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Rothe Time-Semidiscretization for Doubly Nonlinear Parabolic Problems in Musielak–Orlicz–Sobolev Spaces

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Abstract: This paper establishes the existence of an entropy solution for a doubly nonlinear parabolic problem set within the framework of Musielak-Orlicz-Sobolev spaces, without imposing the Δ_2 condition. The problem involves a Leray-Lions operator and a general Lipschitz, strictly increasing nonlinearity in the time derivative term, with L^1 source data. Our approach employs Rothe's time-semidiscretization method, reducing the evolution problem to a sequence of elliptic entropy subproblems at discrete time steps. We derive uniform a priori estimates in the modular topology associated with the Musielak-Orlicz function, which remain valid in the absence of the Δ_2 assumption. Using these estimates, we prove compactness for the Rothe sequence in $W_0^{1,x}L_\Psi(Q_T)$ and in $C([0,T];L^1(\Omega))$. The limit is then identified via monotonicity techniques, confirming that it satisfies the entropy formulation of the original problem. This work unifies and extends previous existence results from standard Sobolev, variable-exponent, and Orlicz-Sobolev settings to the fully Musielak-Orlicz case with general nonlinearities and low-regularity data.

Keywords: Semi-discretization method, Musielak-Orlicz spaces, truncations, parabolic equations, entropy solution, existence, stability.

1. INTRODUCTION

Let Ω be a bounded open set in $\mathbb{R}^d$ with $d \geq 2$, and let $T > 0$. Set $Q_T = (0,T) \times \Omega$. In this work we prove the existence of an *entropy solution* u to the doubly nonlinear parabolic problem

$$\begin{cases} \frac{\partial b(u)}{\partial t} - div\Big(a(x,t,\nabla u)\Big) = f \quad \text{in } Q_T, \\ u = 0 \quad \text{on } (0,T) \times \partial\Omega, \\ b(u)(t=0) = 0 \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

posed in the Musielak–Orlicz framework *without* imposing the Δ_2 condition. Our approach relies on Rothe's time–semidiscretization: we take $f \in L^1(Q_T)$, consider the Leray–Lions operator $Au = -\text{div}(a(x,t,\nabla u))$ acting on $W_0^{1,x}L_\Psi(Q_T)$, and assume $b : \mathbb{R} \to \mathbb{R}$ is Lipschitz, strictly increasing, with $b(0) = 0$.

Rothe's scheme, interpreted as a backward Euler discretization in Banach spaces, reduces the evolution problem to a chain of elliptic entropy problems at discrete times. We derive estimates in the modular topology associated with Ψ, which remain valid beyond the Δ_2 setting, and then pass to the limit to recover an entropy solution of (1.1).

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In standard Sobolev spaces, entropy solutions for problems of type (1.1) with $A(u) = -\Delta_p$ and $b(u) = u$ were obtained in [4]. In the variable–exponent setting, Jamea et al. established existence for $A(u) = -\Delta_{p(x)}$ with $b(u) = u$ [12]. In Orlicz–Sobolev spaces, existence and uniqueness for (1.1) with $b(u) = u$ were proved in [11]. Within Musielak spaces, [3] addressed the *obstacle* problem (variational case), again with $b(u) = u$. Broader context on parabolic problems with nonstandard growth can be found in [6, 16–19].

Relative to the above literature, this paper addresses a *doubly nonlinear* evolution (with $\partial_t b(u)$ for a general Lipschitz, strictly increasing b) under Musielak–Orlicz growth with merely L^1–data f. Moreover, it develops a Rothe-type discrete entropy formulation and derives *a priori* bounds directly in the modular setting, thereby avoiding any Δ_2 assumption on Ψ. Furthermore, it establishes compactness and stability for the Rothe sequence in $W_0^{1,x}L_\Psi(Q_T)$ and $C([0,T]; L^1(\Omega))$, which in turn yields the existence of an entropy solution of (1.1). Finally, it unifies and extends the approaches of [3, 4, 11, 12] to the fully Musielak–Orlicz case with general $b(\cdot)$ and Leray–Lions structure.

Section 2 reviews the necessary material on Musielak–Orlicz–Sobolev spaces. Section 3 states the assumptions and the main theorem. Section 4 implements the Rothe semidiscretization, proves existence and uniqueness for the discrete problems, derives the discrete estimates, and gathers the convergence and compactness results to complete the limit passage and the proof of the main result.

2. PRELIMINARIES

This section gathers the basic notions and tools for Musielak–Orlicz–Sobolev spaces that will be used later on. For a comprehensive treatment, see the monograph [13]. We also recall the inhomogeneous (space–time) versions and a few auxiliary lemmas that enter the analysis below.

Musielak–Orlicz–Sobolev spaces. Let $\Omega \subset \mathbb{R}^d$ be bounded, and let $\Psi : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ satisfy:

(a) for a.e. $x \in \Omega$, $\Psi(x, \cdot)$ is a generalized N-function (convex, increasing, continuous, $\Psi(x, 0) = 0$, $\Psi(x, t) > 0$ for $t > 0$, $\frac{\Psi(x,t)}{t} \to 0$ as $t \to 0$, and $\frac{\Psi(x,t)}{t} \to \infty$ as $t \to \infty$);
(b) for every $t \geq 0$, $\Psi(\cdot, t)$ is measurable in x.

Any Ψ fulfilling (a)–(b) is called a Musielak–Orlicz function. For convenience we set $\Psi_x(t) := \Psi(x, t)$ and denote by Ψ_x^{-1} its (nonnegative) inverse in the t-variable, so that

$$\Psi_x^{-1}(\Psi(x, t)) = \Psi(x, \Psi_x^{-1}(t)) = t.$$

Given two Musielak–Orlicz functions Ψ and Φ, we use the following comparison notation:

(c) If there exist $k > 0$ and $t_0 \geq 0$ such that $\Psi(x, t) \leq \Phi(x, kt)$ for a.e. $x \in \Omega$ and all $t \geq t_0$, we write $\Psi \prec \Phi$; in particular, Φ *globally* dominates Ψ when $t_0 = 0$, and *near infinity* when $t_0 > 0$.
(d) We say Φ grows essentially slower than Ψ at 0 (resp. near ∞), denoted $\Phi \prec\prec \Psi$, if for every $k > 0$,

$$\lim_{t\to 0} \sup_{x\in\Omega} \frac{\Phi(x, kt)}{\Psi(x, t)} = 0 \quad \text{(resp. } \lim_{t\to\infty} \sup_{x\in\Omega} \frac{\Phi(x, kt)}{\Psi(x, t)} = 0).$$

We will also use

$$\inf_{x\in\Omega} \frac{\Psi(x, t)}{t} \longrightarrow \infty \quad \text{as } t \to \infty. \tag{2.2}$$

Indeed, by the definition of $\inf_{x\in\Omega}\Psi(x,t)$, for any $\varepsilon>0$ there exists a measurable $\Omega_\varepsilon\subset\Omega$ with

$$\Psi(y,t)\le \inf_{x\in\Omega}\Psi(x,t)+\varepsilon \quad \text{for all } y\in\Omega_\varepsilon.$$

Dividing by t and using (a) together with $\varepsilon/t\to 0$ as $t\to\infty$ yields (2.2).

