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Vector Fields with Non-Isolated Singular Points Viewed from the Inside and Outside

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Abstract: We present a brief survey of recent results about vector fields with non-isolated singular points and their applications. One of the most interesting and promising application is connected with quasi-linear differential equations of the second order, including the equation of geodesics in signature varying (pseudo-Riemannian) metrics.

Keywords: vector fields, singular points, center manifold, normal forms, resonances

1. INTRODUCTION

We start with a general construction, which naturally leads us to vector field with non-isplated singular points (more precisely, singular points fill a manifols of codimension two in the phase space).

Let M be a real smooth (C^∞) manifold of dimension $n+2$. Here and further we use the following standard notations:

- $C^\infty(M)$ is the ring of smooth functions on M
- $\Gamma(TM)$ is the module of smooth vector fields on M
- $\Gamma(T^*M)$ is the module of smooth covector fields (1-forms) on M

Consider a distribution on M defined via $n+1$ differential 1-forms:

$$\omega_1 = 0,\ \ldots,\ \omega_{n+1} = 0,\ \ \omega_i \in \Gamma(T^*M). \tag{1.1}$$

At points of M where the 1-forms $\omega_1, \ldots, \omega_{n+1}$ are linearly independent, system (1.1) defines a one-dimensional distribution, that is, a direction field, which can be a class of collinear vector fields, such points we shall call *regular*. At points of M where the 1-forms $\omega_1, \ldots, \omega_{n+1}$ are nor linearly independent, the distribution has dimension greater then 1, such points we shall call *singular*.

In (local) coordinates $u = (u_1, \ldots, u_{n+2})$ on M system (1.1) yields the Pfaffian system

$$\omega_i = \sum_{j=1}^{n+2} \omega_{ij}(u)\, du_j = 0, \quad i = 1, \ldots, n+1, \tag{1.2}$$

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with $(n+1) \times (n+2)$ matrix $\Omega = (\omega_{ij})$.

Singular points fill a stratified manifold $\Sigma \subset M$ that consists of the strata

$$\Sigma_i = \{u \in M : \operatorname{rg} \Omega(u) = (n+1) - i\}, \ \ i = 1, \dots, n+1.$$

The dimension of Σ_i decreases rapidly with increasing i, namely: $\operatorname{codim} \Sigma_i = i(i+1)$; see, for example, [2]. The maximal stratum Σ_1 of codimension 2 is stable with respect to small perturbation of ω_i, while the strata of higher codimension are, generally speaking, not.

The goal of the paper is to study integral curve of the distribution (1.1) entering its singular points of the maximal stratum Σ_1. We also consider some application of the obtained results for studying singularities of differential equations of special types, which are interesting due to various applications.

2. THE MAIN RESULTS

The distribution (1.2) can be determined (at least locally) by a smooth vector field $\bar{\mathrm{V}} \in \Gamma(TM)$ that is zero at points of Σ. This field is unique up to multiplying by a scalar factor:

$$\bar{\mathrm{V}} = \sum_{j=1}^{n+2} v_j(u) \, \frac{\partial}{\partial u_j}, \ \ v_j = (-1)^{j+1} \Delta_j, \ \ j = 1, \dots, n+2, \tag{2.3}$$

where Δ_j is the minor of Ω obtained by elimination of the j-th column.

Example 2.1:
Consider the lowest dimension case: $n = 1$. Then the distribution (1.2) in 3-dimensional space is the intersection of two fields of planes:

$$\omega_i = \omega_{i1} du_1 + \omega_{i2} du_2 + \omega_{i3} du_3 = 0, \ \ i = 1, 2.$$

Here Ω is a 2×3 matrix

$$\Omega = \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \end{pmatrix}$$

and $\bar{\mathrm{V}}$ is the vector product of its columns:

$$v_1 = \begin{vmatrix} \omega_{12} & \omega_{13} \\ \omega_{22} & \omega_{23} \end{vmatrix}, \ \ v_2 = -\begin{vmatrix} \omega_{11} & \omega_{13} \\ \omega_{21} & \omega_{23} \end{vmatrix}, \ \ v_3 = \begin{vmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{vmatrix}.$$

The maximal stratum Σ_1 is defined by the condition $\operatorname{rg} \Omega = 1$, that is, the matrix Ω has at least one non-zero element. Without loss of generality, assume that $\omega_{i3} \neq 0$. Then the equality $\omega_{i1} v_1 + \omega_{i2} v_2 + \omega_{i3} v_3 = 0$ yields the expression

$$v_3 = -\frac{\omega_{i1}}{\omega_{i3}} v_1 - \frac{\omega_{i2}}{\omega_{i3}} v_2, \tag{2.4}$$

which is valid locally, in a neighborhood of a point where $\omega_{i3} \neq 0$.

Expression (2.4) shows that the components of the vector field $\bar{\mathrm{V}}$ are connected by functional relations: in a neighborhood of a point of Σ_1 they belong to the ideal in the ring $C^\infty(M)$ generated by two of them.

