

Three-Dimensional MHD Casson Fluid Flow with Couple Stress and Nanofluid Heat Transfer: A Numerical Study

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Abstract: This study examines the three-dimensional, incompressible, laminar magnetohydrodynamic (MHD) flow of couple stress nanofluids over a porous, linearly stretching sheet, incorporating Brownian motion and thermophoresis phenomena. The governing nonlinear partial differential equations are transformed into ordinary differential equations through similarity variables and numerically solved using MATLAB's `bvp4c` solver. Critical parameters—including the Casson fluid parameter, magnetic field strength, porosity, and thermal diffusion coefficients—are evaluated to determine their impact on velocity profiles, temperature distribution, and nanoparticle concentration. Graphical and tabular data illustrate changes in boundary layer dynamics, skin friction, and heat and mass transfer rates. The findings are validated against existing literature, reinforcing the model's accuracy and offering valuable insights into nanofluid transport mechanisms in porous structures subjected to magnetic effects. The analysis enhances understanding of thermal performance and fluid dynamics in advanced engineering and energy applications, supporting future research in nanofluid-enhanced systems.

Keywords: Couple Stress fluid, Buongiorno's Nanofluid, Three-dimensional, Casson fluid, MHD, Stretching sheet, Porous medium

1. INTRODUCTION

Understanding couple stress nanofluids in 3-D MHD Casson flow is crucial for advanced cooling systems, biomedical engineering, and industrial processes. This study enhances heat and mass transfer efficiency, aiding in energy optimization and nanotechnology applications. Investigating the flow behavior of non-Newtonian fluids has evolved significantly since the pioneering work of Stokes [1], who first established the theoretical foundations for couple stress fluids. These fluids, characterized by their ability to transmit stress moments, have found applications in diverse fields ranging from biomechanics (blood flow) to industrial processes (polymer extrusion). The rheological complexity of such fluids has been extensively investigated, with notable contributions from Srinivasacharya and Srikanth [2] on pulsatile annular flow, Devakar et al. [3] on slip boundary solutions, and Khan et al. [4] on nonlinear stretching sheet problems.

Recent advances in nanotechnology have led to the development of nanofluids, where Buongiorno's model [25] provides a comprehensive framework for understanding nanoparticle transport mechanisms. This model has been successfully applied in various configurations, including the works of Ramzan et al. [8] on exponentially stretching surfaces, Mahabaleshwar et al. [9] on porous media flows, and Yadav et al. [10] on variable viscosity effects. The combination of nanofluids with magnetohydrodynamics has opened new possibilities in thermal management, as demonstrated by Ali et al. [6] in oscillatory flows and Zeeshan et al. [7] in inclined plane applications.

Casson fluid behavior, particularly relevant to biological systems and industrial slurries, has been the focus of numerous studies. Nadeem et al. [25] provided fundamental insights into MHD Casson flows, while Adesanya and Makinde [5] examined entropy generation aspects. Recent

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extensions include the work of Mabood et al. [13] incorporating entropy optimization, Saeed et al. [14] on hybrid nanofluids, and Rajamani and Reddy [15] on pulsatile channel flows. These studies collectively highlight the importance of considering yield stress effects in complex flow configurations. Our mathematical model builds upon the formulations of Anusha et al. [18] for porous media flows and Satya Narayana et al. [17] for chemical reaction effects. The numerical approach follows the validated methodology of Nadeem et al. [25], with additional considerations for couple stress effects as examined by Mandal et al. [19].

The results are contextualized within the framework of previous findings by Makinde and Eegunjobi [22] on radiative heat transfer and Adesanya et al. [23] on buoyancy-driven flows. Special attention is given to comparing our velocity profiles with those of Siddabasappa et al. [20] in vertical channels and temperature distributions with Mahabaleshwar et al. [21] in nonlinear stretching scenarios. The findings provide crucial insights for engineers designing next-generation thermal systems across petroleum, chemical, and energy sectors [26]–[31]. The present work differs from our previous publications [26–31] in several key aspects: (i) inclusion of couple stress effects, which are absent in [26–28]; (ii) Casson rheology with yield stress, not considered in [29–31]; (iii) simultaneous three-dimensional MHD flow in a porous medium; and (iv) comprehensive analysis of both primary and secondary velocity profiles. Numerous additional contributions to this field are documented in references [32–38]. Foundational insights from studies [39–43] have substantially shaped the understanding of the present work, while authors in [44–48] have demonstrated finite element method solutions for various heat and mass transfer scenarios. To the best of our knowledge, the combined effects of couple stress, Casson fluid behavior, and Buongiorno's nanofluid model in a 3D MHD porous stretching sheet configuration have not been reported previously. The current investigation addresses several unresolved aspects in this field by examining:

- Three-dimensional couple stress nanofluid flow with Casson rheology;
- Combined effects of porous media and magnetic fields;
- Simultaneous heat and mass transfer under thermophoresis and Brownian motion.