Define the modular

$$\varrho_{\Psi,\Omega}(u)=\int_\Omega \Psi(x,|u(x)|)\,dx,$$

for measurable $u:\Omega\to\mathbb{R}$. The Musielak–Orlicz class is

$$\mathcal{L}_\Psi(\Omega)=\{u \text{ measurable on } \Omega:\ \varrho_{\Psi,\Omega}(u)<\infty\}.$$

The Musielak–Orlicz space $L_\Psi(\Omega)$ is the linear hull of $\mathcal{L}_\Psi(\Omega)$, equivalently

$$L_\Psi(\Omega)=\Big\{u \text{ measurable}:\ \varrho_{\Psi,\Omega}\Big(\frac{|u|}{\alpha}\Big)<\infty \text{ for some } \alpha>0\Big\}.$$

The Young conjugate of Ψ is

$$\overline{\Psi}(x,s)=\sup_{t\ge 0}\{st-\Psi(x,t)\}.$$

On $L_\Psi(\Omega)$ we use the Luxemburg norm

$$\|u\|_{\Psi,\Omega}=\inf\Big\{\lambda>0:\ \int_\Omega \Psi\Big(x,\frac{|u(x)|}{\lambda}\Big)\,dx\le 1\Big\},$$

and the (Orlicz) dual norm

$$\|u\|_{(\Psi),\Omega}=\sup_{\|v\|_{\overline{\Psi},\Omega}\le 1}\int_\Omega |u(x)v(x)|\,dx,$$

where $\overline{\Psi}$ is the Young conjugate of Ψ. We say $u_n\to u$ *modularly* in $L_\Psi(\Omega)$ if there exists $h>0$ such that

$$\lim_{n\to\infty}\varrho_{\Psi,\Omega}\Big(\frac{u_n-u}{h}\Big)=0.$$

For $m\in\mathbb{N}_0$,

$$W^mL_\Psi(\Omega)=\Big\{u\in L_\Psi(\Omega): D^\alpha u\in L_\Psi(\Omega) \text{ for all } |\alpha|\le m\Big\},$$

with multi-index $\alpha=(\alpha_1,\dots,\alpha_d)$ and distributional derivatives $D^\alpha u$. The space $W^mL_\Psi(\Omega)$ is the Musielak–Orlicz–Sobolev space. Set

$$\overline{\varrho}_{\Psi,\Omega}(u)=\sum_{|\alpha|\le m}\varrho_{\Psi,\Omega}(D^\alpha u),\qquad \|u\|^m_{\Psi,\Omega}=\inf\Big\{\lambda>0:\ \overline{\varrho}_{\Psi,\Omega}\Big(\frac{u}{\lambda}\Big)\le 1\Big\}.$$

Then $\overline{\varrho}_{\Psi,\Omega}$ is a convex modular and $\|\cdot\|^m_{\Psi,\Omega}$ a norm. Moreover, if there exists $c>0$ with $\inf_{x\in\Omega}\Psi(x,1)\ge c$, the pair $\big(W^mL_\Psi(\Omega),\|\cdot\|^m_{\Psi,\Omega}\big)$ is a Banach space. One may regard $W^mL_\Psi(\Omega)$ as a subspace of the product $\prod_{|\alpha|\le m}L_\Psi(\Omega)$, closed for the topology $\sigma(\prod L_\Psi,\prod E_{\overline{\Psi}})$. Denote by $W^m_0L_\Psi(\Omega)$ the $\sigma(\prod L_\Psi,\prod E_{\overline{\Psi}})$-closure of $D(\Omega)$ in $W^mL_\Psi(\Omega)$. Likewise, set

$$W^mE_\Psi(\Omega)=\{u:\ u,\ D^\alpha u\in E_\Psi(\Omega) \text{ for } |\alpha|\le m\},$$

and let $W_0^m E_\Psi(\Omega)$ be the (norm) closure of $D(\Omega)$ in $W^m E_\Psi(\Omega)$.

For complementary Ψ and Φ we recall Young's and Hölder's inequalities:

$$ts \le \Psi(x,t) + \Phi(x,s) \quad \text{for } t, s \ge 0,\ x \in \Omega,$$

$$\int_\Omega |u(x)v(x)|\,dx \le \|u\|_{\Psi,\Omega}\,\|v\|_{\overline{\Psi},\Omega} \quad (u \in L_\Psi,\ v \in L_{\overline{\Psi}}).$$

The conjugates $\overline{\Psi}$ and $\overline{\Phi}$ satisfy

$$\lim_{|\xi|\to\infty} \operatorname*{ess\,inf}_{x\in\Omega} \frac{\overline{\Psi}(x,\xi)}{|\xi|} = \infty \quad \text{and} \quad \lim_{|\xi|\to\infty} \operatorname*{ess\,inf}_{x\in\Omega} \frac{\overline{\Phi}(x,\xi)}{|\xi|} = \infty. \tag{2.3}$$

Remark 2.1:
As noted in [15, Remark 2.1], (2.3) *implies the local boundedness*

$$\sup_{\xi\in B(0,R)} \operatorname*{ess\,sup}_{x\in\Omega} \Psi(x,\xi) < \infty \quad \textit{for all } 0 < R < \infty, \tag{2.4}$$

and similarly

$$\sup_{\xi\in B(0,R)} \operatorname*{ess\,sup}_{x\in\Omega} \Phi(x,\xi) < \infty \quad \textit{for all } 0 < R < \infty. \tag{2.5}$$

We say (u_n) converges modularly to u in $W^1 L_\Psi(\Omega)$ (resp. in $W_0^1 L_\Psi(\Omega)$) if there exists $h > 0$ such that

$$\lim_{n\to\infty} \overline{\varrho}_{\Psi,\Omega}\Big(\frac{u_n - u}{h}\Big) = 0.$$

Lemma 2.1:
[3] If $\Phi \ll \Psi$ and $u_n \to u$ modularly in $L_\Psi(\Omega)$, then $u_n \to u$ strongly in $L_\Phi(\Omega)$.

Lemma 2.2:
[17] Let $w_n, w \in L_\Psi(\Omega)$ and $v_n, v \in L_{\overline{\Psi}}(\Omega)$. If $w_n \to w$ modularly in $L_\Psi(\Omega)$ and $v_n \to v$ modularly in $L_{\overline{\Psi}}(\Omega)$, then

$$\int_\Omega w_n v_n\,dx \longrightarrow \int_\Omega wv\,dx \quad \textit{as } n \to \infty.$$

Inhomogeneous Musielak–Orlicz–Sobolev spaces. Let $\Omega \subset \mathbb{R}^d$ be bounded and $Q_T = \Omega \times (0,T)$ for some $T > 0$. For $\alpha \in \mathbb{N}^d$, denote by D_x^α the distributional derivative with respect to x. The first-order inhomogeneous spaces are

$$W^{1,x} L_\Psi(Q_T) = \big\{u \in L_\Psi(Q_T) : D_x^\alpha u \in L_\Psi(Q_T) \text{ for all } |\alpha| \le 1\big\},$$

$$W^{1,x} E_\Psi(Q_T) = \big\{u \in E_\Psi(Q_T) : D_x^\alpha u \in E_\Psi(Q_T) \text{ for all } |\alpha| \le 1\big\}.$$

Clearly $W^{1,x} E_\Psi(Q_T) \subset W^{1,x} L_\Psi(Q_T)$, and each is a Banach space with

$$\|u\| = \sum_{|\alpha|\le 1} \|D_x^\alpha u\|_{\Psi,Q_T}.$$

We view these as subspaces of the product space $\Pi L_\Psi(Q_T)$ (with $d+1$ copies) and use the weak topologies $\sigma(\Pi L_\Psi(Q_T), \Pi E_{\overline{\Psi}}(Q_T))$ and $\sigma(\Pi L_\Psi(Q_T), \Pi L_{\overline{\Psi}}(Q_T))$. For $u \in W^{1,x} L_\Psi(Q_T)$, the map $t \mapsto u(t)$ takes values in $W^1 L_\Psi(\Omega)$; if $u \in W^{1,x} E_\Psi(Q_T)$, then $t \mapsto u(t)$ is $W^1 E_\Psi(\Omega)$-valued and strongly measurable. In general $W^{1,x} L_\Psi(Q_T)$ need not be separable, hence $t \mapsto u(t)$ need not be (Bochner) measurable; however $t \mapsto \|u(t)\|_{\Psi,\Omega}$ belongs to $L^1(0,T)$. Define $W_0^{1,x}E_\Psi(Q_T)$ as the (norm) closure of $D(Q_T)$ in $W^{1,x}E_\Psi(Q_T)$.

