $\beta = 1$ means a power-law growth of the potential q, for which E. C. Titchmarsh obtained the asymptotics (0.1). For the parameter $\beta > 2$, the potential $q \in \mathfrak{Q}_{\beta,\mu}$ satisfies the condition:

$$\frac{\ln q(x)}{\ln^2 x} \to +\infty, \quad x \to +\infty.$$

Under such conditions, the spectrum of the operator $\mathbb{L}_q$ has an asymptotics (A. I. Kozko [3])

$$\lambda_n \sim (\pi n)^2 p^{-2}((\pi n)^2), \quad n \to +\infty,$$

where p is the inverse function to q. The following result of A. I. Kozko [3] establishes the asymptotics of the spectrum of the operator $\mathbb{L}_q$ for potentials of the class $\mathfrak{Q}_{\beta,\mu}$ in the case of the parameter value $\beta = 2$:

$$\lambda_n \sim (\pi n)^2 p^{-2}((\pi n)^2) \exp\left(\frac{2}{\mu}\right), \quad n \to +\infty. \tag{1.6}$$

Later, A. Yu. Kiseleva (personal communication) found asymptotic expansions for the eigenvalues of the Sturm–Liouville operator in the problem under consideration for the potential of class $\mathfrak{Q}_{\beta,\mu}$ and values of the parameter $\beta \in (3/2, 2]$:

$$\lambda_n \sim (\pi n)^2 p^{-2}((\pi n)^2) \exp\left(\frac{4}{\mu\beta} \ln^{2-\beta} p((\pi n)^2)\right), \quad n \to +\infty. \tag{1.7}$$

and $\beta \in (4/3, 3/2]$:

$$\lambda_n \sim (\pi n)^2 p^{-2}((\pi n)^2) \exp\left(\frac{4}{\mu\beta} \ln^{2-\beta} p((\pi n)^2) - \frac{4}{\mu^2\beta}\left(\frac{3}{\beta} - 1\right) \ln^{3-2\beta} p((\pi n)^2)\right), \; n \to +\infty. \tag{1.8}$$

The study of the asymptotics of the eigenvalues of the operator $\mathbb{L}_q$ was continued by I. G. Nasrtdinov [7] for values of the parameter β closer to unity. Thus, for potential $q \in \mathfrak{Q}_{\beta,\mu}$, parameter values $\beta \in (5/4, 4/3]$ and $\nu = \delta^{\frac{1}{\beta}}$ the following holds:

$$\lambda_n \sim (\pi n)^2 \exp\Bigg(-\Bigg(2\nu^{\beta} \ln^{\frac{1}{\beta}}((\pi n)^2) - \frac{4}{\beta}\nu^{2\beta} \ln^{\frac{2}{\beta}-1}((\pi n)^2) + $$
$$+ \frac{4(3-\beta)}{\beta^2}\nu^{3\beta} \ln^{\frac{3}{\beta}-2}((\pi n)^2) - \frac{16(8-6\beta+\beta^2)}{3\beta^3}\nu^{4\beta} \ln^{\frac{4}{\beta}-3}((\pi n)^2)\Bigg)\Bigg), \quad n \to +\infty. \tag{1.9}$$

Using Lemma 1.2, we can rewrite this result in terms of the inverse function p. We obtain the following form of asymptotics:

$$\lambda_n \sim (\pi n)^2 p^{-2}((\pi n)^2) \exp\Bigg(\frac{4}{\mu\beta} \ln^{2-\beta} p((\pi n)^2) - \frac{4}{\mu^2\beta}\left(\frac{3}{\beta} - 1\right) \ln^{3-2\beta} p((\pi n)^2) + $$
$$+ \frac{16(8-6\beta+\beta^2)}{3\mu^3\beta^3} \ln^{4-3\beta}(p(\pi n)^2)\Bigg), \quad n \to +\infty.$$

2. THE MAIN RESULT AND ITS PROOF

Let us denote $c_n = (\pi n)^2$, $n \in \mathbb{N}$. In paper [3] it is proved that in the case of $q \in \widetilde{\mathfrak{Q}}$ the asymptotics $n \sim \frac{1}{\pi}\lambda_n^{1/2} p(\lambda_n)$, $n \to +\infty$ holds. From here we get that $\lambda_n \sim \frac{c_n}{p^2(\lambda_n)}$, $n \to +\infty$, that is, for some sequence $\alpha_n \to 1$, $n \to +\infty$ the following equality holds:

$$\lambda_n = \alpha_n \frac{c_n}{p^2(\lambda_n)},\ n \in \mathbb{N}.$$

Then $\lim\limits_{n\to+\infty} \frac{\lambda_n}{c_n} = \lim\limits_{n\to+\infty} \frac{\alpha_n}{p^2(\lambda_n)} = 0$ due to the unlimited monotonic growth of the function p. This means that $\lambda_n = o(c_n)$, $n \to +\infty$. Therefore, starting from some number, the inequality $\lambda_n < c_n$ is satisfied.