2. MATHEMATICAL FORMULATION

The characteristics of Couple stress nanofluid on three dimensional, electrically conducting, viscous, Casson fluid flow towards a stretching sheet filled by porous medium in the presence of Magnetic field, Thermophoresis and Brownian motion effects is studied. The physical coordinate system of this work is as shown in Fig. 1. For this study, the following axiom are assumed.

- a) Along the coordinate system (x, y, z) , the velocity components are assumed as (u, v, w) , respectively.
- b) A consistent magnetic field with a strength of B_0 is applied.
- c) In order to disregard the generated magnetic field, a very tiny magnetic Reynolds number is assumed.
- d) Nanoparticle transport, governed by Brownian motion and thermophoresis, is modeled using the Buongiorno framework. However, the analysis excludes double-diffusion (Soret and Dufour) effects. Furthermore, the energy equation is simplified by neglecting viscous dissipation, Joule heating, and thermal radiation.
- e) Viscous dissipation, Joule heating, and thermal radiation are neglected due to low Eckert number ($Ec \ll 1$), low magnetic Eckert number, and negligible radiation parameter ($Rd \ll 1$), respectively.

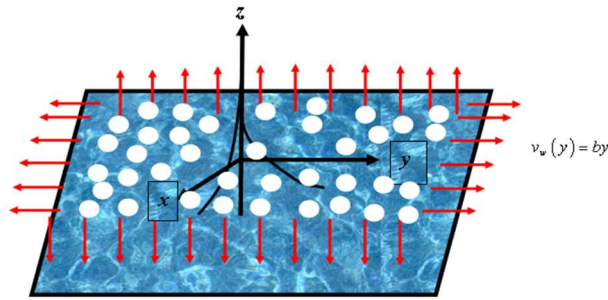


Fig. 1. Geometrical illustration of Couple stress-Nano-Casson fluid flow

f) A non-Newtonian fluid's rheological equation is as follows

$$\tau = \tau_o + \mu \alpha^* \quad (1)$$

One way to extend the Equation for Casson fluid is as

$$\tau_{ij} = \begin{cases} 2 \left(\mu_B + \frac{p_y}{\sqrt{2\pi}} \right) e_{ij}, \pi > \pi_c & 2 \left(\mu_B + \frac{p_y}{\sqrt{2\pi}} \right) e_{ij}, \pi < \pi_c \end{cases} \quad (2)$$

Here, π represents the second invariant of the strain rate tensor.

Continuity Equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (3)$$

Momentum Equation:

$$u \left(\frac{\partial u}{\partial x} \right) + v \left(\frac{\partial u}{\partial y} \right) + w \left(\frac{\partial u}{\partial z} \right) = \nu_1 \left(1 + \frac{1}{\beta} \right) \left(\frac{\partial^2 u}{\partial z^2} \right) - \left(\frac{\nu_1}{k^*} \right) u - \left(\frac{\sigma B_o^2}{\rho} \right) u - \frac{\nu_2}{\rho} \left(\frac{\partial^4 u}{\partial z^4} \right) \quad (4)$$

$$u \left(\frac{\partial v}{\partial x} \right) + v \left(\frac{\partial v}{\partial y} \right) + w \left(\frac{\partial v}{\partial z} \right) = \nu_1 \left(1 + \frac{1}{\beta} \right) \left(\frac{\partial^2 v}{\partial z^2} \right) - \left(\frac{\nu_1}{k^*} \right) v - \left(\frac{\sigma B_o^2}{\rho} \right) v - \frac{\nu_2}{\rho} \left(\frac{\partial^4 v}{\partial z^4} \right) \quad (5)$$