The letter property established for $n = 1$, is valid for arbitrary n:

Lemma 2.1:
In a neighborhood of a point of Σ_1, all components of the field (2.3) *belong to the ideal (in $C^\infty(M)$) generated by two of them. Therefore, one can choose coordinates $x, y, z = (z_1, \ldots, z_n)$ such that the germ of the field* (2.3) *has the form*

$$\dot{x} = v, \ \ \dot{y} = w, \ \ \dot{z}_i = a_i v + b_i w, \ \ i = 1, \ldots, n, \tag{2.5}$$

where $v, w, a_i, b_i \in C^\infty(M)$.

Proof
The proof of the first statement of the lemma is similar to those for the case $n = 1$ considered above. The second statement obviously follows from the first one. □

Lemma 2.1 establish some important properties of vector fields defined by Pfaffian systems (1.2). The main distinctive feature is that singular points of such fields are not isolated, but filled a manifold of codimension two. For instance, singular points of the field (2.5) are given by two equations

$$v(x, y, z) = 0, \ \ w(x, y, z) = 0. \tag{2.6}$$

Therefore, the spectrum of the linear part of field (2.5) at every singular points $T \in \Sigma_1$ has the form

$$\operatorname{spec}(T) = (\lambda_1(T), \lambda_2(T), 0, \ldots, 0), \ \ \#0 = n,$$

where the eigenvalues $\lambda_{1,2}(T)$ continuously depend on $T \in \Sigma_1$. Further we shall use the notation

$$\lambda_{1,2} = \lambda_{1,2}(T_0).$$

Further we shall consider the germ (2.3) at a *generic* singular point $T_0 \in \Sigma_1$, where

$$\operatorname{Re} \lambda_{1,2} \neq 0. \tag{2.7}$$

In this case, the set of singular points Σ_1 is the (unique) center manifold W^c of the field (2.5) passing through the point T_0. This allows us to apply the reduction principle [1, 6], which yields the following result:

Theorem 2.1:
The germ of the field (2.5) *at a generic singular point T_0 satisfying the condition* (2.7) *is topologically equivalent to the field*

$$\dot{\xi} = a_1 \xi, \ \ \dot{\eta} = a_2 \eta, \ \ \dot{\zeta}_i = 0, \ \ i = 1, \ldots, n, \tag{2.8}$$

where $a_j = \operatorname{sgn}(\operatorname{Re} \lambda_j)$.

Proof
This is a trivial corollary of the reduction principle, which states that the germ of the field (2.5) at T_0 is topologically equivalent to the product of the field $\dot{\xi} = a_1 \xi$, $\dot{\eta} = a_2 \eta$ and the restriction of (2.5) to its center manifold W^c. Since W^c consists of singular points of the field (2.5), this yields normal form (2.8). □

The next step in the study of fields (2.5) is their smooth local classification. For this, we need to consider two types of *resonances* (integer relations) between the non-zero eigenvalues:

$$s_1 \lambda_1 + s_2 \lambda_2 = 0, \ \ s_i \in \mathbb{Z}_+, \ \ i = 1, 2, \tag{2.9}$$

$$s_1 \lambda_1 + s_2 \lambda_2 = \lambda_j, \ \ s_i \in \mathbb{Z}_+, \ \ i, j = 1, 2. \tag{2.10}$$

From them we have to exclude *trivial* resonances, which always exist: resonance (2.9) with $s_1 = s_2 = 0$ and resonance (2.10) with $s_1 = 1$, $s_2 = 0$, $j = 1$ or $s_1 = 0$, $s_2 = 1$, $j = 2$.

The number $|s| = s_1 + s_2$ is called the *order* of resonance (2.9) or (2.10).

Remark 2.1:
1. The absence of resonances (2.10) implies the absence of the resonances (2.9):
2. In the absence of resonances (2.9), resonances (2.10) may have only the simplest form

$$\lambda_1 = m\lambda_2 \ \text{ or } \ \lambda_2 = m\lambda_1 \tag{2.11}$$

with integer $m \geq 1$.

Theorem 2.2:
If between the eigenvalues $\lambda_{1,2}(T)$ there are no non-trivial resonances (2.9) of any order $|s| \geq 1$ for all $T \in W^c$ sufficiently close to T_0, then the germ of the field (2.5) at T_0 is C^∞-smoothly equivalent to

$$\dot{\xi} = X(\xi, \eta, \zeta), \quad \dot{\eta} = Y(\xi, \eta, \zeta), \quad \dot{\zeta}_j = 0, \ \ j = 1, \ldots, n, \tag{2.12}$$

where X and Y are smooth functions vanishing on the center manifold.
If, in addition, $|\lambda_1| > |\lambda_2|$ at T_0, then

$$X = \lambda_1(\zeta)\xi + \varphi(\zeta)\eta^m, \ \ Y = \lambda_2(\zeta)\eta, \tag{2.13}$$

where $\varphi(\zeta) \not\equiv 0$ only if $\lambda_1 = m\lambda_2$ with some integer $m > 1$.