In the previously adopted notation, we set by definition:

$$Y_n = \frac{c_n}{p^2(c_n)}, \quad Z_n = \frac{c_n}{p^2(Y_n\alpha_n)}, \quad W_n = \frac{c_n}{p^2(Z_n\alpha_n)},$$
$$V_n = \frac{c_n}{p^2(W_n\alpha_n)}, \quad F_n = \frac{c_n}{p^2(V_n\alpha_n)}, \quad G_n = \frac{c_n}{p^2(F_n\alpha_n)}.$$

Lemma 2.1:
For $\beta > 1$ and $\mu > 0$, consider an arbitrary function $q \in \mathfrak{Q}_{\beta,\mu}$ and its inverse function p. Let us use the notation for the sequences introduced above. Then, in these notation, starting from some number, the inequalities hold

$$Y_n\alpha_n < W_n\alpha_n < F_n\alpha_n < \lambda_n < G_n\alpha_n < V_n\alpha_n < Z_n\alpha_n < c_n.$$

As has been proved, $\lambda_n < c_n$, starting from some number N. Due to the strict increase of the function p, for all numbers $n > N$ the inequality $p^2(\lambda_n) < p^2(c_n)$ is satisfied, and therefore the following chain of relations holds:

$$Y_n\alpha_n = \alpha_n \frac{c_n}{p^2(c_n)} < \alpha_n \frac{c_n}{p^2(\lambda_n)} = \lambda_n < c_n, \quad n > N.$$

Thus, for all numbers $n > N$ the double inequality $Y_n\alpha_n < \lambda_n < c_n$ is proved. The inequality $p^2(Y_n\alpha_n) < p^2(\lambda_n)$ for $n > N$ implies that

$$\lambda_n = \alpha_n \frac{c_n}{p^2(\lambda_n)} < \alpha_n \frac{c_n}{p^2(Y_n\alpha_n)} = Z_n\alpha_n, \quad n > N.$$

From this and the previously obtained inequalities we can state that $Y_n\alpha_n < \lambda_n < Z_n\alpha_n$ for $n > N$.

Let us establish that $Z_n\alpha_n < c_n$, starting from some number. Since by Lemma 1.1 the function p grows slower than any power function, in particular, $p^2(c_n) = o(c_n)$, $n \to +\infty$, we obtain that $Y_n\alpha_n = \frac{c_n\alpha_n}{p^2(c_n)} \to +\infty$, $n \to +\infty$. Hence,

$$\frac{Z_n\alpha_n}{c_n} = \frac{\alpha_n}{p^2(Y_n\alpha_n)} \to 0, \quad n \to +\infty.$$

Thus, $Z_n\alpha_n = o(c_n)$, $n \to +\infty$, which implies the inequality $Z_n\alpha_n < c_n$ from some number. Without loss of generality, we will assume that this number is equal to N. Thus, we obtain a chain of inequalities $Y_n\alpha_n < \lambda_n < Z_n\alpha_n < c_n$ for $n > N$.

With use of the strict increase of the function p and the inequality $Z_n\alpha_n < c_n$ established for $n > N$ we obtain the required inequality

$$Y_n = \frac{c_n}{p^2(c_n)} < \frac{c_n}{p^2(Z_n\alpha_n)} = W_n, \quad n > N.$$

Combining this inequality with the previously obtained relations we get $Y_n\alpha_n < W_n\alpha_n < \lambda_n < Z_n\alpha_n < c_n$ for $n > N$. Next, all necessary inequalities on $V_n\alpha_n$, $F_n\alpha_n$ and $G_n\alpha_n$ are established in a similar manner. □

Theorem 2.1:
Let $q \in \mathfrak{Q}_{\beta,\mu}$, $\beta \in (6/5, 5/4]$. *Then for the spectrum of the operator* $\mathbb{L}_q$ *the following holds:*

$$\lambda_n \sim c_n \exp\left(-2\delta\left(\ln^{\frac{1}{\beta}} c_n - \frac{2\delta}{\beta}\ln^{\frac{2}{\beta}-1} c_n + (2\delta)^2\frac{3-\beta}{2\beta^2}\ln^{\frac{3}{\beta}-2} c_n - \right.\right.$$
$$\left.\left. - (2\delta)^3\frac{8-6\beta+\beta^2}{3\beta^3}\ln^{\frac{4}{\beta}-3} c_n + (2\delta)^4\frac{125-150\beta+55\beta^2-6\beta^3}{24\beta^4}\ln^{\frac{5}{\beta}-4} c_n\right)\right),\ n \to +\infty.$$

Using the formula from lemma 1.2, we can rewrite this result in terms of the inverse function p. We obtain the following form of asymptotics:

$$\lambda_n \sim (\pi n)^2 p^{-2}((\pi n)^2)\exp\left(4\frac{1}{\mu\beta}\ln^{2-\beta} p((\pi n)^2) - \frac{4}{\mu^2\beta}\left(\frac{3}{\beta}-1\right)\ln^{3-2\beta} p((\pi n)^2) + \right.$$
$$\left. + \frac{16(8-6\beta+\beta^2)}{3\mu^3\beta^3}\ln^{4-3\beta}(p(\pi n)^2) - \frac{4(125-150\beta+55\beta^2-6\beta^3)}{3\mu^4\beta^4}\ln^{5-4\beta} p((\pi n)^2)\right),$$
$$n \to +\infty.$$