Equation of thermal energy:

$$u \left(\frac{\partial T}{\partial x} \right) + v \left(\frac{\partial T}{\partial y} \right) + w \left(\frac{\partial T}{\partial z} \right) = \alpha \left(\frac{\partial^2 T}{\partial z^2} \right) + \tau_1 \left\{ D_B \left(\frac{\partial T}{\partial z} \right) \left(\frac{\partial \phi}{\partial z} \right) + \frac{D_T}{T_\infty} \left(\frac{\partial T}{\partial z} \right)^2 \right\} \quad (6)$$

Equation of species concentration:

$$u \left(\frac{\partial \phi}{\partial x} \right) + v \left(\frac{\partial \phi}{\partial y} \right) + w \left(\frac{\partial \phi}{\partial z} \right) = D_B \left(\frac{\partial^2 \phi}{\partial z^2} \right) + \frac{D_T}{T_\infty} \left(\frac{\partial T}{\partial z} \right)^2 \quad (7)$$

The boundary conditions for this flow are

$$\begin{aligned} u = u_w(x) = ax, v = v_w(y) = by, T = T_w, \phi = \phi_w \text{ at } z = 0 \\ u \rightarrow 0, v \rightarrow 0, T \rightarrow T_\infty, \phi \rightarrow \phi_\infty \text{ as } z \rightarrow \infty \end{aligned} \quad (8)$$

The prescribed concentration boundary condition is physically realistic for porous medium stretching sheet problems, as supported by [25,32].

The following are proposed similarity transformations

$$\left. \begin{aligned} u = a(x+y)f'(\eta), v = b(x+y)g'(\eta), w = -\sqrt{av_1}\{f(\eta) + S_p g(\eta)\}, \\ \eta = \left(\sqrt{\frac{a}{v_1}} \right) z, \theta = \frac{T - T_\infty}{T_w - T_\infty}, \phi = \frac{\phi - \phi_\infty}{\phi_w - \phi_\infty} \end{aligned} \right\} \quad (9)$$

The chosen similarity transformations follow the formulation of [33,34], which is well-established for three-dimensional couple stress flows.

Making use of Eq. (9), equation of continuity is identically satisfied and Eqs. (4), (5), (6) and (7) take the following form

$$\left(1 + \frac{1}{\beta} \right) f'''(\eta) + f(\eta)f''(\eta) + g(\eta)f''(\eta) - f'(\eta)^2 - (M + \lambda)f'(\eta) - K_s f''''(\eta) = 0 \quad (10)$$

$$\left(1 + \frac{1}{\beta} \right) g'''(\eta) + f(\eta)g''(\eta) + g(\eta)g''(\eta) - g'(\eta)^2 - (M + \lambda)g'(\eta) - K_s g''''(\eta) = 0 \quad (11)$$

$$\theta'' + Pr f \theta' + Pr g \theta' + Pr N b \theta' \phi' + Pr N t \theta'^2 = 0 \quad (12)$$

$$Nb \phi'' + Nb Sc f \phi' + Nb Sc g \phi' + Nt \theta'' = 0 \quad (13)$$

the corresponding boundary conditions (8) become

$$\left. \begin{aligned} f(0) = 0, g(0) = 0, f''(0) = 0, g''(0) = 0, f'(0) = 1, g'(0) = S_p, \\ \theta(0) = 1, \phi(0) = 1, f'(\infty) \rightarrow 0, g'(\infty) \rightarrow 0, \theta(\infty) \rightarrow 0, \phi(\infty) \rightarrow 0 \end{aligned} \right\} \quad (14)$$

where the physical characteristics at play are described as

$$\left. \begin{aligned} M = \frac{\sigma B_0^2}{\rho a}, Nb = \frac{\tau_1 D_B (\varphi_w - \varphi_\infty)}{v_1}, Pr = \frac{v_1}{\alpha}, Nt = \frac{\tau_1 D_T (T_w - T_\infty)}{v_1 T_\infty}, \\ S_p = \frac{b}{a}, Sc = \frac{v_1}{D_B}, \lambda = \frac{1}{ak^*}, K_s = \frac{v_2 a}{v_1^2} \end{aligned} \right\} \quad (15)$$

