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Multiple Capture of Coordinated Evaders in the Linear Group Pursuit Problem with a Simple Matrix and Phase Restrictions

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Abstract: In a finite-dimensional Euclidean space, the pursuit problem of two evaders by a group of pursuers with equal opportunities for all players is considered. The motion of each participant is described by a linear system of differential equations with a simple matrix. It is assumed that the evaders use the same control and do not move out of a convex polyhedral set. The pursuers use counterstrategies based on information about the initial positions and the prehistory of the evaders' control. The set of admissible controls V is a sphere of unit radius with its center at the origin, and the goal sets are the origin. The goal of the group of pursuers is the capture of at least one evader by a given number of pursuers. In terms of the initial positions and parameters of the game, a sufficient condition for capture is obtained. The method of resolving functions, which is used as a basis for analysis, provides sufficient conditions for solvability of the problem of pursuit in some guaranteed time.

Keywords: differential games, pursuer, evader, capture, multiple capture, conflict-controlled processes

1. INTRODUCTION

Mathematical control theory and dynamical game theory provide a fundamental framework for investigating controlled processes of different nature. The schools of N.N. Krasovskii and L.S. Pontryagin played a key role in establishing these theories and developing a number of classical methods in this area.

The methods for investigating dynamical games include those aimed at constructing optimal strategies [1] and methods that ensure a guaranteed result [2]. The main condition in this case is to achieve the goal in hand and to accomplish the task under specific conditions. To obtain the guaranteed result, use is often made of the first method of L.S. Pontryagin [2] and the method of resolving functions [3–5]. These methods give sufficient conditions to complete the game in finite time from given initial positions. Two-player differential game theory was developed further by studying the problem of pursuit of one evader by a group of pursuers and the problem of evasion of one evader from a group of pursuers [6–10]. B.N. Pshenichny obtained [11] necessary and sufficient conditions for capture of the evader. N.L. Grigorenko introduced the notion of multiple capture. For the problem with simple motions and equal opportunities he presented necessary and sufficient conditions [4] for multiple capture of the evader. In [12], sufficient conditions were obtained for multiple capture of the evader in L.S. Pontryagin's example with equal opportunities for all participants. The problem of multiple capture of an evader in the presence of defenders was discussed in [13].

A natural generalization of these group pursuit problems is the situation of conflict interaction of a group of pursuers and a group of evaders. The goal of the group of pursuers

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is to capture a given number of evaders, and the goal of the group of evaders is the opposite one [14–16].

The authors of [17] addressed the problem of pursuit of a group of evaders by a group of pursuers under the condition that all evaders use the same control. Sufficient conditions for capture of at least one evader were obtained. In what follows, we will call the pursuit problem in which all evaders use the same control the *problem of pursuing coordinated evaders*. The study presented in [17] was developed, in particular, in [18–20], where sufficient, and in some cases necessary, conditions for capture of at least one evader were obtained under the condition that all participants have equal opportunities and all evaders use the same control.

The problem of pursuing two coordinated evaders in which the goal of the group of pursuers was the capture of at least one evader by two pursuers was treated in [21, 22].

This paper addresses the linear problem of pursuit of two coordinated evaders by a group of pursuers in a differential game with a simple matrix and phase restrictions on the states of the evaders. The goal of the group of pursuers is the capture of at least one evader by a given number of pursuers.

2. FORMULATION OF THE PROBLEM

In the space $\mathbb{R}^k$ $(k \geqslant 2)$ we consider a differential game $\Gamma(n+2)$ involving $n+2$ players: n pursuers $P_1, \ldots, P_n$ and two evaders E_1, E_2.

The law of motion of each of pursuers P_i has the form

$$\dot{x}_i = \alpha x_i + u_i, \quad x_i(0) = x_i^0, \quad u_i \in V. \tag{2.1}$$

The law of motion of each of evaders E_j has the form

$$\dot{y}_j = \alpha y_j + v, \quad y_j(0) = y_j^0, \quad v \in V. \tag{2.2}$$

Here $i \in I = \{1, \ldots, n\},\ j \in \{1, 2\},\ x_i, y_j, u_i, v \in \mathbb{R}^k,\ V = \left\{v \mid v \in \mathbb{R}^k, \|v\| \leqslant 1\right\}$, $\alpha \in \mathbb{R}^1$, $\alpha \leqslant 0$. In addition, $x_i^0 \neq y_j^0$ for all $i \in I$, $j = 1, 2$.

Additionally, it is assumed that each evader $E_j, j \in \{1, 2\}$ does not move out of a convex polyhedral set

$$\Omega = \{y \in \mathbb{R}^k \mid (p_l, y) \leqslant \beta_l,\ l = 1, \ldots, r\},$$

where $p_1, \ldots, p_r$ are the unit vectors $\mathbb{R}^k$, $\beta_1, \ldots, \beta_r$ are the real numbers, and (u, v) is the scalar product. Assume that $\Omega = \mathbb{R}^k$ for $r = 0$.