To prove the theorem, it suffices to establish that

$$F_n \sim G_n \sim c_n \exp\left(-2\delta\left(\ln^{\frac{1}{\beta}} c_n - \frac{2\delta}{\beta}\ln^{\frac{2}{\beta}-1} c_n + (2\delta)^2\frac{3-\beta}{2\beta^2}\ln^{\frac{3}{\beta}-2} c_n - \right.\right.$$
$$\left.\left. - (2\delta)^3\frac{8-6\beta+\beta^2}{3\beta^3}\ln^{\frac{4}{\beta}-3} c_n + (2\delta)^4\frac{125-150\beta+55\beta^2-6\beta^3}{24\beta^4}\ln^{\frac{5}{\beta}-4} c_n\right)\right),\ n \to +\infty.$$

Then by Lemma 2.1 we will obtain that $\lambda_n \sim F_n \sim G_n$, $n \to +\infty$. Let us find asymptotic expansions for the sequences F_n and G_n.

1. We write the relation using the expression for the inverse function from the lemma 1.2:

$$\ln p(Y_n\alpha_n) = \delta \ln^{\frac{1}{\beta}}(Y_n\alpha_n) + o(1) = \delta(\ln Y_n + \ln\alpha_n)^{\frac{1}{\beta}} + o(1), \quad n \to +\infty.$$

Since $\ln\alpha_n = o(1)$, $n \to +\infty$, we obtain

$$\ln p(Y_n\alpha_n) = \delta \ln^{\frac{1}{\beta}} Y_n\left(1 + \frac{o(1)}{\ln Y_n}\right)^{\frac{1}{\beta}} + o(1), \quad n \to +\infty.$$

Taking into account Taylor's formula and the inequality $\dfrac{1}{\beta} - 1 < 0$, the term on the right-hand side of the equality can be written as

$$\delta \ln^{\frac{1}{\beta}} Y_n\left(1 + \frac{1}{\beta}o(\ln^{-1} Y_n)\right) = \delta \ln^{\frac{1}{\beta}} Y_n + o(1), \quad n \to +\infty.$$

Thus, the relation $\ln p(Y_n\alpha_n) = \delta \ln^{\frac{1}{\beta}} Y_n + o(1)$, $n \to +\infty$ is established. By definition of Y_n and in view of Lemma 1.2, we have the following relation

$$\ln^{\frac{1}{\beta}} Y_n = (\ln c_n - 2(\delta \ln^{\frac{1}{\beta}} c_n + o(1)))^{\frac{1}{\beta}}, \quad n \to +\infty.$$

We take the multiplier $\ln^{\frac{1}{\beta}} c_n$ out of the brackets and expand the last expression $(\ln^{\frac{1}{\beta}} c_n) \cdot (1 - \frac{1}{\ln c_n}\big(2\delta \ln^{\frac{1}{\beta}} c_n + o(1)\big)^{\frac{1}{\beta}}$, $n \to +\infty$ into a Taylor series. The second factor, using Taylor's formula and the Pochhammer symbol $(x)_n = x(x-1)\dots(x-(n-1))$, can be written as follows:

$$1 + \frac{1}{\beta}(-2\delta \ln^{\frac{1}{\beta}-1} c_n + o(\ln^{-1} c_n)) + \frac{1}{2!}\left(\frac{1}{\beta}\right)_2 ((2\delta)^2 \ln^{\frac{2}{\beta}-2} c_n + o(\ln^{\frac{1}{\beta}-2} c_n) +$$
$$o(\ln^{-2} c_n)) + \frac{1}{3!}\left(\frac{1}{\beta}\right)_3 (-(2\delta)^3 \ln^{\frac{3}{\beta}-3} c_n + o(\ln^{\frac{2}{\beta}-3} c_n) + o(\ln^{\frac{1}{\beta}-3} c_n) + o(\ln^{-3} c_n)) +$$
$$+ \frac{1}{4!}\left(\frac{1}{\beta}\right)_4 ((2\delta)^4 \ln^{\frac{4}{\beta}-4} c_n + o(\ln^{\frac{3}{\beta}-4} c_n) + o(\ln^{\frac{2}{\beta}-4} c_n) + o(\ln^{\frac{1}{\beta}-4} c_n) + o(\ln^{-4} c_n)) +$$
$$+ O(\ln^{\frac{5}{\beta}-5} c_n + o(\ln^{\frac{4}{\beta}-5} c_n) + o(\ln^{\frac{3}{\beta}-5} c_n) + o(\ln^{\frac{2}{\beta}-5} c_n) + o(\ln^{\frac{1}{\beta}-5} c_n) +$$
$$+ o(\ln^{-5} c_n)), n \to +\infty.$$