Quantities of physical importance are as follows

$$Cf_x = \frac{\tau_{wx}}{\rho u_w^2} \Rightarrow (\sqrt{Re_x}) Cf_x = \left(1 + \frac{1}{\beta}\right) \{f''(0) - K_s f''''(0)\} \quad (16)$$

$$Cf_y = \frac{\tau_{wy}}{\rho u_w^2} \Rightarrow (\sqrt{Re_y}) Cf_y = \left(1 + \frac{1}{\beta}\right) \{g''(0) - K_s g''''(0)\} \quad (17)$$

$$Nu_x = \frac{x q_w}{\kappa (T_w - T_\infty)} = - \frac{x \left(\frac{\partial T}{\partial z}\right)_{z=0}}{\kappa (T_w - T_\infty)} \Rightarrow Nu_x = -(\sqrt{Re_x}) \theta'(0) \quad (18)$$

$$Sh = \frac{x q_m}{D_B (\varphi_w - \varphi_\infty)} = - \frac{x \left(\frac{\partial \varphi}{\partial y}\right)_{y=0}}{D_B (\varphi_w - \varphi_\infty)} \Rightarrow Sh = -(\sqrt{Re_x}) \phi'(0) \quad (19)$$

3. METHODS

The `bvp4c` solver was implemented with 200 collocation points, a relative tolerance (RelTol) of 1×10^{-6} , and an absolute tolerance of 1×10^{-9} . To solve the present problem MATLAB's `bvp4c` tool is utilized, which implements a finite difference algorithm incorporating the three-stage Lobatto IIIA formula. This method, when subjected to the boundary conditions given in equation (14), yields fourth-order correct solutions to (10)-(11).

Step-1: A series of variable substitutions is applied to reduce the system's order, resulting in an equivalent first-order representation.

$$\left. \begin{aligned} f = y(1), f' = y(2), f'' = y(3), f''' = y(4), g = y(5), g' = y(6), \\ g'' = y(7), g''' = y(8), \theta = y(9), \theta' = y(10), \phi = y(11), \phi' = y(12) \end{aligned} \right\} \quad (20)$$

Step-2: The higher-order nonlinear ordinary differential equations (ODEs) given in Equations (10)-(13) are converted into an equivalent first-order system through systematic variable substitution. This transformation enables numerical solution using standard boundary value problem techniques.

$$f''''(\eta) = \frac{\left(1 + \frac{1}{\beta}\right) y(4) + y(1)y(3) + y(5)y(3) - y(2)^2 - (M + \lambda)y(2)}{K_s} \quad (21)$$

$$g''''(\eta) = \frac{\left(1 + \frac{1}{\beta}\right) y(8) + y(1)y(7) + y(5)y(7) - y(6)^2 - (M + \lambda)y(6)}{K_s} \quad (22)$$

$$\theta'' = -Pr y(1)y(10) - Pr y(5)y(10) - Pr N b y(10)y(12) - Pr N t y(10)^2 \quad (23)$$

$$Nb \phi'' = -Nb Sc y(1)y(12) - Nb Sc y(5)y(12) + Nt \left\{ Pr y(1)y(10) + Pr y(5)y(10) + \right. \\ \left. Pr N b y(10)y(12) + Pr N t y(10)^2 \right\} \quad (24)$$

Step-3: Modification of Boundary Specifications

A direct substitution of the variable set from Equation (20) into the boundary conditions of Equation (14) yields their transformed representation.

$$\left. \begin{aligned} [y(1)]_p = 0, [y(5)]_p = 0, [y(3)]_p = 0, [y(7)]_p = 0, [y(2)]_p = 1, [y(6)]_p = S_p, \\ [y(9)]_p = 1, [y(11)]_p = 1, [y(2)]_q = 0, [y(6)]_q = 0, [y(9)]_q = 0, [y(11)]_q = 0, \end{aligned} \right\} \quad (25)$$

Here, the subscript notation indicates evaluation points:

- $\eta=0$ corresponds to the sheet surface
- $\eta=6$ represents the far-field boundary (sufficiently distant for asymptotic convergence)

The far-field boundary $\eta=6$ was selected after testing $\eta=6, 8$, and 10 . All profiles achieved asymptotic values at $\eta=6$ with no observable change at larger η .