The condition that between $\lambda_{1,2}(T)$ there are no non-trivial resonances (2.9) of any order $|s| \geq 1$ for all $T \in W^c$ appears when we consider infinitely smooth classification of vector fields with non-isolated singular points. First it was formulated in the paper [17] and then it is often named after him.

Remark 2.2:
1. If the pair $\lambda_{1,2} = \lambda_{1,2}(T_0)$ belongs to the Poincaré domain, that is, $\lambda_{1,2}$ are real and of the same sign or complex conjugate with the condition (2.7), then the Roussarie condition holds true, whence Theorem 2.2 is valid.
2. If the pair $\lambda_{1,2} = \lambda_{1,2}(T_0)$ belongs to the Siegel domain, that is, $\lambda_{1,2}$ are real and of different signs, the Roussarie condition holds true if and only if

$$\lambda_1(T) : \lambda_2(T) \equiv \text{const} \notin \mathbb{Q}, \ \ \forall T \in W^c.$$

If we consider C^k-smooth equivalence with $k < \infty$, the absence of resonances of all orders $|s|$ at all points $T \in W^c$ can be replaced with the absence of resonances of the orders $|s| \leq N(k)$ at T_0, where

$$N(k) = 2\left[(2k+1)\frac{\max|\operatorname{Re}\lambda_{1,2}|}{\min|\operatorname{Re}\lambda_{1,2}|}\right] + 2, \tag{2.14}$$

the square brackets denote the integer part of a number. The estimation (2.14) is taken from [18].

Theorem 2.3:
If between the eigenvalues $\lambda_{1,2} = \lambda_{1,2}(T_0)$ there are no non-trivial resonances (2.9) of any order $1 \leq |s| \leq N(k)$, then the germ of the field (2.5) at T_0 is C^k-smoothly equivalent to

$$\dot{\xi} = X(\xi, \eta, \zeta), \quad \dot{\eta} = Y(\xi, \eta, \zeta), \quad \dot{\zeta}_j = 0, \ \ j = 1, \ldots, n,$$

where X and Y are smooth functions vanishing on the center manifold.
If, in addition, $|\lambda_1| > |\lambda_2|$ at T_0, then

$$X = \lambda_1(\zeta)\xi + \varphi(\zeta)\eta^m, \ \ Y = \lambda_2(\zeta)\eta,$$

where $\varphi(\zeta) \not\equiv 0$ only if $\lambda_1 = m\lambda_2$ with positive integer $m \leq N(k)$.

For the proofs of Theorems 2.2, 2.3, see [5, 13].

Remark 2.3:
Geometrically, Theorems 2.2, 2.3 state that the vector field (2.5) has 2-dimensional invariant foliation such that the restriction of the field to its leaves has the spectrum $\lambda_{1,2}(T)$, that is, it is a node or a saddle od a focus. See, for example, Fig. 2.1.

From the analytical viewpoint, Theorems 2.2, 2.3 are generalization of the Poincare – Dulac normal form for vector fields that have zero eigenvalues.

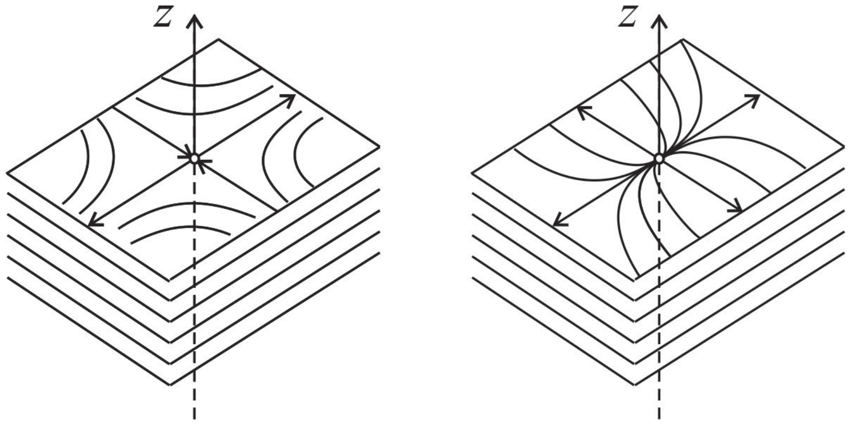

Fig. 2.1. Illustration of Theorems 2.1 – 2.3: local phase portraits of the vector field (2.5). Here $n = 1$ and W^c coincides with the z-axis.