After all the transformations, taking into account that parameter values are $\beta \in \left(\frac{6}{5}, \frac{5}{4}\right]$, and therefore the relation $O(\ln^{\frac{6}{\beta}-5} c_n) = o(1)$, $n \to +\infty$, we obtain that

$$\ln p(Y_n\alpha_n) = \delta(\ln^{\frac{1}{\beta}} c_n - \frac{2\delta}{\beta} \ln^{\frac{2}{\beta}-1} c_n + \frac{(2\delta)^2}{2!}\left(\frac{1}{\beta}\right)_2 \ln^{\frac{3}{\beta}-2} c_n -$$
$$- \frac{(2\delta)^3}{3!}\left(\frac{1}{\beta}\right)_3 \ln^{\frac{4}{\beta}-3} c_n + \frac{(2\delta)^4}{4!}\left(\frac{1}{\beta}\right)_4 \ln^{\frac{5}{\beta}-4} c_n) + o(1), \ n \to +\infty.$$

2. Similar to the previous step, we have

$$\ln p(Z_n\alpha_n) = \delta \ln^{\frac{1}{\beta}} Z_n + o(1) = \delta\big(\ln c_n - 2\ln p(Y_n\alpha_n) + o(1)\big)^{\frac{1}{\beta}} + o(1), \quad n \to +\infty.$$

After substituting the expression obtained above for $\ln p(Y_n\alpha_n)$, we obtain that

$$\ln p(Z_n\alpha_n) = \delta\Bigg(\ln c_n - 2\delta\Bigg(\ln^{\frac{1}{\beta}} c_n - \frac{2\delta}{\beta} \ln^{\frac{2}{\beta}-1} c_n + \frac{(2\delta)^2}{2!}\left(\frac{1}{\beta}\right)_2 \ln^{\frac{3}{\beta}-2} c_n -$$
$$- \frac{(2\delta)^3}{3!}\left(\frac{1}{\beta}\right)_3 \ln^{\frac{4}{\beta}-3} c_n + \frac{(2\delta)^4}{4!}\left(\frac{1}{\beta}\right)_4 \ln^{\frac{5}{\beta}-4} c_n + o(1)\Bigg)\Bigg)^{\frac{1}{\beta}} + o(1), \quad n \to +\infty.$$

To simplify the calculations, we set $\xi_{\beta,n} = 2\delta \ln^{\frac{1}{\beta}-1} c_n$, $n \in \mathbb{N}$. Then, using the introduced notation, we have

$$\ln p(Z_n\alpha_n) = \delta \ln^{\frac{1}{\beta}} c_n\Bigg(1 - \Bigg(\xi_{\beta,n} - \frac{1}{\beta}\xi^2_{\beta,n} + \frac{1}{2!}\left(\frac{1}{\beta}\right)_2 \xi^3_{\beta,n} - \frac{1}{3!}\left(\frac{1}{\beta}\right)_3 \xi^4_{\beta,n} +$$
$$+ \frac{1}{4!}\left(\frac{1}{\beta}\right)_4 \xi^5_{\beta,n}\Bigg) + o(1)\Bigg)^{\frac{1}{\beta}} + o(1), n \to +\infty,$$

from which, using Taylor's formula, we obtain

$$
\begin{aligned}
&\ln p(Z_n\alpha_n) = \\
&= \delta \ln^{\frac{1}{\beta}} c_n \Bigg(1 - \frac{1}{\beta}\Bigg(\xi_{\beta,n} - \frac{1}{\beta}\xi_{\beta,n}^2 + \frac{1}{2!}\left(\frac{1}{\beta}\right)_2 \xi_{\beta,n}^3 - \frac{1}{3!}\left(\frac{1}{\beta}\right)_3 \xi_{\beta,n}^4 + \frac{1}{4!}\left(\frac{1}{\beta}\right)_4 \xi_{\beta,n}^5\Bigg) + \\
&+ \frac{1}{2!}\left(\frac{1}{\beta}\right)_2 \left(\xi_{\beta,n}^2 - \frac{2}{\beta}\xi_{\beta,n}^3 + \frac{1}{\beta^2}\xi_{\beta,n}^4 + 2\frac{1}{2!}\left(\frac{1}{\beta}\right)_2 \xi_{\beta,n}^4\right) - \frac{1}{3!}\left(\frac{1}{\beta}\right)_3 \left(\xi_{\beta,n}^3 - \frac{3}{\beta}\xi_{\beta,n}^4\right) + \\
&\qquad + \frac{1}{4!}\left(\frac{1}{\beta}\right)_4 \xi_{\beta,n}^4 + O(\ln^{\frac{5}{\beta}-5} c_n)\Bigg) + o(1), \quad n \to +\infty.
\end{aligned}
$$