As in [13], when Ω is Lipschitz, any u in the $\sigma(\Pi L_\Psi, \Pi E_{\overline{\Psi}})$-closure of $D(Q_T)$ is the modular limit in $W^{1,x}L_\Psi(Q_T)$ of some subsequence $(u_n) \subset D(Q_T)$: there exists $\lambda > 0$ such that, for all $|\alpha| \leq 1$,

$$\int_{Q_T} \Psi\Big(x, \frac{D_x^\alpha u_n - D_x^\alpha u}{\lambda}\Big)\, dx\, dt \;\longrightarrow\; 0 \quad \text{as } n \to \infty.$$

Consequently $(u_n) \to u$ in $W^{1,x}L_\Psi(Q_T)$ for the topology $\sigma(\Pi L_\Psi, \Pi L_{\overline{\Psi}})$, and thus

$$\overline{D(Q_T)}^{\sigma(\Pi L_\Psi, \Pi L_{\overline{\Psi}})} = \overline{D(Q_T)}^{\sigma(\Pi L_\Psi, \Pi E_{\overline{\Psi}})} =: W_0^{1,x}L_\Psi(Q_T).$$

Moreover $W_0^{1,x}E_\Psi(Q_T) = W_0^{1,x}L_\Psi(Q_T) \cap \Pi E_{\overline{\Psi}}(Q_T)$. A Poincaré-type inequality holds in $W_0^{1,x}L_\Psi(Q_T)$ (see [20]): there exists $C > 0$ such that, for all $u \in W_0^{1,x}L_\Psi(Q_T)$,

$$\sum_{|\alpha| \leq 1} \|D_x^\alpha u\|_{\Psi,Q_T} \;\leq\; C \sum_{|\alpha|=1} \|D_x^\alpha u\|_{\Psi,Q_T}. \tag{2.6}$$

We will use the dual pair

$$\begin{pmatrix} W_0^{1,x}L_\Psi(Q_T) & F \\ W_0^{1,x}E_\Psi(Q_T) & F_0 \end{pmatrix},$$

where $F = (W_0^{1,x}E_\Psi(Q_T))'$ can be identified with

$$W^{-1,x}L_{\overline{\Psi}}(Q_T) = \Big\{ f = \sum_{|\alpha| \leq 1} D_x^\alpha f_\alpha \,:\, f_\alpha \in L_{\overline{\Psi}}(Q_T) \Big\},$$

endowed with the usual quotient norm

$$\|f\| = \inf \sum_{|\alpha| \leq 1} \|f_\alpha\|_{\overline{\Psi},Q_T}.$$

Similarly

$$F_0 = W^{-1,x}E_{\overline{\Psi}}(Q_T) = \Big\{ f = \sum_{|\alpha| \leq 1} D_x^\alpha f_\alpha \,:\, f_\alpha \in E_{\overline{\Psi}}(Q_T) \Big\}.$$

Lemma 2.3:
[3] Let $Q_T = [0,T] \times \Omega$, Ψ a Musielak–Orlicz function, $E_\Psi(\Omega)$ the Musielak–Orlicz space on Ω, and $E_\Psi(Q_T)$ the inhomogeneous Musielak–Orlicz space on Q_T. Then

$$E_\Psi(Q_T) \subseteq L^1(0,T; E_\Psi(\Omega)). \tag{2.7}$$

Lemma 2.4:
[3] Let $Q_T = [0,T] \times \Omega$, Ψ a Musielak–Orlicz function, $W^1E_\Psi(\Omega)$ the Sobolev space on Ω, and $W^1E_\Psi(Q_T)$ its inhomogeneous counterpart on Q_T. Then

$$W^1E_\Psi(Q_T) \subset L^1(0,T; W^1E_\Psi(\Omega)), \tag{2.8}$$

$$W^{-1}E_{\overline{\Psi}}(Q_T) \subset L^1(0,T; W^{-1}E_{\overline{\Psi}}(\Omega)), \tag{2.9}$$

and both embeddings are continuous.

Theorem 2.1:
[7] Let Ψ be a Musielak function. If $F \subset W_0^{1,x}L_\Psi(Q_T)$ is bounded and, for each $u \in F$, $\partial_t u$ is bounded in $W^{-1,x}L_{\overline{\Psi}}(Q_T)$, then F is relatively compact in $L^1(Q_T)$.

Corollary 2.1:
[3] Let Ψ be a Musielak–Orlicz function and $(u_n) \subset W^{1,x}L_\Psi(Q_T)$ satisfy

$$u_n \rightharpoonup u \quad \text{in } W^{1,x}L_\Psi(Q_T) \text{ for } \sigma(\Pi L_\Psi, \Pi E_{\overline{\Psi}})$$

and

$$\frac{\partial u_n}{\partial t} = h_n + k_n \quad \text{in } \mathcal{D}'(Q_T),$$

with (h_n) bounded in $W^{-1,x}L_{\overline{\Psi}}(Q_T)$ and (k_n) bounded in the space $L^1(Q_T)$ of measures on Q_T. Then

$$u_n \to u \quad \text{strongly in } L^1_{loc}(Q_T).$$

If in addition $u_n \in W_0^{1,x}L_\Psi(Q_T)$, then $u_n \to u$ in $L^1(Q_T)$.

3. FRAMEWORK AND MAIN EXISTENCE THEOREM

For the analysis of (1.1), we collect here the structural hypotheses on the data. Throughout, $\Omega \subset \mathbb{R}^d$ is a bounded open set with Lipschitz boundary $\partial\Omega$, $T > 0$, and $Q_T = (0, T) \times \Omega$. Let Ψ and Φ be N-functions with $\Phi \ll \Psi$. We work with the Leray–Lions operator

$$A(u) = -\,div\big(a(x, t, \nabla u)\big) \quad \text{on } W_0^{1,x}L_\Psi(Q_T),$$

where $a : Q_T \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory vector field (measurable in (x, t) and continuous in ξ) satisfying, for a.e. $(x, t) \in Q_T$ and all $\xi, \xi^* \in \mathbb{R}^d$ with $\xi \neq \xi^*$:

$$|a(x, t, \xi)| \le \gamma\Big[c(x, t) + \overline{\Psi}^{-1}\big(\Psi(|\xi|)\big)\Big], \tag{3.10}$$

$$\big(a(x, t, \xi) - a(x, t, \xi^*)\big) \cdot (\xi - \xi^*) > 0, \tag{3.11}$$

$$a(x, t, \xi) \cdot \xi \;\ge\; \alpha\, \Psi\big(x, |\xi|\big), \tag{3.12}$$

with $c(\cdot, \cdot) \in E_{\overline{\Psi}}(Q_T)$ and constants $\alpha, \gamma > 0$.† We assume furthermore

$$f \in L^1(Q_T), \tag{3.13}$$

and

$$b : \mathbb{R} \to \mathbb{R} \text{ is Lipschitz, strictly increasing, with } b(0) = 0. \tag{3.14}$$

We adopt the following notion of entropy solution for (1.1).