3. ROUSSARIE VECTOR FIELDS

Now we consider vector fields of the form (2.5) satisfying the additional condition:

$$\mathrm{spec}\,(T) \equiv (\lambda_1(T), \lambda_2(T), 0, \ldots, 0), \quad \forall T \in W^c,$$

where $\lambda_{1,2}(T)$ are non-zero real numbers such that

$$q\lambda_1(T) + p\lambda_2(T) = 0, \quad p, q \in \mathbb{Z}_+, \quad \gcd\,(p, q) = 1, \quad \forall T \in W^c. \tag{3.15}$$

Such vector fields are named after Robert Roussarie, who studied the partial case

$$p = q = 1 \tag{3.16}$$

in his [17]. This study was motivated by the degeneracy of closed differential 2-forms. A generic closed 2-form ω on the 4-dimensional real space degenerates on a smooth 3-dimensional manifold Σ. At a generic point of Σ, the 2-dimensional kernel of the form ω is transversal to Σ, and the germ of ω can be reduced to

$$p_1 dp_1 \wedge dq_1 + dp_2 \wedge dq_2.$$

However, the kernel of ω is tangent to Σ at some points, which generically fill a curve $S \subset \Sigma$, the normal form of ω at points of S is more complicated. The kernel of ω cuts out a direction field on Σ, which can be given (uniquely up to a scalar factor) by a vector field of the form (2.5), whose singular points fill the curve S. The condition that ω is closed implies that the trace of the linear part of the vector field at every its singular point is zero, that is, the non-zero eigenvalues $\lambda_{1,2}$ satisfy the resonance (3.15) with $p = q = 1$.

Actually, in many problems we are interested not in vector fields themselves, but in the corresponding direction fields. Therefore, when bringing a vector field to its normal forms, one can multiply it by a non-vanishing scalar function in addition to changes of the phase variables. Normal forms thus obtained are called *orbital*. Orbital normal forms allow us to get rid of a redundant module, and therefore, to simplify the classification.

Theorem 3.1:
The germ of every Roussarie vector field is C^∞-smoothly orbitally equivalent to

$$\begin{gathered} \dot{x} = px(1+\Phi_1(r,z)), \;\; \dot{y} = qy(-1+\Phi_2(r,z)), \\ \dot{z}_i = r\Psi_i(r,z), \;\; i=1,\ldots,n, \end{gathered} \tag{3.17}$$

where $r = x^q y^p$ is the resonant monomial of the resonance (3.15)*, $\Phi_{1,2}, \Psi_i \in C^\infty(M)$ and* $\Phi_1(0,0) = \Phi_2(0,0) = 0$.

Theorem 3.1 establishes a generalization of the Poincare – Dulac normal form for Roussarie vector fields: the functions $\Phi_{1,2}$ and Ψ_i contain all resonant terms. It is well-known that the Poincare – Dulac normal form allows further simplification. To obtain such simplification for (3.17), we shall use the notion of the *quotient* vector field.

The field (3.17) generates the field in the (r,z)-space:

$$\begin{aligned} \dot{r} &= \dot{(x^q y^p)} = qx^{q-1}y^p\dot{x} + px^q y^{p-1}\dot{y} = \\ &\quad qx^{q-1}y^p px(1+\Phi_1) + px^q y^{p-1} qy(-1+\Phi_2) = pqr(\Phi_1+\Phi_2), \\ \dot{z}_i &= r\Psi_i(r,z), \;\; i=1,\ldots,n. \end{aligned}$$

Reducing the common factor r, we get the *quotient* vector field for (3.17):

$$\dot{r} = pq\Phi(r,z), \;\; \dot{z}_i = \Psi_i(r,z), \;\; i=1,\ldots,n, \tag{3.18}$$

where

$$\Phi(r,z) = \Phi_1(r,z) + \Phi_2(r,z).$$

Remark 3.1:
From the resonant relation (3.15) it follows that $\Phi(0,z) \equiv 0$ for all z, i.e., the restriction of $\Phi(r,z)$ to the center manifold is identically zero.

Generically, for almost all points $T_0 \in W^c$ the field (3.17) satisfies the following condition:

$$\exists i \in \{1,\ldots,n\}: \;\; \Psi_i(0,0) \neq 0. \tag{3.19}$$

Equivalently, 0 is not a singular point of its quotient field (3.18). This allows us to establish the existence of n independent first integrals of the field (3.18), which are obviously first integrals of the field (3.17). For details, see [5, 13].

Using these first integrals, one can prove the following theorem:

Theorem 3.2:
If the condition (3.19) *for* (3.17) *holds true, the germ of the Roussarie vector field is C^∞-smoothly orbitally equivalent to*

$$\dot{x} = px, \;\; \dot{y} = -qy, \;\; \dot{z}_1 = x^q y^p, \;\; \dot{z}_i = 0, \;\; i=2,\ldots,n. \tag{3.20}$$

Moreover, it is C^{k-1}-smoothly orbitally equivalent to

$$\dot{x} = px, \;\; \dot{y} = -qy, \;\; \dot{z}_i = 0, \quad i=1,\ldots,n, \tag{3.21}$$

where $k = \max\{p,q\}$, but in general it is not C^k-smoothly equivalent.