Hence, expanding the Pochhammer symbols and calculating the coefficients, and also in view of the relation $O(\ln^{\frac{6}{\beta}-5} c_n) = o(1)$, $n \to +\infty$, we obtain that

$$
\begin{aligned}
\ln p(Z_n\alpha_n) = \delta \ln^{\frac{1}{\beta}} c_n \Bigg(1 - \frac{1}{\beta}\xi_{\beta,n} + \frac{3-\beta}{2\beta^2}\xi_{\beta,n}^2 - \frac{5 - 6\beta + \beta^2}{3\beta^3}\xi_{\beta,n}^3 + \\
+ \frac{41 - 90\beta + 55\beta^2 - 6\beta^3}{24\beta^4}\xi_{\beta,n}^4\Bigg) + o(1), n \to +\infty.
\end{aligned}
$$

3. Next, using similar calculations, we obtain the relation $\ln p(W_n\alpha_n) = \delta\big(\ln c_n - 2\ln p(Z_n\alpha_n) + o(1)\big)^{\frac{1}{\beta}} + o(1)$, $n \to +\infty$. Using the expression for $\ln p(Z_n\alpha_n)$ obtained in the previous paragraph, we have

$$
\begin{aligned}
\ln p(W_n\alpha_n) = \delta\Bigg(\ln c_n - 2\delta \ln^{\frac{1}{\beta}} c_n \Bigg(1 - \frac{1}{\beta}\xi_{\beta,n} + \frac{3-\beta}{2\beta^2}\xi_{\beta,n}^2 - \frac{5 - 6\beta + \beta^2}{3\beta^3}\xi_{\beta,n}^3 + \\
+ \frac{41 - 90\beta + 55\beta^2 - 6\beta^3}{24\beta^4}\xi_{\beta,n}^4\Bigg) + o(1)\Bigg)^{\frac{1}{\beta}} + o(1), \quad n \to +\infty.
\end{aligned}
$$

For convenience, let's put $\ln^{\frac{1}{\beta}} c_n$ out of brackets again:

$$
\begin{aligned}
\ln p(W_n\alpha_n)) = \delta \ln^{\frac{1}{\beta}} c_n \Bigg(1 - \Bigg(\xi_{\beta,n} - \frac{1}{\beta}\xi_{\beta,n}^2 + \frac{3-\beta}{2\beta^2}\xi_{\beta,n}^3 - \frac{5 - 6\beta + \beta^2}{3\beta^3}\xi_{\beta,n}^4 + \\
+ \frac{41 - 90\beta + 55\beta^2 - 6\beta^3}{24\beta^4}\xi_{\beta,n}^5 + o(\ln^{-1} c_n)\Bigg)\Bigg)^{\frac{1}{\beta}} + o(1), \quad n \to +\infty.
\end{aligned}
$$

Then, using Taylor's formula:

$$
\begin{aligned}
&\ln p(W_n\alpha_n)) = \delta \ln^{\frac{1}{\beta}} c_n \Bigg(1 - \frac{1}{\beta}\Bigg(\xi_{\beta,n} - \frac{1}{\beta}\xi_{\beta,n}^2 + \frac{3-\beta}{2\beta^2}\xi_{\beta,n}^3 - \frac{5 - 6\beta + \beta^2}{3\beta^3}\xi_{\beta,n}^4 + \\
&+ \frac{41 - 90\beta + 55\beta^2 - 6\beta^3}{24\beta^4}\xi_{\beta,n}^5\Bigg) + \frac{1}{2!}\left(\frac{1}{\beta}\right)_2 \left(\xi_{\beta,n}^2 + \frac{1}{\beta^2}\xi_{\beta,n}^4 - \frac{2}{\beta}\xi_{\beta,n}^3 + 2\frac{3-\beta}{2\beta^2}\xi_{\beta,n}^4\right) - \\
&- \frac{1}{3!}\left(\frac{1}{\beta}\right)_3 \left(\xi_{\beta,n}^3 - \frac{3}{\beta}\xi_{\beta,n}^4\right) + \frac{1}{4!}\left(\frac{1}{\beta}\right)_4 \xi_{\beta,n}^4 + O(\ln^{\frac{5}{\beta}-5} c_n)\Bigg) + o(1), \quad n \to +\infty.
\end{aligned}
$$

Hence, taking into account the relation $O(\ln^{\frac{6}{\beta}-5} c_n) = o(1)$, $n \to +\infty$, expanding the Pochhammer symbols and calculating the coefficients, we obtain that

$$\ln p(W_n\alpha_n) = \delta \ln^{\frac{1}{\beta}} c_n\left(1 - \frac{1}{\beta}\xi_{\beta,n} + \frac{3-\beta}{2\beta^2}\xi_{\beta,n}^2 - \frac{8-6\beta+\beta^2}{3\beta^3}\xi_{\beta,n}^3 + \right. \\ \left. + \frac{101-150\beta+55\beta^2-6\beta^3}{24\beta^4}\xi_{\beta,n}^4\right) + o(1), n \to +\infty.$$