Definition 3.1:
A measurable $u : Q_T \to \mathbb{R}$ is an entropy solution of (1.1) if:

- $b(u) \in L^\infty\big([0, T]; L^1(\Omega)\big)$*;*
- $T_k(u) \in W_0^{1,x}L_\Psi(Q_T)$ *for every $k > 0$;*

†Compared with the draft, (3.12) corrects a typographical slip: since $\xi \in \mathbb{R}^d$, one must use $a(x, t, \xi) \cdot \xi$ and $\Psi(x, |\xi|)$ rather than $\Psi(x, |\nabla\xi|)$.

• *for all* $k > 0$ *and every* $\varphi \in L^\infty(Q_T) \cap W_0^{1,x} L_\Psi(Q_T)$ *with* $\varphi(T) = 0$ *and* $\partial_s \varphi \in W^{-1,x} L_{\overline{\Psi}}(Q_T) + L^1(Q_T)$*, one has*

$$\int_0^T \Big\langle \frac{\partial \varphi}{\partial s}\,,\, \int_0^u \frac{\partial b(z)}{\partial s}\, T_k'(z-\varphi)\, dz \Big\rangle ds \;+\; \int_{Q_T} a(x,t,\nabla u)\cdot \nabla T_k(u-\varphi)\, dx\, ds$$
$$\leq \int_{Q_T} f\, T_k(u-\varphi)\, dx\, ds.$$

Our main result is the following existence theorem.

Theorem 3.1:
Under assumptions (3.10)–(3.14)*, the nonlinear parabolic problem* (1.1) *admits an entropy solution in the sense of Definition 3.1.*

4. PROOF OF THE MAIN RESULT

We now prove the main theorem through a sequence of standard steps. First, we replace the evolution (1.1) by a family of elliptic problems obtained via time semi–discretization (Rothe's method) and show existence/uniqueness of discrete entropy solutions. Next, we construct the Rothe interpolants and derive stability estimates that are uniform in the time step. Finally, we pass to the limit along the Rothe sequence and recover an entropy solution of the nonlinear degenerate parabolic problem (1.1).

4.1. The Rothe Problem

To apply the semi–discretization in time, fix $n \in \mathbb{N}$ and partition $[0,T]$ into equidistant nodes $t_j = j\tau$ for $j = 0,\dots,n$, where $\tau = T/n$. Replacing the time derivative $\partial_t b(u)$ by the backward difference $\big(b(u_j) - b(u_{j-1})\big)/\tau$ leads to

$$\begin{cases} b(u_j) \;-\; \tau\,\mathrm{div}\Big(a(x,t_j,\nabla u_j)\Big) \;=\; \tau f_j \;+\; b(u_{j-1}) \quad \text{in } \Omega, \quad j = 1,\dots,n, \\ \qquad\qquad\qquad\qquad u_j = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.15}$$

Here

$$f_j(\cdot) = \frac{1}{\tau}\int_{t_{j-1}}^{t_j} f(s,\cdot)\, ds, \qquad f_0 = f(0).$$

For brevity we will sometimes write $a(x,\nabla u_j)$ to denote $a(x,t_j,\nabla u_j)$ when no confusion can arise.

We first formulate the discrete entropy notion and then establish existence/uniqueness for (4.15).

Definition 4.1:
A measurable function u_j *on* Ω *is an entropy solution of* (4.15) *if* $T_k(u_j) \in W_0^1 L_\Psi(\Omega)$ *for every* $k > 0$ *and, for all* $\varphi \in W_0^1 L_\Psi(\Omega) \cap L^\infty(\Omega)$ *and every* $k > 0$*, one has*

$$\begin{aligned} &\int_\Omega b(u_j)\, T_k(u_j-\varphi)\, dx \;+\; \tau \int_\Omega a(x,\nabla u_j)\cdot \nabla T_k(u_j-\varphi)\, dx \\ &\leq \int_\Omega \big(\tau f_j + b(u_{j-1})\big)\, T_k(u_j-\varphi)\, dx. \end{aligned} \tag{4.16}$$

Lemma 4.1:
For all $k>0$, $j=1,\dots,n$ *and* $h>0$,

$$b(u_j)\in L^1(\Omega), \tag{4.17}$$

$$\int_{\{h\le |u_j|\le h+k\}} \Psi\big(x,|\nabla u_j|\big)\,dx \;\longrightarrow\; 0 \quad \textit{as } h\to+\infty. \tag{4.18}$$

Proof
Taking $\varphi=0$ in (4.16) with $j=1$ gives

$$\int_\Omega b(u_1)\,T_k(u_1)\,dx+\tau\int_\Omega a(x,\nabla u_1)\cdot\nabla T_k(u_1)\,dx \;\le\; \int_\Omega (\tau f_1)\,T_k(u_1)\,dx. \tag{4.19}$$

Since

$$\sum_{j=1}^n \tau\|f_j\|_{L^1(\Omega)}\le \|f\|_{L^1(Q_T)}$$

and

$$\int_\Omega a(x,\nabla u_1)\cdot\nabla T_k(u_1)\,dx=\int_{\{|u_1|\le k\}} a(x,\nabla u_1)\cdot\nabla u_1\,dx \;\ge\; \alpha\int_{\{|u_1|\le k\}}\Psi\big(x,|\nabla u_1|\big)\,dx \;\ge 0,$$

(4.19) yields $0\le\int_\Omega b(u_1)T_k(u_1)\,dx\le kC$, with C independent of k. Hence

$$0\le\int_\Omega b(u_1)\,\frac{T_k(u_1)}{k}\,dx\le C,$$

and by Fatou's lemma $\|b(u_1)\|_{L^1(\Omega)}\le C$.

Assume by induction that $b(u_i)\in L^1(\Omega)$ for all $i<j$; this gives (4.17). Now take $\varphi=T_h(u_j)$ in (4.16). We obtain

$$\begin{aligned}&\int_\Omega b(u_j)\,T_k\big(u_j-T_h(u_j)\big)\,dx \;+\; \tau\int_\Omega a(x,\nabla u_j)\cdot\nabla T_k\big(u_j-T_h(u_j)\big)\,dx\\ &\le\int_\Omega\big(\tau f_j+b(u_{j-1})\big)\,T_k\big(u_j-T_h(u_j)\big)\,dx.\end{aligned} \tag{4.20}$$

Using that $T_k\big(u_j-h\,\operatorname{sgn}u_j\big)$ has the same sign as u_j on $\{|u_j|\ge h\}$, we get

$$\int_\Omega b(u_j)\,T_k\big(u_j-T_h(u_j)\big)\,dx \;\ge\; 0. \tag{4.21}$$

Moreover,

$$\int_\Omega a(x,\nabla u_j)\cdot\nabla T_k\big(u_j-T_h(u_j)\big)\,dx=\int_{\{h\le|u_j|\le h+k\}} a(x,\nabla u_j)\cdot\nabla u_j\,dx. \tag{4.22}$$

Combining (4.20)–(4.22) gives

$$\tau\int_{\{h\le|u_j|\le h+k\}} a(x,\nabla u_j)\cdot\nabla u_j\,dx \;\le\; k\int_{\{|u_j|\ge h\}}\tau|f_j|\,dx \;+\; k\int_{\{|u_j|\ge h\}}|b(u_{j-1})|\,dx.$$

Since $f_j,\ b(u_{j-1})\in L^1(\Omega)$ and $|\{|u_j|\ge h\}|\to 0$ as $h\to\infty$, we conclude that

$$\int_{\{h\le|u_j|\le h+k\}}\Psi\big(x,|\nabla u_j|\big)\,dx \;\longrightarrow\; 0 \quad \text{as } h\to+\infty,$$

which proves the lemma. □

We now prove existence and uniqueness for the discrete problem (4.15).