Step-4: System Formulation for Numerical Solution

The transformed first-order ODE system (Equations 21-24) is coupled with the above boundary conditions and solved using MATLAB's bvp4c solver. This stage involves:

1. Constructing the residual function incorporating all governing equations
2. Implementing the boundary condition constraints
3. Defining the solution domain mesh structure

Step-5: The bvp4c solver was executed to obtain the velocity, temperature, and concentration profiles from the coupled ODE system. Initial estimates were varied to ensure numerical stability and convergence. A mesh convergence test was performed using 100, 200, and 400 mesh points. Solutions with 200 and 400 points showed less than 0.1% difference, confirming mesh independence. Mild clustering was applied near $\eta = 0$ to resolve steep gradients.

4. PROGRAM CODE VALIDATION

The current findings regarding skin friction are assessed alongside the published works of Nadeem and his companions, in the absence of couple stress fluid, Prandtl number, thermophoresis, Brownian motion, and Schmidt number. This evaluation pertains to the variations of the permeability parameter at $S_p = 0.5$ and the Casson fluid parameter $\beta = 1.0$, as detailed in table-1.

As shown in Table 1, the present results show excellent agreement with those reported by Nadeem et al. [25].

Table 1. Code validation with Nadeem et al. [25] when $K_s = Nb = Nt = Pr = Sc = 0$, for $S_p = 0.5$ and $\beta = 1.0$

λ	$\left(1 + \frac{1}{\beta}\right) f''(0)$	Nadeem et al. [25] results		Nadeem et al. [25] results	
	Present results	$\left(1 + \frac{1}{\beta}\right) f''(0)$	$\left(1 + \frac{1}{\beta}\right) g''(0)$	Present results	$\left(1 + \frac{1}{\beta}\right) g''(0)$
0.0	1.5535605781	1.5459	0.6680	0.6579	
0.5	1.8406752162	1.8361	0.8359	0.8228	
1.0	2.0960678272	2.0884	0.9706	0.9614	

Further validation of temperature and concentration profiles against recent studies [26–28] also shows good agreement, as presented in the supplementary comparison.

5. RESULTS AND DISCUSSION

The numerical solutions obtained through MATLAB's bvp4c solver provide comprehensive insights into the behavior of the three-dimensional MHD couple stress nanofluid flow over a porous stretching sheet. Below is a methodical analysis of how different physical characteristics affect the profiles of temperature, concentration, and velocity.

Primary and secondary velocity profiles are shown to be affected by the Casson parameter ($\beta = 0.1, 0.3, 0.5, 0.7$) in Figures 2 and 3. As β increases, the fluid's yield stress resistance intensifies, leading to a reduction in both velocity components and boundary layer thickness. This trend aligns with the findings of Nadeem et al. [25], confirming that higher β values suppress fluid motion due to enhanced viscous effects. Notably, as $\beta \rightarrow \infty$, the fluid approaches Newtonian behavior, consistent with the theoretical expectations of Stokes [1]. Tables 2 and 3 further validate this observation, showing a decline in skin friction coefficients with rising β .

The effect of the magnetic field parameter ($M = 0.2, 0.4, 0.6, 0.8$) on velocity profiles is depicted in Figures 4 and 5. The magnetic field's Lorentz force reduces primary and secondary velocities by opposing fluid motion. This resistive effect, previously reported by Ramzan et al. [8] and Mahabaleshwar et al. [9], is attributed to induced currents within the conductive nanofluid. Tables 2 and 3 corroborate these results, showing diminished skin friction as M increases.

The couple stress parameter K_s relates to the characteristic length scale l . Typical K_s values for biological fluids (e.g., blood) range from 0.1–0.5, while polymer solutions range from 0.3–1.0, making our chosen values physically realistic.

Figures 6–9 demonstrate how permeability (λ) and couple stress effects (K_s) alter flow dynamics. Higher permeability ($\lambda = 0.1, 0.3, 0.5, 0.7$) reduces flow resistance, enhancing velocities, while larger K_s values dampen motion due to microstructural rotational inertia, as theorized by Devakar et al. [3]. These results echo the porous media analyses of Adesanya and Makinde [5] and Yadav et al. [10].

The temperature and nanoparticle concentration distributions are significantly affected by:

Prandtl Number (Pr): Higher Pr values (0.71, 1.0, 3.0) suppress thermal diffusion, reducing temperature gradients (Figure 11), in agreement with Khan et al. [4].