4. Similar to the previous step, we have $\ln p(V_n\alpha_n) = \delta\big(\ln c_n - 2\ln p(W_n\alpha_n) + o(1)\big)^{\frac{1}{\beta}} + o(1)$, $n \to +\infty$. Substituting the resulting expression for $\ln p(W_n\alpha_n)$, we have

$$\ln p(V_n\alpha_n) = \delta\left(\ln c_n - 2\delta \ln^{\frac{1}{\beta}} c_n\left(1 - \frac{1}{\beta}\xi_{\beta,n} + \frac{3-\beta}{2\beta^2}\xi_{\beta,n}^2 - \frac{8-6\beta+\beta^2}{3\beta^3}\xi_{\beta,n}^3 + \right.\right. \\ \left.\left. + \frac{101-150\beta+55\beta^2-6\beta^3}{24\beta^4}\xi_{\beta,n}^4\right) + o(1)\right)^{\frac{1}{\beta}} + o(1), \quad n \to +\infty.$$

We take the multiplier $\ln^{\frac{1}{\beta}} c_n$ out of the brackets and expand the last expression into a Taylor series. We obtain the following relationship:

$$\ln p(V_n\alpha_n) = \delta \ln^{\frac{1}{\beta}} c_n\left(1 - \frac{1}{\beta}\left(\xi_{\beta,n} - \frac{1}{\beta}\xi_{\beta,n}^2 + \frac{3-\beta}{2\beta^2}\xi_{\beta,n}^3 - \frac{8-6\beta+\beta^2}{3\beta^3}\xi_{\beta,n}^4 + \right.\right. \\ \left. + \frac{101-150\beta+55\beta^2-6\beta^3}{24\beta^4}\xi_{\beta,n}^5\right) + \frac{1}{2!}\left(\frac{1}{\beta}\right)_2\left(\xi_{\beta,n}^2 + \frac{1}{\beta^2}\xi_{\beta,n}^4 - \frac{2}{\beta}\xi_{\beta,n}^3 + 2\frac{3-\beta}{2\beta^2}\xi_{\beta,n}^4\right) - \\ \left. - \frac{1}{3!}\left(\frac{1}{\beta}\right)_3\left(\xi_{\beta,n}^3 - \frac{3}{\beta}\xi_{\beta,n}^4\right) + \frac{1}{4!}\left(\frac{1}{\beta}\right)_4\xi_{\beta,n}^4 + O(\ln^{\frac{5}{\beta}-5} c_n)\right) + o(1), \quad n \to +\infty.$$

After calculating all the coefficients and taking into account the relation $O(\ln^{\frac{6}{\beta}-5} c_n) = o(1)$, $n \to +\infty$, we obtain that

$$\ln p(V_n\alpha_n) = \delta \ln^{\frac{1}{\beta}} c_n\left(1 - \frac{1}{\beta}\xi_{\beta,n} + \frac{3-\beta}{2\beta^2}\xi_{\beta,n}^2 - \frac{8-6\beta+\beta^2}{3\beta^3}\xi_{\beta,n}^3 + \right. \\ \left. + \frac{125-150\beta+55\beta^2-6\beta^3}{24\beta^4}\xi_{\beta,n}^4\right) + o(1), n \to +\infty. \quad (2.10)$$

5. Using the same reasoning, we obtain that $\ln p(F_n\alpha_n) = \delta\big(\ln c_n - 2\ln p(V_n\alpha_n) + o(1)\big)^{\frac{1}{\beta}} + o(1)$, $n \to +\infty$. From where, using the expression found (2.10), we have

$$\ln p(F_n\alpha_n) = \delta\left(\ln c_n - 2\delta \ln^{\frac{1}{\beta}} c_n\left(1 - \frac{1}{\beta}\xi_{\beta,n} + \frac{3-\beta}{2\beta^2}\xi_{\beta,n}^2 - \frac{8-6\beta+\beta^2}{3\beta^3}\xi_{\beta,n}^3 + \right.\right. \\ \left.\left. + \frac{125-150\beta+55\beta^2-6\beta^3}{24\beta^4}\xi_{\beta,n}^4\right) + o(1)\right)^{\frac{1}{\beta}} + o(1), \quad n \to +\infty.$$

We take the multiplier $\ln^{\frac{1}{\beta}} c_n$ out of the brackets again and expand the last expression into a Taylor series. We obtain the following relationship:

$$\ln p(F_n\alpha_n) = \delta \ln^{\frac{1}{\beta}} c_n\Bigg(1 - \frac{1}{\beta}\Bigg(\xi_{\beta,n} - \frac{1}{\beta}\xi^2_{\beta,n} + \frac{3-\beta}{2\beta^2}\xi^3_{\beta,n} - \frac{8-6\beta+\beta^2}{3\beta^3}\xi^4_{\beta,n} + \\ + \frac{125-150\beta+55\beta^2-6\beta^3}{24\beta^4}\xi^5_{\beta,n}\Bigg) + \frac{1}{2!}\left(\frac{1}{\beta}\right)_2\Bigg(\xi^2_{\beta,n} + \frac{1}{\beta^2}\xi^4_{\beta,n} - \frac{2}{\beta}\xi^3_{\beta,n} + 2\frac{3-\beta}{2\beta^2}\xi^4_{\beta,n}\Bigg) - \\ - \frac{1}{3!}\left(\frac{1}{\beta}\right)_3\Bigg(\xi^3_{\beta,n} - \frac{3}{\beta}\xi^4_{\beta,n}\Bigg) + \frac{1}{4!}\left(\frac{1}{\beta}\right)_4\xi^4_{\beta,n} + O(\ln^{\frac{5}{\beta}-5} c_n)\Bigg) + o(1), \quad n\to+\infty.$$