Theorem 4.1:
Under (3.10)–(3.14)*, for each* $j = 1, \dots, n$ *the problem* (4.15) *admits a unique entropy solution* u_j*.*

Proof
Fix $j \in \{1, \dots, n\}$. By (3.13), (3.14) and Lemma 4.1, the functions

$$F := f_j + \frac{1}{\tau} b(u_{j-1}) \in L^1(\Omega), \qquad g(s) := \frac{1}{\tau} b(s)$$

satisfy the hypotheses of Theorem (1) in [1, 2]; hence the elliptic problem (4.15) has an entropy solution u_j.

For uniqueness, let u, v solve (4.15) with $j = 1$ (the general case is identical). Testing the inequality for u with $\varphi = T_h(v)$ and the one for v with $\varphi = T_h(u)$, summing, and letting $h \to \infty$ (using the Lipschitz continuity of b and the dominated convergence theorem) yield

$$\int_\Omega \big(b(u) - b(v)\big)\, T_k(u - v)\, dx \;+\; \tau \lim_{h\to\infty} I_{k,h} \;\le\; 0,$$

where

$$I_{k,h} = \int_\Omega a(x, \nabla u) \cdot \nabla T_k\big(u - T_h(v)\big)\, dx + \int_\Omega a(x, \nabla v) \cdot \nabla T_k\big(v - T_h(u)\big)\, dx.$$

As in the proof of Theorem (4.6) in [3], $\lim_{h\to\infty} I_{k,h} \ge 0$, so

$$\int_\Omega \big(b(u) - b(v)\big)\, T_k(u - v)\, dx \;\le\; 0.$$

Letting $k \downarrow 0$ and using $\frac{1}{k} T_k(\cdot) \to \operatorname{sgn}(\cdot)$ gives $\|b(u) - b(v)\|_{L^1(\Omega)} \le 0$, hence $b(u) = b(v)$ a.e., whence $u = v$ by strict monotonicity of b. An induction in j completes the proof. □

We next derive estimates that are uniform in n.

Proposition 4.1:
Under (3.10)–(3.14)*, there exists* $C = C(f) > 0$*, independent of* j, h, n*, such that for all* $j = 1, \dots, n$*,*

$$\|b(u_j)\|_{L^1(\Omega)} \le C(f), \tag{4.23}$$

$$\sum_{i=1}^{j} \|b(u_i) - b(u_{i-1})\|_{L^1(\Omega)} \le C(f), \tag{4.24}$$

$$\sum_{i=1}^{j} \tau \int_\Omega \Psi\big(x, \nabla T_k(u_i)\big)\, dx \;\le\; C(f). \tag{4.25}$$

Proof
Fix $i \in \{1, \dots, j\}$ and take $\varphi = 0$ in (4.16):

$$\int_\Omega b(u_i)\, T_k(u_i)\, dx + \tau \int_\Omega a(x, \nabla u_i) \cdot \nabla T_k(u_i)\, dx \;\le\; \int_\Omega (\tau f_i)\, T_k(u_i)\, dx + \int_\Omega b(u_{i-1})\, T_k(u_i)\, dx.$$

Since

$$\int_\Omega a(x, \nabla u_i) \cdot \nabla T_k(u_i)\, dx = \int_{\{|u_i| \le k\}} a(x, \nabla u_i) \cdot \nabla u_i\, dx \;\ge\; 0,$$

we get

$$\int_\Omega b(u_i)\, T_k(u_i)\, dx \;\le\; k\tau\|f_i\|_{L^1(\Omega)} + k\|b(u_{i-1})\|_{L^1(\Omega)}.$$

Because $\operatorname{sgn}(b(u_i)) = \operatorname{sgn}(u_i)$ by (3.14),

$$\lim_{k\downarrow 0} b(u_i)\,\frac{T_k(u_i)}{k} = |b(u_i)| \quad \text{a.e.}$$

so by Fatou's lemma

$$\|b(u_i)\|_{L^1(\Omega)} \;\le\; \tau\|f_i\|_{L^1(\Omega)} + \|b(u_{i-1})\|_{L^1(\Omega)}. \tag{4.26}$$

Summing (4.26) from $i = 1$ to j yields (4.23).

For (4.24), test (4.16) with $\varphi = T_h(u_i) - \operatorname{sgn}\big(T_h(u_i) - T_h(u_{i-1})\big)$ to obtain

$$\begin{aligned}
&\int_\Omega \big(b(u_i) - b(u_{i-1})\big)\, T_k\Big(u_i - T_h(u_i) + \operatorname{sgn}\big(T_h(u_i) - T_h(u_{i-1})\big)\Big)\, dx \\
&\qquad + \tau \int_\Omega a(x, \nabla u_i)\cdot \nabla T_k\Big(u_i - T_h(u_i) + \operatorname{sgn}\big(T_h(u_i) - T_h(u_{i-1})\big)\Big)\, dx \\
&\le \int_\Omega \tau f_i\, T_k\Big(u_i - T_h(u_i) + \operatorname{sgn}\big(T_h(u_i) - T_h(u_{i-1})\big)\Big)\, dx.
\end{aligned}$$

The second term is nonnegative since it reduces to

$$\int_{\Omega_{k,h}} a(x, \nabla u_i)\cdot \nabla u_i\, dx \;\ge\; 0, \quad \Omega_{k,h} = \Big\{\big|u_i - T_h(u_i - \operatorname{sgn}(u_i - u_{i-1}))\big| \le k\Big\} \cap \{|u_i| > h\}.$$

Letting $h \to \infty$ and taking $k = 1$ gives

$$\|b(u_i) - b(u_{i-1})\|_{L^1(\Omega)} \;\le\; \tau\|f_i\|_{L^1(\Omega)}. \tag{4.27}$$

Summing (4.27) from $i = 1$ to j yields (4.24).

Finally, to obtain (4.25), test (4.16) with $\varphi = 0$ and rewrite as

$$\int_\Omega \big(b(u_i) - b(u_{i-1})\big)\, T_k(u_i)\, dx + \tau \int_\Omega a(x, \nabla u_i)\cdot \nabla T_k(u_i)\, dx \;\le\; \int_\Omega (\tau f_i)\, T_k(u_i)\, dx.$$

Hence,

$$\alpha\,\tau \int_{\{|u_i|\le k\}} \Psi\big(x, |\nabla u_i|\big)\, dx \;\le\; \tau k\, \|f_i\|_{L^1(\Omega)} + k\, \|b(u_i) - b(u_{i-1})\|_{L^1(\Omega)}. \tag{4.28}$$

Summing (4.28) over $i = 1, \dots, j$ and invoking (4.24) gives (4.25). □

We now introduce the Rothe interpolants built from the discrete solutions u_j.

$$\begin{cases} b(u^n)(0) := 0, \\ b(u^n)(t) := b(u_{j-1}) + \big(b(u_j) - b(u_{j-1})\big)\dfrac{t - t_{j-1}}{\tau}, \quad t \in (t_{j-1}, t_j], \text{ in } \Omega, \end{cases} \tag{4.29}$$

and the piecewise constant companion

$$\begin{cases} b(\overline{u}^n)(0) := 0, \\ b(\overline{u}^n)(t) := b(u_j), \qquad t \in (t_{j-1}, t_j], \text{ in } \Omega. \end{cases} \tag{4.30}$$

By Theorem 4.1, each u_j is uniquely determined; hence u^n and $\overline{u}^n$ are well defined. Using Proposition 4.1 we infer the following uniform estimates for the Rothe functions.