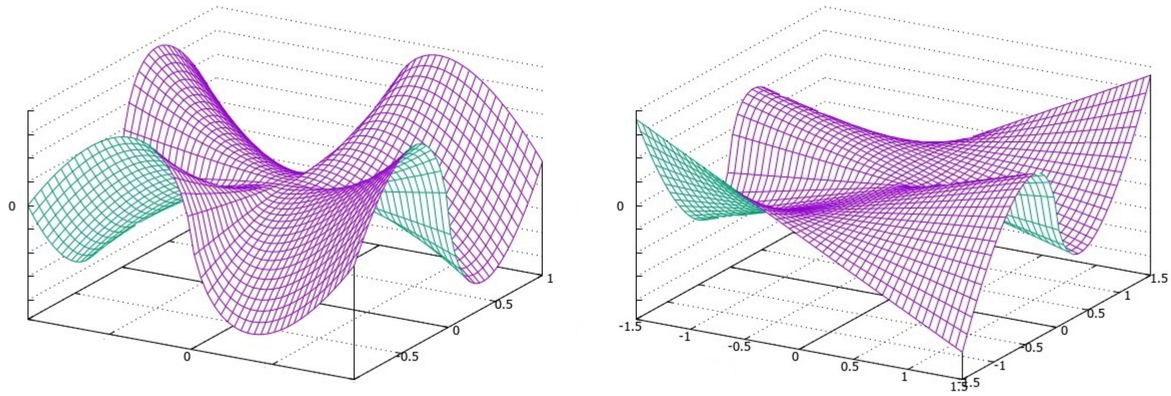

Fig. 3.2. Two examples of C^0 saddle surfaces of the vector field $\dot{x} = x$, $\dot{y} = -y$, $\dot{z} = xy$: $z = -\frac{1}{2}xy \ln|y/x|$ (left) and $z = -xy \ln|y|$ (right).

Proof
The first statement of the theorem is proved in the case $p = q = 1$ by Roussarie in 1975. Other statements are proved in [10]. □

Example 3.1:
Consider the vector field

$$\dot{x} = x, \ \ \dot{y} = -y, \ \ \dot{z} = xy, \ \ (x, y, z) \in \mathbb{R}^3.$$

Any saddle surface of this field has the form $z = -\frac{1}{2}F(x, y)$, where

$$F(x, y) = f(xy) + xy \ln\left|\frac{y}{x}\right|, \ \text{ if } \ xy \neq 0,$$

and $F(x, y) = 0$, if $xy = 0$. Here f is an arbitrary continuous function.

The obtained formula shows that F is continuous (for continuous f), but it is never C^1. See examples in Fig. 3.2.

4. APPLICATIONS: QUASI-LINEAR ODES OF THE SECOND ORDER

Consider the differential equation

$$\Delta(x, y)\frac{dp}{dx} = M(x, y, p), \ \ p = dy/dx, \tag{4.22}$$

where $\Delta(x, y)$, $M(x, y, p)$ are smooth functions, M is analytic in p. Generically, the set of singular points of equation (4.22) is a regular curve

$$\Gamma = \{(x, y) : \Delta(x, y) = 0\}.$$

If the point $q_0 = (x_0, y_0) \notin \Gamma$, then for every direction p_0 equation (4.22) has a unique solution satisfying the initial condition

$$y(x_0) = y_0, \ \ p(x_0) = p_0.$$

The situation is more complicated if $q_0 = (x_0, y_0) \in \Gamma$. For example, there may be solutions whose oscillations accumulate near a singular point.

Example 4.1:
The equation $x^4 dp/dx = 2x^3 p - (2x^2+1)y$ has a family of solutions

$$y(x) = x^2(\alpha \cos x^{-1} + \beta \sin x^{-1}), \quad \alpha, \beta = \text{const},$$

Except for $\alpha = \beta = 0$, all solutions are oscillating at $x = 0$:

$$\exists \lim_{x \to 0} y(x) = 0, \quad \nexists \lim_{x \to x_0} y'(x) \ \ (\textit{but}\ y'(0) = 0).$$

Example 4.2:
The equation $x^2 dp/dx = xp - 2y$ has a family of solutions

$$y = x(\alpha \cos \ln |x| + \beta \sin \ln |x|), \quad \alpha, \beta = \text{const},$$

Except for $\alpha = \beta = 0$, all solutions are oscillating at $x = 0$:

$$\exists \lim_{x \to 0} y(x) = 0, \quad \nexists \lim_{x \to x_0} y'(x) \ \ (\textit{and}\ \nexists\, y'(0)).$$

The above examples motivate the following formal definition:

Definition 4.1:
Oscillating solutions entering q_0 are solutions such that

$$\exists \lim_{x \to x_0} y(x) = y_0, \quad \nexists \lim_{x \to x_0} p(x).$$

Here $x \to x_0$ may be either two-sided or one-sided limit.