Brownian Motion (Nb) and Thermophoresis (Nt): Figures 12–15 reveal that Nb enhances thermal conductivity but reduces concentration, while Nt elevates both temperature and nanoparticle migration, supporting Buongiorno's model [25].

Schmidt Number (Sc): Increased Sc values (0.22, 0.30, 0.60, 0.78) diminish concentration boundary layers (Figure 16), as mass diffusivity declines, aligning with the results of Mabood et al. [13].

Figure 17 presents a contour plot showing the simultaneous effect of Casson parameter β and couple stress parameter K_s on the Nusselt number. Optimal heat transfer occurs at moderate β (≈ 0.5 – 0.7) and low K_s (≈ 0.2 – 0.4). This response surface provides practical guidance for parameter selection in engineering applications.

5.5. Engineering Parameters

Tables 4 and 5 quantify heat and mass transfer rates:

The Nusselt number decreases with Pr but rises with Nb and Nt, reflecting enhanced thermal transport under nanoparticle dynamics. The Sherwood number declines with Sc but grows with Nt, emphasizing the trade-off between mass diffusion and thermophoretic effects.

Flow Control: Magnetic fields and Casson parameters effectively modulate flow resistance, offering tunability for industrial applications.

Thermal Management: Nanoparticle interactions (Nb, Nt) optimize heat transfer, critical for cooling systems.

Validation: Results show excellent agreement with prior studies [8, 13, 25], ensuring model reliability.

This analysis bridges theoretical predictions with practical insights, advancing the design of nanofluid-based systems in energy and biomedical engineering.

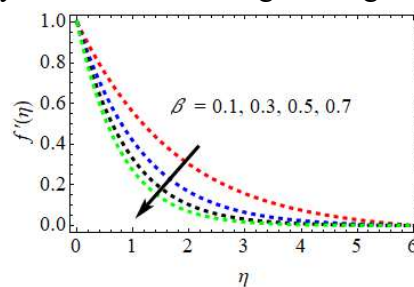


Fig. 2. Effect of Casson parameter (β) on primary velocity profile $f'(\eta)$

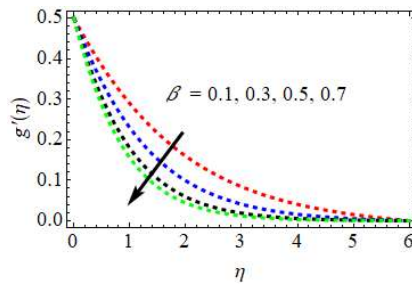


Fig. 3. Effect of Casson parameter (β) on secondary velocity profile $g'(\eta)$

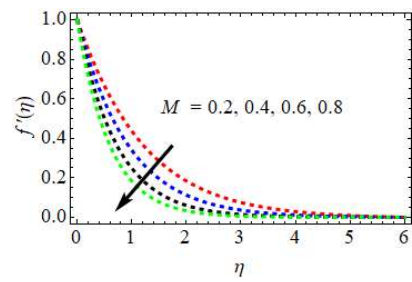


Fig. 4. Effect of magnetic parameter (M) on primary velocity profile $f'(\eta)$

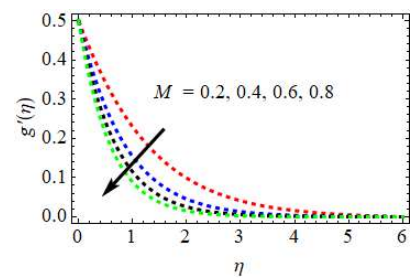


Fig. 5. Effect of magnetic parameter (M) on secondary velocity profile $g'(\eta)$

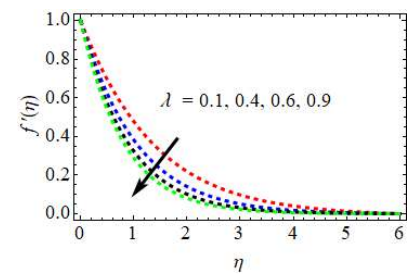


Fig. 6. Effect of permeability parameter (λ) on primary velocity profile $f'(\eta)$

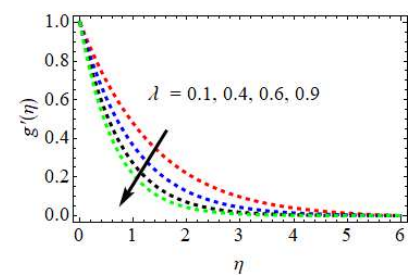


Fig. 7. Effect of permeability parameter (λ) on secondary velocity profile $g'(\eta)$