Having calculated all the coefficients taking into account the relation $O(\ln^{\frac{6}{\beta}-5} c_n) = o(1)$, $n\to+\infty$, we obtain that

$$\ln p(F_n\alpha_n) = \delta \ln^{\frac{1}{\beta}} c_n\Bigg(1 - \frac{1}{\beta}\xi_{\beta,n} + \frac{3-\beta}{2\beta^2}\xi^2_{\beta,n} - \frac{8-6\beta+\beta^2}{3\beta^3}\xi^3_{\beta,n} + \\ + \frac{125-150\beta+55\beta^2-6\beta^3}{24\beta^4}\xi^4_{\beta,n}\Bigg) + o(1), n\to+\infty. \quad (2.11)$$

6. Using the formula (2.10), with the back substitution $\xi_{\beta,n} = 2\delta\ln^{\frac{1}{\beta}-1} c_n$, $n\in\mathbb{N}$, we have:

$$\ln p(V_n\alpha_n) = \delta\Bigg(\ln^{\frac{1}{\beta}} c_n - \frac{2\delta}{\beta}\ln^{\frac{2}{\beta}-1} c_n + (2\delta)^2\frac{3-\beta}{2\beta^2}\ln^{\frac{3}{\beta}-2} c_n - \\ - (2\delta)^3\frac{8-6\beta+\beta^2}{3\beta^3}\ln^{\frac{4}{\beta}-3} c_n + \\ + (2\delta)^4\frac{125-150\beta+55\beta^2-6\beta^3}{24\beta^4}\ln^{\frac{5}{\beta}-4} c_n\Bigg) + o(1),\ n\to+\infty.$$

By definition $F_n = c_n\exp\big(-2\ln p(V_n\alpha_n)\big)$, $n\in\mathbb{N}$, therefore the relation

$$F_n = c_n\exp\Bigg(-2\delta\Bigg(\ln^{\frac{1}{\beta}} c_n - \frac{2\delta}{\beta}\ln^{\frac{2}{\beta}-1} c_n + (2\delta)^2\frac{3-\beta}{2\beta^2}\ln^{\frac{3}{\beta}-2} c_n - \\ - (2\delta)^3\frac{8-6\beta+\beta^2}{3\beta^3}\ln^{\frac{4}{\beta}-3} c_n + \\ + (2\delta)^4\frac{125-150\beta+55\beta^2-6\beta^3}{24\beta^4}\ln^{\frac{5}{\beta}-4} c_n\Bigg) + o(1)\Bigg), \quad n\to+\infty.$$

From here we conclude that

$$F_n \sim c_n\exp\Bigg(-2\delta\Bigg(\ln^{\frac{1}{\beta}} c_n - \frac{2\delta}{\beta}\ln^{\frac{2}{\beta}-1} c_n + (2\delta)^2\frac{3-\beta}{2\beta^2}\ln^{\frac{3}{\beta}-2} c_n - \\ - (2\delta)^3\frac{8-6\beta+\beta^2}{3\beta^3}\ln^{\frac{4}{\beta}-3} c_n + \\ + (2\delta)^4\frac{125-150\beta+55\beta^2-6\beta^3}{24\beta^4}\ln^{\frac{5}{\beta}-4} c_n\Bigg)\Bigg), \quad n\to+\infty,$$

since $e^{o(1)} = 1 + o(1)$, $n \to +\infty$.
Similarly, using the formula (2.11) and in view of the definition $G_n = c_n \exp\big(-2\ln p(F_n\alpha_n)\big)$, $n \in \mathbb{N}$, we have:

$$G_n = c_n \exp\Bigg(-2\delta\Bigg(\ln^{\frac{1}{\beta}} c_n - \frac{2\delta}{\beta}\ln^{\frac{2}{\beta}-1} c_n + (2\delta)^2\frac{3-\beta}{2\beta^2}\ln^{\frac{3}{\beta}-2} c_n - \\ - (2\delta)^3\frac{8-6\beta+\beta^2}{3\beta^3}\ln^{\frac{4}{\beta}-3} c_n + \\ + (2\delta)^4\frac{125-150\beta+55\beta^2-6\beta^3}{24\beta^4}\ln^{\frac{5}{\beta}-4} c_n\Bigg) + o(1)\Bigg), \quad n \to +\infty.$$

Therefore,

$$F_n \sim G_n \sim c_n \exp\Bigg(-2\delta\Bigg(\ln^{\frac{1}{\beta}} c_n - \frac{2\delta}{\beta}\ln^{\frac{2}{\beta}-1} c_n + (2\delta)^2\frac{3-\beta}{2\beta^2}\ln^{\frac{3}{\beta}-2} c_n - \\ - (2\delta)^3\frac{8-6\beta+\beta^2}{3\beta^3}\ln^{\frac{4}{\beta}-3} c_n + \\ + (2\delta)^4\frac{125-150\beta+55\beta^2-6\beta^3}{24\beta^4}\ln^{\frac{5}{\beta}-4} c_n\Bigg)\Bigg), \quad n \to +\infty,$$

which completes the proof theorems. □

This theorem generalizes previously obtained results for the values of the parameter β from the segment $[5/4, 2]$.