The above examples show that oscillating solutions exist. However, the following theorem states that oscillating solutions do not exist generically.

Theorem 4.1:
Let $q_0 \in \Gamma$ and $M(q_0, p)$ is an analytic function not identically zero. Then equation (4.22) *has no oscillating solutions entering the point q_0. Moreover, solutions can enter q_0 at so-called admissible directions p that correspond to real roots of $M(q_0, p)$ only.*

The proof of Theorem 4.1 is based on the analysis of the distribution defined by equation (4.22) in the (x, y, p)-space:

$$\omega_1 = \Delta dp - M dx = 0, \quad \omega_2 = dy - p dx = 0,$$

which generates the vector field

$$\dot{x} = \Delta(x, y), \quad \dot{y} = p\Delta(x, y), \quad \dot{p} = M(x, y, p). \tag{4.23}$$

All components of the field (4.23) belong to the ideal generated by Δ and M. Given $q_0 \in \Gamma$, the *admissible directions* p correspond to singular points (q_0, p) of the vector field (4.23) given by two equations:

$$\Delta(q_0) = 0, \quad M(q_0, p) = 0.$$

For more details about oscillating solutions see [11].

Further, we shall analyse vector filed (4.23) in order to study non-oscillating solutions of equation (4.22) entering its singular points. First, remark that vector filed (4.23) has the form (2.5) with $n = 1$, where the variables x, p in (4.23) correspond to x, y in (2.5), the variable y in (4.23) corresponds to z_1 in (2.5).

Let $q_0 \in \Gamma$ and $M(q_0, p_*) = 0$, i.e., p_* is an admissible direction at q_0. The spectrum of the linear part of the field (4.23) at singular point (q_0, p_*) is

$$\text{spec}\,(q_0, p_*) = (\lambda_1, \lambda_2, 0),$$

where the eigenvalues

$$\lambda_1 = (\Delta_x + p\Delta_y)(q_0, p_*), \ \ \lambda_2 = M_p(q_0, p_*).$$

Assume that $\lambda_{1,2} \neq 0$. Then we define the value $\lambda = \lambda_2 : \lambda_1$, which determines the number and the behavior of solutions of equation (4.22) that enter q_0 with the direction p_*. Namely, the following result is obtained in [15].

Theorem 4.2:
If $\lambda < 0$, then equation (4.22) *has only one solutions passing though the point q_0 with the tangential direction p_*.*

If $\lambda > 0$, then equation (4.22) *has an infinite number of solutions entering the point q_0 with the direction p_*. In appropriate local coordinates, these solutions have one of two following forms:*

$$y = F(x, c|x|^{\lambda}), \ \textit{if} \ \lambda \notin \mathbb{N},$$
$$y = F(x, x^{\lambda}(c + \varepsilon \ln |x|)), \ \ \varepsilon \in \{0, 1\}, \ \textit{if} \ \lambda \in \mathbb{N},$$

where F is a smooth function, $c = \text{const}$.

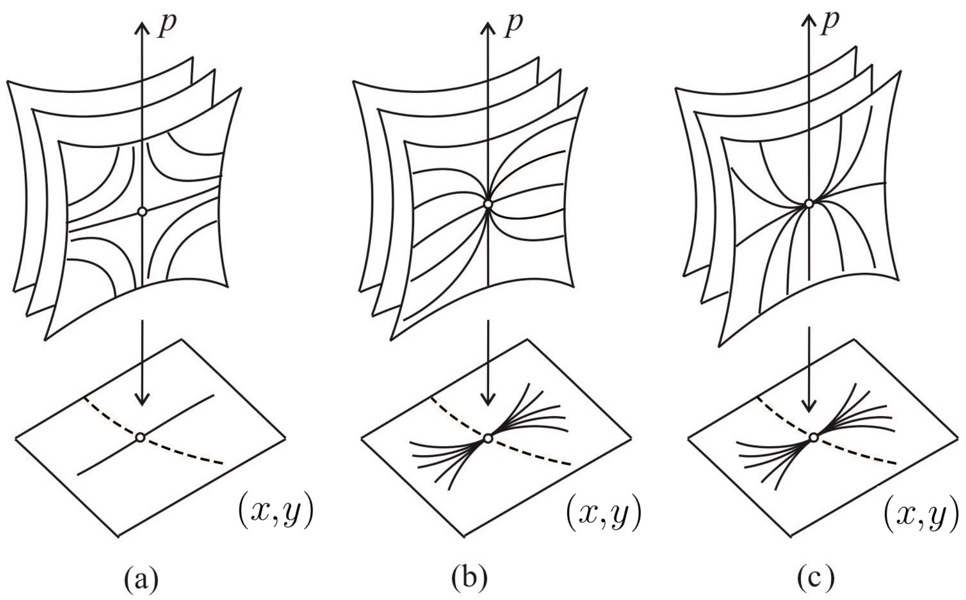

Fig. 4.3. Illustration for Theorem 4.2: integral curves of the field (4.23) and their projections to the (x, y)-plane – solutions of equation (4.22). From left to right: $\lambda < 0$ (a), $0 < \lambda < 1$ (b), $\lambda > 1$ (c). The curve Γ is depicted as a dashed curve on the (x, y)-plane.