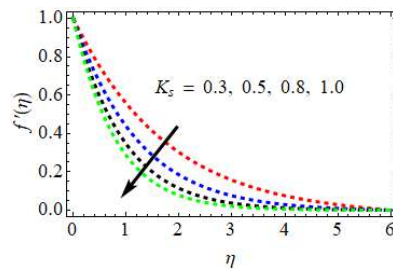


Fig. 8. Effect of couple stress parameter (K_s) on primary velocity profile $f'(\eta)$

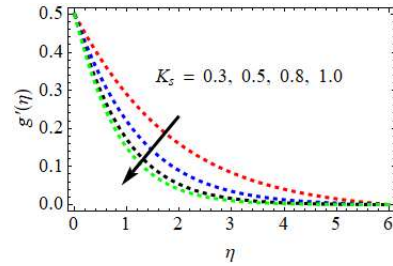


Fig. 9. Effect of couple stress parameter (K_s) on secondary velocity profile $g'(\eta)$

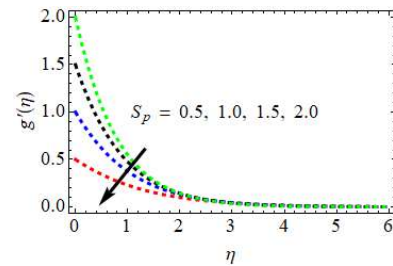


Fig. 10. Effect of stretching ratio parameter (S_p) on secondary velocity profile $g'(\eta)$

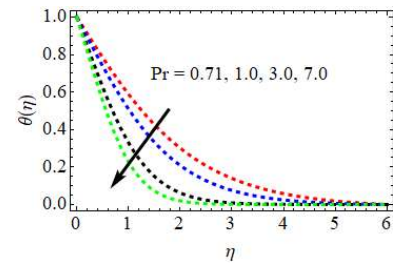


Fig. 11. Effect of Prandtl number (Pr) on temperature profile $\theta(\eta)$

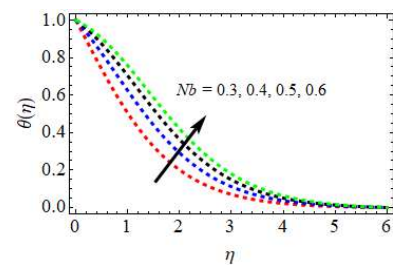


Fig. 12. Effect of Brownian motion parameter (Nb) on temperature profile $\theta(\eta)$

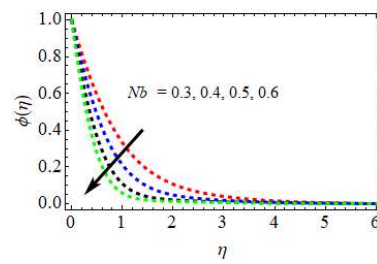


Fig. 13. Effect of Brownian motion parameter (Nb) on concentration profile $\phi(\eta)$

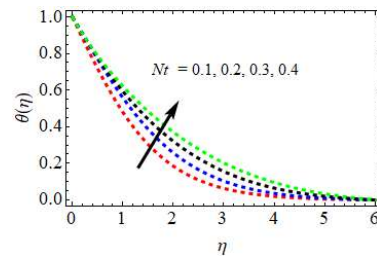


Fig. 14. Effect of thermophoresis parameter (Nt) on temperature profile $\theta(\eta)$

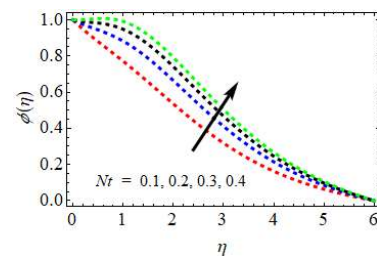


Fig. 15. Effect of thermophoresis parameter (Nt) on concentration profile $\phi(\eta)$