REFERENCES

1. Titchmarsh, E. C. (1960) *Decomposition by eigenfunctions related to second-order differential equations, vol.1*. Oxford, UK: Clarendon Press.
2. Titchmarsh, E. C. (1980) *Theory of Functions*. Moscow, USSR: Nauka.
3. Kozko, A. I. (2005) Asymptotics of the spectrum of the differential operator $-y" + q(x)y$ with a boundary condition at zero and a rapidly growing potential, *Differential Equations*, **41**(5), 636–648.
4. Molchanov, A. M. (1953) On the discreteness conditions of the spectrum of self-adjoint differential equations of the second order, *Trudy Moskovskogo Matematicheskogo Obshchestva*, **2**, 169–200.
5. Murtazin, X. H. & Amangildin T. G. (1979) Asymptotics of the operator spectrum Sturm–Liouville, *Mathematical Collection*, **110**(1), 135–149.
6. Ishkin, X. K. (1995) The asymptotics of the spectrum and the regularized trace of singular differential operators of higher order, *Differential equations*, **31**(10), 1658–1668.
7. Nasrtdinov, I. G. (2018) Asymptotics of the spectrum of the Sturm–Liouville operator in $L_2(\mathbb{R}_+)$ with the boundary condition $y(0)\cos(\alpha) + y'(0)\sin(\alpha) = 0$, *Proc. of the XV International Conference "Algebra, number theory and discrete geometry: modern problems and applications", dedicated to the centenary of the birth of Professor Nikolai Mikhailovich Korobov* (Tula, Russia).
8. Kachkina, A. V. (2022) Asymptotics of the spectrum of the Sturm—Liouville operator with a boundary condition at zero and a rapidly growing potential", *Proc. of the XXI International Conference "Algebra, Number theory, discrete geometry and multiscale modeling: modern problems, applications and problems of history", dedicated to the 85th anniversary of the birth of A.A. Karatsuba*, (Tula, Russia).

9. Savchuk, A. M., Shkalikov & A. A. (1999) Sturm–Liouville operators with singular potentials, *Mathematical notes*, **66**, 897–912.
10. Giertz, M. (1962) On the solution in $L^2(-\infty, +\infty)$ of $y'' + (\lambda - q(x))y = 0$ when q is rapidly increasing, *Proceedings of the London Mathematical society*, **3-14**(1), 53–73.
11. Kozko, A. I. & Pechentsov, A. S. (2011) Regularized traces of singular differential operators with canonical boundary conditions, *Moscow University Bulletin . Series 1. Matematical Mechanics*, **4**, 11–17.
12. Kozko, A. I. & Pechentsov, A. S. (2010) The spectral function of a singular differential operator of the order of $2m$, *Proceedings of the Russian Academy of Sciences. Series: Mathematics*, **74**(6), 107–126.
13. Kozko, A. I. & Pechentsov, A. S. (2008) Regularized traces of singular differential operators of higher orders, *Mathematical notes*, **83**(1), 39–49.
14. Sadovnichiy, V. A., Pechentsov, A. S. & Kozko, A. I. (2009) Regularized traces of singular differential operators, *RAS reports*, **427**, 461–465.
15. Ying Yang, G. W. (2018) Inverse scattering problems for Sturm–Liouville operators with boundary conditions depending on the spectral parameter, *Matematical notes*, **103**(1), 65–74.
16. Ishkin, H. K. (2023) Spectral properties of the nonsectorial Sturm–Liouville operator on the semiaxis, *Mathematical notes*, **113**(5), 703–722.
17. Labovsky, S. M. (2015). On the Discreteness of the Spectrum of a Functional Differential Operator, *Control Theory and Mathematical Modeling. Abstracts of the All-Russian Conference with international participation, dedicated to the memory of Professor N.V. Azbelev and Professor E.L. Tonkov. Udmurt State University*, 73–75.
18. Labovskiy, S., Getimane, M. F. (2014). On discreteness of spectrum and positivity of the Green's function for a second order functional-differential operator on semiaxes, *Boundary Value Problems*, 102.
19. Pechentsov, A. S. (2022). Distribution of the Spectrum of a singular positive Sturm–Liouville operator Perturbed by the Dirac delta function, *Differential Equations*, **53**(8), 1029–1034.
20. Pechentsov, A. S. (2021). Spectral Distribution of the Weber Operator by the Dirac Delta Function, *Differential Equations*, **57**(8), 1003–1009.
21. Pechentsov, A. S. (2022). Distribution of the Spectrum of Airy Operator Perturbed by Delta Interactions, *Russian Journal of Mathematical Physics*, **29**, 115–118.