4.1. Quasi-Linear ODEs of the Second Order Cubic in p

Consider an important class of equations (4.22), where $M(x, y, p)$ is a cubic polynomial in p:

$$M = \mu_0 + \mu_1 p + \mu_2 p^2 + \mu_3 p^3, \ \ \mu_i = \mu_i(x, y).$$

An attention to such equations is motivated by their role in physics and geometry, for instance, the description of various geometric structures (geodesic flows in affine or projective connection, etc.). Equations of this class were studies by Sophus Lie, A. Tresse, J. Liouville, E. Cartan, etc. See also the recent papers [7, 19, 20].

For a generic cubic polynomial $M(q_0, p)$ and almost all points $q_0 \in \Gamma$ there are 4 possible cases (the abbreviation AD below means admissible direction):

- 1 AD p_0 with $\lambda(q_0, p_0) > 0$;
- 3 ADs p_0, p_1, p_2 with $\lambda(q_0, p_0) > 0$ and $\lambda(q_0, p_i) < 0$, $i = 1, 2$;
- 1 AD p_0 with $\lambda(q_0, p_0) < 0$;
- 3 ADs p_0, p_1, p_2 with $\lambda(q_0, p_0) < 0$ and $\lambda(q_0, p_i) > 0$, $i = 1, 2$.

Example 4.3:
Consider the equation

$$x\frac{dp}{dx} = \alpha p(p^2 - 1), \quad \alpha \neq 0, \tag{4.24}$$

whose singular points fill the curve $\Gamma = \{x = 0\}$. For every point $q_0 \in \Gamma$ the cubic polynomial

$$M(p) = \alpha p(p^2 - 1)$$

has three different real roots: $p_0 = 0$ and $p_{1,2} = \pm 1$, where $\lambda(q_0, 0) = -\alpha$ and $\lambda(q_0, \pm 1) = 2\alpha$. Solution of equation (4.24) entering the origin are presented in Fig. 4.4.

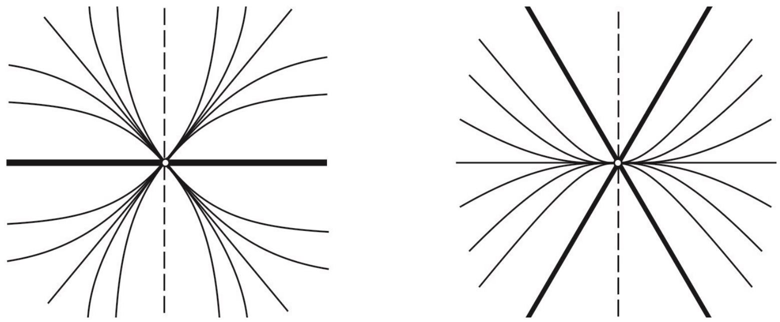

Fig. 4.4. On the left: the case $\alpha > 0$, on the right: the case $\alpha < 0$. The bold lines are solutions that are unique with given AD.

Example 4.4:
Consider the equation

$$x\frac{dp}{dx} = \alpha p(p^2 + 1), \quad \alpha \neq 0, \tag{4.25}$$

whose singular points fill the curve $\Gamma = \{x = 0\}$. For every point $q_0 \in \Gamma$ the cubic polynomial

$$M(p) = \alpha p(p^2 + 1)$$

has one real root $p_0 = 0$ and $\lambda(q_0, 0) = \alpha$.

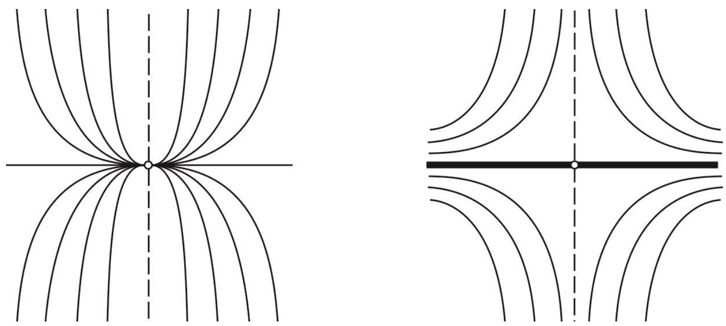

Fig. 4.5. On the left: the case $\alpha > 0$, on the right: the case $\alpha < 0$. The bold lines are solutions that are unique with given AD.