Proposition 4.2:
For every $n \in \mathbb{N}^*$ there exists $C = C(T, f)$, independent of n, such that

$$\| b(\overline{u}^n) - b(u^n) \|_{L^1(Q_T)} \leq \frac{C}{n}, \tag{4.31}$$

$$\| b(u^n) \|_{L^1(Q_T)} \leq C, \tag{4.32}$$

$$\| b(\overline{u}^n) \|_{L^1(Q_T)} \leq C, \tag{4.33}$$

$$\Big\| \partial_t b(u^n) \Big\|_{L^1(Q_T)} \leq C, \tag{4.34}$$

$$\int_{Q_T} \Psi\big(x, \nabla T_k(\overline{u}^n)\big)\, dx\, dt \leq k\, C. \tag{4.35}$$

Proof
From (4.29)–(4.30),

$$b(\overline{u}^n)(t) - b(u^n)(t) = \big(b(u_j) - b(u_{j-1})\big)\Big(1 - \frac{t - t_{j-1}}{\tau}\Big) = \frac{t_j - t}{\tau}\, b(u_j) - \frac{t_j - t}{\tau}\, b(u_{j-1}),$$

whence

$$\begin{aligned}\|b(\overline{u}^n) - b(u^n)\|_{L^1(Q_T)} &= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \|b(u_j) - b(u_{j-1})\|_{L^1(\Omega)} \frac{t_j - t}{\tau}\, dt \\ &= \frac{\tau}{2} \sum_{j=1}^{n} \|b(u_j) - b(u_{j-1})\|_{L^1(\Omega)},\end{aligned}$$

and (4.31) follows from (4.24). Next, using (4.29),

$$\begin{aligned}\|b(u^n)\|_{L^1(Q_T)} &= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \Big(\|b(u_{j-1})\|_{L^1(\Omega)} \frac{t_j - t}{\tau} + \|b(u_j)\|_{L^1(\Omega)} \frac{t - t_{j-1}}{\tau}\Big) dt \\ &= \frac{\tau}{2} \sum_{j=1}^{n} \big(\|b(u_{j-1})\|_{L^1} + \|b(u_j)\|_{L^1}\big),\end{aligned}$$

which gives (4.32) by (4.23). Estimate (4.33) follows directly from (4.30) and (4.23).
For (4.34), observe that on $(t_{j-1}, t_j]$,

$$\partial_t b(u^n) = \frac{b(u_j) - b(u_{j-1})}{\tau},$$

so

$$\big\|\partial_t b(u^n)\big\|_{L^1(Q_T)} \leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \frac{1}{\tau}\, \|b(u_j) - b(u_{j-1})\|_{L^1(\Omega)}\, dt = \sum_{j=1}^{n} \|b(u_j) - b(u_{j-1})\|_{L^1(\Omega)} \leq C.$$

Finally, for (4.35),

$$\begin{aligned}\int_{Q_T} \Psi\big(x, \nabla T_k(\overline{u}^n)\big)\, dx\, dt &= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \int_\Omega \Psi\big(x, \nabla T_k(u_j)\big)\, dx\, dt \\ &= \tau \sum_{j=1}^{n} \int_\Omega \Psi\big(x, \nabla T_k(u_j)\big)\, dx \leq k\, C\end{aligned}$$

by (4.25). This completes the proof. □

4.2. Convergence Results

Proposition 4.3:
Assume that (3.10)–(3.14) *hold. Then there exists a measurable function* u *such that, for every* $k > 0$,

$$T_k(\overline{u}^n) \rightharpoonup T_k(u) \quad \textit{weakly in } W_0^{1,x}L_\Psi(Q_T), \tag{4.36}$$
$$\nabla\overline{u}^n \to \nabla u \quad \textit{a.e. in } Q_T, \tag{4.37}$$
$$a\big(x,t,\nabla T_k(\overline{u}^n)\big) \rightharpoonup a\big(x,t,\nabla T_k(u)\big) \quad \textit{weakly in } \big(L_{\overline{\Psi}}(Q_T)\big)^d, \tag{4.38}$$
$$\Psi\big(x,|\nabla T_k(\overline{u}^n)|\big) \to \Psi\big(x,|\nabla T_k(u)|\big) \quad \textit{strongly in } L^1(Q_T), \tag{4.39}$$
$$b(u^n) \to b(u) \quad \textit{in } C\big([0,T];L^1(\Omega)\big)\,. \tag{4.40}$$

as $n \to \infty$.

Proof
Proof of (4.36). From (4.32) and (3.14) we know that $\{b(u^n)\}$ is bounded in $L^1(Q_T)$; combining this with standard truncation estimates and the boundedness of Q_T yields uniform integrability for $\{u^n\}$. Hence, up to a subsequence,

$$u^n \overset{\sigma(L^1,L^\infty)}{\rightharpoonup} u \quad \text{in } L^1(Q_T) \quad \text{and} \quad u^n \to u \text{ a.e. in } Q_T.$$

Next we prove tail control for the piecewise constant interpolant $\overline{u}^n$. Using (3.12) and (4.35) we obtain

$$\begin{aligned}\inf_{x\in\Omega}\Psi(x,k)\ \mathrm{meas}\big\{(x,t)\in Q_T:\ |\overline{u}^n|>k\big\} \ &\le\ \int_{\{|\overline{u}^n|>k\}}\Psi\big(x,|\nabla T_k(\overline{u}^n)|\big)\,dx\,dt\\ &\le\ C''(T,f)\,k,\end{aligned}$$

whence, by (2.2),

$$\mathrm{meas}\big\{(x,t)\in Q_T:\ |\overline{u}^n|>k\big\} \ \longrightarrow\ 0 \quad \text{as } k\to\infty, \text{ uniformly in } n.$$

For any $\gamma > 0$,

$$\begin{aligned}\mathrm{meas}\big\{|\overline{u}^p-\overline{u}^q|>\gamma\big\} \le \mathrm{meas}\big\{|\overline{u}^p|>k\big\} + \mathrm{meas}\big\{|\overline{u}^q|>k\big\}\\ + \mathrm{meas}\big\{|T_k(\overline{u}^p)-T_k(\overline{u}^q)|>\gamma\big\}.\end{aligned} \tag{4.41}$$

Since $T_k(\overline{u}^n)$ is bounded in $W_0^{1,x}L_\Psi(Q_T)$ for each fixed $k > 0$, there is $v_k \in W_0^{1,x}L_\Psi(Q_T)$ and a subsequence (not relabeled) with $T_k(\overline{u}^n) \rightharpoonup v_k$ weakly in $W_0^{1,x}L_\Psi(Q_T)$. Hence $\{T_k(\overline{u}^n)\}$ is Cauchy in measure. Using (4.41) and the uniform tail bound, we infer that $\{\overline{u}^n\}$ is Cauchy in measure, thus $\overline{u}^n \to v$ a.e. in Q_T for some v.

We now show $v = u$. By (4.31) and the Lipschitz property of b,

$$\|v-u\|_{L^1(Q_T)} \ \le\ \liminf_n\ \|\overline{u}^n-u^n\|_{L^1(Q_T)} \ =\ 0,$$

hence $v = u$. Therefore $T_k(\overline{u}^n) \rightharpoonup T_k(u)$ weakly in $W_0^{1,x}L_\Psi(Q_T)$, which proves (4.36).

Proofs of (4.37)–(4.39). These follow by the standard Minty–Browder/monotonicity argument for Leray–Lions operators in the Musielak–Orlicz setting applied to the truncations, exactly as in Proposition (5.5) of [3]; we omit the repetition.