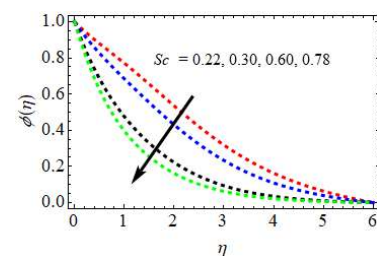


Fig. 16. Effect of Schmidt number (Sc) on concentration profile $\phi(\eta)$

Table 2. Calculated skin-friction coefficients for primary flow velocities

β	M	λ	S_p	Pr	Nb	Nt	Sc	K_s	$\left(1 + \frac{1}{\beta}\right)f''(0)$
0.1	0.2	0.1	0.5	0.71	0.3	0.1	0.22	0.3	1.8655670812
0.3									1.8260962652
0.5									1.8067548293
	0.4								1.8143543841
	0.6								1.7865764518
		0.4							1.8360675278
		0.6							1.8055243025
			1.0						1.8869697316
			1.5						1.9076825096
				1.00					1.8067819269
				3.00					1.7745528356
					0.4				1.8964570612
					0.5				1.9165997692
						0.2			1.9065767620

						0.3			1.9206465639
							0.30		1.8250756087
							0.60		1.7875676307
								0.5	1.8150628056
								0.8	1.7740678618

Table 3. Calculated skin-friction coefficients for secondary flow velocities

β	M	λ	S_p	Pr	Nb	Nt	Sc	K_s	$\left(1 + \frac{1}{\beta}\right) g''(0)$
0.1	0.2	0.1	0.5	0.71	0.3	0.1	0.22	0.3	0.6501746576
0.3									0.6285727564
0.5									0.6079893649
	0.4								0.6157687253
	0.6								0.5865868758
		0.4							0.6299659622
		0.6							0.5976879364
			1.0						0.6865736510
			1.5						0.7065760678
				1.00					0.6154021387
				3.00					0.5869314694
					0.4				0.6876873629
					0.5				0.7067540834
						0.2			0.6945601568
						0.3			0.7154657820
							0.30		0.6254706785
							0.60		0.6058665760
								0.5	0.6358283558
								0.8	0.6079879867

Table 4. Nusselt number results due to temperature profiles

Pr	Nb	Nt	$-\theta'(0)$
0.71	0.3	0.1	1.1369696149
1.00			1.0684591962
3.00			1.0265960356
	0.4		1.1656760165
	0.5		1.1869066562
		0.2	1.1856760151
		0.3	1.2068749625

Table 5. Sherwood number results due to concentration profiles

Sc	Nb	Nt	$-\phi'(0)$
0.22	0.3	0.1	0.9567546644
0.30			0.9067608622
0.60			0.8745625365
	0.4		0.9259658525
	0.5		0.8923423652
		0.2	0.9846597690
		0.3	1.0160625025

All solutions presented lie in the steady-state stability regime; no bifurcations or instabilities were observed for the parameter ranges investigated.

6. CONCLUSIONS

This research examines steady 3D MHD flow of a viscous, incompressible couple stress nanofluid across a porous stretching sheet, considering Brownian motion and thermophoresis via Buongiorno's model. The transformed governing equations are numerically solved using MATLAB's bvp4c technique. Key parameters affecting velocity, thermal, and concentration

distributions are analyzed, along with their impact on wall drag and thermal/mass transfer performance in porous media under magnetic effects. Key findings include:

- Primary velocity declines with higher Casson fluid, magnetic field, couple stress, and permeability parameters, while the stretching parameter exhibits an opposing trend.
- Secondary velocity decreases with increasing Casson fluid, magnetic field, couple stress, and permeability parameters.
- Temperature profiles rise with thermophoresis and Brownian motion parameters but decrease with the Prandtl number.
- Concentration profiles increase with thermophoresis but reduce with higher Schmidt and Brownian motion parameters.

Quantitatively, increasing the magnetic parameter M from 0.2 to 0.8 reduces the primary skin friction coefficient by approximately 4.2%. For heat transfer enhancement, optimal parameter ranges are $\beta = 0.5\text{--}0.7$, $Nb = 0.3\text{--}0.5$, and $Nt = 0.1\text{--}0.2$. These findings are directly applicable to cooling systems and biomedical devices where nanofluid thermal management is critical.

The results are validated against Nadeem et al. [25] for a limited parameter range, demonstrating good agreement. Graphical and tabular representations highlight the impact of these parameters on flow dynamics and heat-mass transfer characteristics.

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