4.2. Equation of Geodesics in Signature Varying Metrics

Consider a pseudo-Riemannian metric

$$ds^2 = a(x,y)\,dx^2 + 2b(x,y)\,dxdy + c(x,y)\,dy^2,$$

where $a, b, c \in C^\infty(M)$ and the *discriminant function*

$$\Delta = ac - b^2$$

vanishes and changes its sign on the regular curve $\Gamma \subset M$. The latter condition implies that the coefficients a, b, c do not vanish simultaneously, therefore, the square polynomial

$$a(q_0) + 2b(q_0)p + c(q_0)p^2$$

at every point $q_0 \in Gamma$ has the double root

$$p_0 = -\frac{a}{b} = -\frac{b}{c},$$

which defines the *isotropic* direction of the metric at the point q_0.

The equation of (unparametrized) geodesics in this metric has the form

$$\Delta(x,y)\frac{dp}{dx} = \frac{1}{2}\,M(x,y,p), \quad p = dy/dx,$$

where M is a cubic polynomial

$$M = \mu_0 + \mu_1 p + \mu_2 p^2 + \mu_3 p^3$$

with the coefficients

$$\begin{aligned}
\mu_3 &= c(2b_y - c_x) - bc_y,\\
\mu_2 &= b(2b_y - 3c_x) + 2a_y c - ac_y,\\
\mu_1 &= b(3a_y - 2b_x) + a_x c - 2ac_x,\\
\mu_0 &= a(a_y - 2b_x) + a_x b.
\end{aligned}$$

This is a special case of quasi-linear equation of the second order.

In this case, the isotropic direction p_0 is always admissible, that is, $M(q_0, p_0) = 0$. At almost all points of the curve Γ, the cubic polynomial M has either one or three real roots, and there are only two possible combinations:

- one AD p_0 with $\lambda(q_0, p_0) = \frac{1}{2}$,
- three ADs p_0, p_1, p_2 with $\lambda(q_0, p_0) = \frac{1}{2}$ and $\lambda(q_0, p_i) = -1$, $i = 1, 2$.

A deep explanation of this surprising fact can be found in [5].

Finally, we remark one more interesting property established recently. Let $q_0 \in \Gamma$ and the polynomial M at q_0 has three different real roots p_0, p_1, p_2, that is, three different admissible directions at q_0. Let p_Γ be the tangent direction to the curve Γ at q_0. Then the relative position of the directions p_0, p_1, p_2, p_Γ is determined by their cross-ratio:

$$\mathrm{DV}(p_0, p_1, p_\Gamma, p_2) = 2.$$

The proof can be found in [12].

In Fig. 4.6, we present three families of geodesics issuing from a point $q_0 \in \Gamma$. Here the curve Γ coincides with the horizontal axis (dashed line) and the isotropic admissible direction p_0 at all points $q_0 \in \Gamma$ is vertical. The families on the left and in the center correspond to points q_0 with unique admissible direction p_0. The family on the right corresponds to a point q_0 with three admissible direction p_0, p_1, p_2; geodesics with non-isotropic tangential directions p_1, p_2 are depicted as bold lines.

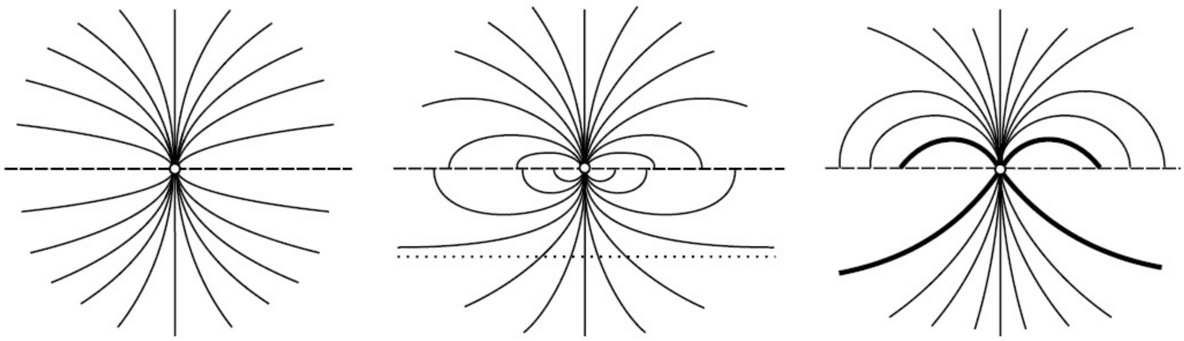

Fig. 4.6. Three families of geodesics issuing from a point $q_0 \in \Gamma$.

5. CONCLUSION

We presented a survey of recent results about vector fields with non-isolated singular points and some their applications. One of the most interesting and promising application is connected with quasi-linear differential equations of the second order, including the equation of geodesics in signature varying (pseudo-Riemannian) metrics. This subject motivates research in various directions, see, for example, the papers [9,14,16]. Applications of different types can be also found in [3,4], [7,19], and [8].

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