Proof of (4.40). Let f^n be the piecewise constant in time reconstruction of f:

$$f^n(t,x) := f_j(x) \quad \text{for } t\in(t_{j-1},t_j],\ j=1,\ldots,n.$$

The discrete entropy inequality (4.16) for (u_j) rewrites, for the interpolants, as

$$\int_0^T \Big\langle \partial_t b(u^n),\, T_k(\overline{u}^n - \varphi)\Big\rangle dt + \int_{Q_T} a(x,t,\nabla \overline{u}^n)\cdot \nabla T_k(\overline{u}^n - \varphi)\,dx\,dt \le \int_{Q_T} f^n\, T_k(\overline{u}^n - \varphi)\,dx\,dt, \tag{4.42}$$

for all $\varphi \in L^\infty(Q_T)\cap W_0^{1,x}L_\Psi(Q_T)$ with $\varphi(T)=0$.

Fix $n,m\in\mathbb{N}$ and choose in (4.42) for $(u^n,\overline{u}^n)$ the test $\varphi = T_h(\overline{u}^m)$, and for $(u^m,\overline{u}^m)$ the test $\varphi = T_h(\overline{u}^n)$. Summing the resulting inequalities gives

$$\int_0^T \Big\langle \partial_t\big(b(u^n)-b(u^m)\big),\, T_k(\overline{u}^n - \overline{u}^m)\Big\rangle dt + \lim_{h\to\infty} II^{n,m}_{k,h} \;\le\; \|f^n - f^m\|_{L^1(Q_T)}, \tag{4.43}$$

where

$$II^{n,m}_{k,h} = \int_{Q_T} a(x,t,\nabla\overline{u}^n)\cdot\nabla T_k(\overline{u}^n - T_h(\overline{u}^m)) + a(x,t,\nabla\overline{u}^m)\cdot\nabla T_k(\overline{u}^m - T_h(\overline{u}^n))\;dx\,dt.$$

Adding and subtracting $T_k\big(b(u^n)-b(u^m)\big)$ in the duality term, we obtain

$$\begin{aligned}&\int_0^T \Big\langle \partial_t\big(b(u^n)-b(u^m)\big),\, T_k\big(b(u^n)-b(u^m)\big)\Big\rangle dt + \lim_{h\to\infty} II^{n,m}_{k,h} \\ &\le\; \|f^n - f^m\|_{L^1(Q_T)} + \int_0^T \Big\langle \partial_t\big(b(u^n)-b(u^m)\big),\, T_k\big(b(u^n)-b(u^m)\big) - T_k(\overline{u}^n - \overline{u}^m)\Big\rangle dt.\end{aligned} \tag{4.44}$$

Let $J_k:\mathbb{R}\to\mathbb{R}_+$ be the convex primitive $J_k(s)=\int_0^s T_k(\sigma)\,d\sigma$. Then

$$\Big\langle \partial_t v,\, T_k(v)\Big\rangle = \frac{d}{dt}\int_\Omega J_k(v)\,dx \quad \text{in } L^1(0,T)$$

for $v\in L^1(0,T;L^1(\Omega))$ with $\partial_t v\in L^1(0,T;\mathcal{M}(\Omega))$. Applying this to $v=b(u^n)-b(u^m)$ in (4.44) and integrating in time yields

$$\begin{aligned}&\int_\Omega J_k\big(b(u^n)(t)-b(u^m)(t)\big)\,dx + \lim_{h\to\infty} II^{n,m}_{k,h} \\ &\le \left|\int_0^T \Big\langle \partial_t\big(b(u^n)-b(u^m)\big),\, T_k\big(b(u^n)-b(u^m)\big) - T_k(\overline{u}^n - \overline{u}^m)\Big\rangle dt\right| + \|f^n - f^m\|_{L^1(Q_T)}.\end{aligned}$$

Since $\partial_t b(u^n)$ is uniformly bounded in $L^1(Q_T)$ by (4.34), $T_k(\overline{u}^n - \varphi) \overset{*}{\rightharpoonup} T_k(u-\varphi)$ in $L^\infty(Q_T)$ as $n\to\infty$, and $f^n\to f$ in $L^1(Q_T)$, it follows that

$$\begin{aligned}&\lim_{n,m\to\infty}\left|\int_0^T \Big\langle \partial_t\big(b(u^n)-b(u^m)\big),\, T_k\big(b(u^n)-b(u^m)\big) - T_k(\overline{u}^n - \overline{u}^m)\Big\rangle dt\right| = 0,\\ &\lim_{n,m\to\infty}\|f^n - f^m\|_{L^1(Q_T)} = 0.\end{aligned}$$

Moreover, by the same monotonicity argument used in the uniqueness part of Theorem 4.1, $\lim_{h\to\infty} II^{n,m}_{k,h}\ge 0$. Hence, letting $n,m\to\infty$ gives

$$\lim_{n,m\to\infty}\int_\Omega J_k\big(b(u^n)(t)-b(u^m)(t)\big)\,dx = 0 \qquad \text{for all } t\in[0,T]. \tag{4.45}$$

Standard properties of J_k (see, e.g., the proof of Theorem 1.1 in [10]) then yield

$$\lim_{n,m\to\infty}\int_\Omega \big|b(u^n)(t)-b(u^m)(t)\big|\,dx=0,$$

so $\{b(u^n)\}$ is Cauchy in $C([0,T];L^1(\Omega))$. Since $b(u^n)\to b(u)$ in $L^1(Q_T)$, we conclude $b(u^n)\to b(u)$ in $C([0,T];L^1(\Omega))$, i.e., (4.40). □

4.3. Passage to the Limit

We now show that the limit function u obtained in Proposition 4.3 is an entropy solution of (1.1).

Let $v\in W^{1,x}L_\Psi(Q_T)\cap L^\infty(Q_T)$ with $\partial_t v\in W^{-1,x}L_{\overline{\Psi}}(Q_T)+L^1(Q_T)$. By Lemma 5 and Theorem 3 of [9], there exists an extension $\overline{v}$ to $\Omega\times\mathbb{R}$ such that

$$\overline{v}\in W^{1,x}L_\Psi(\Omega\times\mathbb{R})\cap L^1(\Omega\times\mathbb{R})\cap L^\infty(\Omega\times\mathbb{R}),\quad \partial_t\overline{v}\in W^{-1,x}L_{\overline{\Psi}}(\Omega\times\mathbb{R})+L^1(\Omega\times\mathbb{R}),$$

and there exists a sequence $(\omega_j)_{j\in\mathbb{N}}\subset\mathcal{D}(\Omega\times\mathbb{R})$ with $\omega_j(\cdot,T)=0$ such that

$$\omega_j\to\overline{v}\text{ in }W_0^{1,x}L_\Psi(\Omega\times\mathbb{R}),\qquad \partial_t\omega_j\to\partial_t\overline{v}\text{ in }W^{-1,x}L_{\overline{\Psi}}(\Omega\times\mathbb{R})+L^1(\Omega\times\mathbb{R}).\tag{4.46}$$

Fix $k>0$ and $j\in\mathbb{N}$. In the discrete entropy inequality (4.42) for $(u^n,\overline{u}^n)$, choose as test function

$$\varphi=\overline{u}^n-T_k(\overline{u}^n-\omega_j).$$

Since $\omega_j(\cdot,T)=0$, we have $\varphi(T)=0$ and $\varphi\in L^\infty(Q_T)\cap W_0^{1,x}L_\Psi(Q_T)$. Denote $\bar{k}:=k+c\|\omega_j\|_\infty$ (with $c>0$ independent of n,j). We obtain, for every $t\in[0,T]$,

$$\begin{aligned}&\int_0^T\Big\langle\partial_t b(u^n),\,T_k(\overline{u}^n-\omega_j)\Big\rangle dt+\int_{Q_T}a\big(x,t,\nabla T_{\bar{k}}(\overline{u}^n)\big)\cdot\nabla T_k(\overline{u}^n-\omega_j)\,dx\,dt\\&=\int_{Q_T}f^nT_k(\overline{u}^n-\omega_j)\,dx\,dt.\end{aligned}\tag{4.47}$$

Adding and subtracting $T_k(u^n-\omega_j)$ in the time–duality term gives

$$\begin{aligned}&\int_0^T\Big\langle\partial_t b(u^n),\,T_k(u^n-\omega_j)\Big\rangle dt+\int_0^T\Big\langle\partial_t b(u^n),\,T_k(\overline{u}^n-\omega_j)-T_k(u^n-\omega_j)\Big\rangle dt\\&+\int_{Q_T}a\big(x,t,\nabla T_{\bar{k}}(\overline{u}^n)\big)\cdot\nabla T_k(\overline{u}^n-\omega_j)\,dx\,dt\ \le\ \int_{Q_T}f^nT_k(\overline{u}^n-\omega_j)\,dx\,dt.\end{aligned}\tag{4.48}$$

Limit in the first term. Using the chain rule in duality (as in the proof of (4.40)) we can write

$$\begin{aligned}&\int_0^T\Big\langle\partial_t b(u^n),\,T_k(u^n-\omega_j)\Big\rangle dt\\&=\int_0^T\Big\langle\partial_t\omega_j,\,\int_0^{u^n}b'(z)\,T_k'(z-\omega_j)\,dz\Big\rangle dt+\int_\Omega\int_0^{u^n(T)}b'(s)\,T_k\big(s-\omega_j(T)\big)\,ds\,dx.\end{aligned}$$

Since $b'\ge 0$ and $\omega_j(\cdot,T)=0$, the terminal term is nonnegative. Let $M:=k+(d+2)\|\omega_j\|_\infty$. By the strong convergence $T_M(u^n)\to T_M(u)$ in $E_\Psi(Q_T)$ (Proposition 4.3) and (4.46), we pass to the limit:

$$\lim_{n\to\infty}\int_0^T\Big\langle\partial_t\omega_j,\,\int_0^{u^n}b'(z)\,T_k'(z-\omega_j)\,dz\Big\rangle dt=\int_0^T\Big\langle\partial_t\omega_j,\,\int_0^{T_\Psi(u)}b'(z)\,T_k'(z-\omega_j)\,dz\Big\rangle dt.$$

Limit in the second term. From (4.34) we have $\|\partial_t b(u^n)\|_{L^1(Q_T)} \le C$. Moreover, $T_k(\overline{u}^n - \omega_j) \overset{*}{\rightharpoonup} T_k(u-\omega_j)$ and $T_k(u^n - \omega_j) \overset{*}{\rightharpoonup} T_k(u-\omega_j)$ in $L^\infty(Q_T)$ (by (4.40) and (4.36)). Hence

$$\lim_{n\to\infty} \left| \int_0^T \Big\langle \partial_t b(u^n),\, T_k(\overline{u}^n - \omega_j) - T_k(u^n - \omega_j) \Big\rangle dt \right| = 0. \tag{4.49}$$

Limit in the operator term. Write

$$\begin{aligned}&\int_{Q_T} a\big(x,t,\nabla T_{\bar{k}}(\overline{u}^n)\big) \cdot \nabla T_k(\overline{u}^n - \omega_j)\, dx\, dt \\ &= \int_{D_{n,j}} a\big(x,t,\nabla T_{\bar{k}}(\overline{u}^n)\big) \cdot \nabla \overline{u}^n\, dx\, dt - \int_{D_{n,j}} a\big(x,t,\nabla T_{\bar{k}}(\overline{u}^n)\big) \cdot \nabla \omega_j\, dx\, dt,\end{aligned}$$

where $D_{n,j} := \{(x,t) \in Q_T : \, |\, \overline{u}^n - \omega_j \,| \le k\}$. Using (4.37)–(4.39), Fatou's lemma and the standard Minty argument for Leray–Lions operators in Musielak–Orlicz spaces (applied to the truncations), we obtain

$$\begin{aligned}&\liminf_{n\to\infty} \int_{Q_T} a\big(x,t,\nabla T_{\bar{k}}(\overline{u}^n)\big) \cdot \nabla T_k(\overline{u}^n - \omega_j)\, dx\, dt \\ &\ge \int_{Q_T} a\big(x,t,\nabla T_{\bar{k}}(u)\big) \cdot \nabla T_k(u - \omega_j)\, dx\, dt.\end{aligned} \tag{4.50}$$

Limit in the right-hand side. Since $f^n \to f$ in $L^1(Q_T)$ and $T_k(\overline{u}^n - \omega_j) \overset{*}{\rightharpoonup} T_k(u-\omega_j)$ in $L^\infty(Q_T)$,

$$\int_{Q_T} f^n T_k(\overline{u}^n - \omega_j)\, dx\, dt \;\longrightarrow\; \int_{Q_T} f\, T_k(u - \omega_j)\, dx\, dt. \tag{4.51}$$

Combining (4.48), the lower semicontinuity (4.50), the time–duality limits, and (4.51), and then letting $n \to \infty$, we arrive at

$$\begin{aligned}&\int_0^T \Big\langle \partial_t \omega_j,\, \int_0^{T_M(u)} b'(z)\, T_k'(z - \omega_j)\, dz \Big\rangle dt + \int_{Q_T} a\big(x,t,\nabla T_{\bar{k}}(u)\big) \cdot \nabla T_k(u - \omega_j)\, dx\, dt \\ &\le \int_{Q_T} f\, T_k(u - \omega_j)\, dx\, dt.\end{aligned}$$

Finally, sending $j \to \infty$ and using (4.46) (together with the density of $\mathcal{D}(Q_T)$ in the class of admissible tests) yields, for every admissible v,

$$\int_0^T \Big\langle \partial_t v,\, \int_0^u b'(z)\, T_k'(z - v)\, dz \Big\rangle dt + \int_{Q_T} a\big(x,t,\nabla u\big) \cdot \nabla T_k(u - v)\, dx\, dt \;\le\; \int_{Q_T} f\, T_k(u - v)\, dx\, dt.$$

This is precisely the entropy inequality of Definition 3.1. Therefore, u is an entropy solution of (1.1), which completes the proof of Theorem 3.1.

5. CONCLUSION AND PERSPECTIVES

We have proved the existence of an entropy solution to the doubly nonlinear parabolic problem (1.1) in the full Musielak–Orlicz setting without assuming the Δ_2 condition. The proof is built on a Rothe–type semidiscretization, a discrete entropy formulation, and *a priori* bounds derived in the modular topology. These ingredients yield compactness of the Rothe sequence in $W_0^{1,x}L_\Psi(Q_T)$ and $C([0,T];L^1(\Omega))$, and allow for the identification of the limit via monotonicity methods for Leray–Lions operators. The result encompasses general Lipschitz, strictly increasing nonlinearities $b(\cdot)$ and merely L^1 data, thereby unifying and extending earlier contributions obtained in Sobolev, variable–exponent, and Orlicz–Sobolev frameworks.

Beyond offering a streamlined existence theory in a nonstandard growth context, the approach is robust and suggests several directions for future research. On the analytical side, one may investigate conditions ensuring uniqueness and L^1–contraction at the continuous level, finer regularity (e.g., local higher integrability of gradients), or stability under lower–order perturbations. On the modeling and numerical side, the discrete entropy structure lends itself to fully discrete schemes and error analysis; extending the method to inhomogeneous boundary conditions, obstacle problems, measure–valued data, or anisotropic/fractional operators in Musielak–Orlicz spaces also appears within reach. These perspectives highlight the versatility of the Rothe framework for nonlinear evolutions with generalized growth.

CONFLICTS OF INTEREST

The authors declare that they have no conflicts of interest.

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