Introduce new variables $z_{ij} = x_i - y_j$. Then instead of the systems (2.1) and (2.2) we obtain the system

$$\dot{z}_{ij} = \alpha z_{ij} + u_i - v, \quad z_{ij}(0) = z_{ij}^0 = x_i^0 - y_j^0. \tag{2.3}$$

The measurable function $v\colon [0, \infty) \to \mathbb{R}^k$ is called *admissible* if $v(t) \in V$, $y_1(t) \in \Omega$, $y_2(t) \in \Omega$ for all $t \geqslant 0$. Let us call the restriction of the function v to the interval $[0, t]$ the prehistory $v_t(\cdot)$ of the function $v(\cdot)$ at time t.

The actions of the evaders can be interpreted as follows: there is a center that chooses the same control $v(t)$ for evaders E_1 and E_2.

Definition 2.1:
We will say that a quasi-strategy $\mathcal{U}_i$ of pursuer P_i is given if a map $\mathcal{U}_i(t, z^0, v_t(\cdot))$ is defined which associates a measurable function $u_i(t) = \mathcal{U}_i(t, z^0, v_t(\cdot))$ with values in V to the initial position $z^0 = (z_{ij}^0)$, time t and an arbitrary prehistory $v_t(\cdot)$ of admissible control $v(\cdot)$ of evaders E_j.

Definition 2.2:
A q-fold capture (if $q = 1$, then it is a capture) occurs in the game $\Gamma(n+2)$ if there exist time $T_0 = T(z^0)$, quasi-strategies $\mathcal{U}_1, \ldots, \mathcal{U}_n$ of pursuers $P_1, \ldots, P_n$ such that for any admissible measurable function $v(\cdot)$ there are numbers $i_1, \ldots, i_q \in I$, $l \in \{1, 2\}$, and time instants $\tau_{i_1}, \ldots, \tau_{i_q} \in [0, T_0]$ such that $z_{i_s l}(\tau_{i_s}) = 0$ for all $s = 1, \ldots, q$.

Remark 2.1:
The capture condition implies that q pursuers perform a capture of one evader and that the time instants may or may not coincide. Such a situation can arise if the system consists of q blocks, and in order to put it out of operation, one needs to damage all blocks.

3. AUXILIARY RESULTS

Definition 3.1 (see [23])**:**
The vectors $a_1, a_2, \ldots, a_s$ form a positve basis in $\mathbb{R}^k$ if for any $x \in \mathbb{R}^k$ there exist nonnegative real numbers $\alpha_1, \alpha_2, \ldots, \alpha_s$ such that

$$x = \alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_s a_s.$$

We introduce the following notation. $\operatorname{Int} X$ and $\operatorname{co} X$ are the interior and the convex hull of the set $X \subset \mathbb{R}^k$, respectively, and $|J|$ is the number of elements of the finite set J,

$$\lambda(h, v) = \sup\{\lambda \geqslant 0 \mid -\lambda h \in V - v\},$$
$$\Omega_l(K) = \{(i_1, \ldots, i_l) \mid i_1, \ldots, i_l \in K \text{ and are pairwise different}\},$$

where K is a finite set of natural numbers and l is a natural number.

Theorem 3.1 (see [23])**:**
Let $a_1, a_2, \ldots, a_m \in \mathbb{R}^k$. The following statements are equivalent.
1. The vectors $a_1, a_2, \ldots, a_m$ form a positive basis in $\mathbb{R}^k$.
2. $0 \in \operatorname{Intco}\{a_1, \ldots, a_m\}$.
3. For any $p \in \mathbb{R}^k$, $p \neq 0$, there exists a number $l \in \{1, \ldots, m\}$ for which $(a_l, p) > 0$.

Lemma 3.1:
Let $m \geqslant q$, $a_1, \ldots, a_m, c_1, p_1 \in \mathbb{R}^k$ such that, for each $J_0 \subset J = \{1, \ldots, m\}$, $|J_0| = q - 1$, one has

$$0 \in \operatorname{Intco}\{a_s, s \in J \setminus J_0, c_1, p_1\}.$$

Then for any $b_1, \ldots, b_q \in \mathbb{R}^k$ there is $\widehat{\mu} > 0$ such that for all $\mu > \hat{\mu}$ the following inequality holds:

$$\delta(\mu) = \min_{v \in V} \max\Big\{ \max_{\Lambda \in \Omega_q^0(J_1)} \min_{i \in \Lambda} \lambda(\omega_i, v), (p_1, v) \Big\} > 0,$$

where

$$J_1 = J \cup \{m+1, \ldots, m+q\}, \quad \Omega_q^0(J_1) = \Omega_q(J) \cup \{(m+1, \ldots, m+q)\},$$
$$\omega_i = \begin{cases} a_i & \text{if } i \in J, \\ b_{i-m} + \mu c_1 & \text{if } i = m+1, \ldots, m+q. \end{cases}$$

Proof
Assume that the statement of the lemma is not valid. Then there exist $b_1^0, \ldots, b_q^0 \in \mathbb{R}^k$ for which for any $\widehat{\mu} > 0$ there is $\mu > \hat{\mu}$ such that $\delta(\mu) = 0$. It follows from the condition $\delta(\mu) = 0$

that there exists $v_\mu \in V$ such that $(p_1, v_\mu) \leqslant 0$ and

$$\max_{\Lambda \in \Omega_q^0(J_1)} \min_{i \in \Lambda} \lambda(\omega_i, v_\mu) = 0. \tag{3.4}$$

We show that it follows from condition (3.4) that there exist a set $J(\mu) \subset J$, $|J(\mu)| = q-1$, and a number $s(\mu) \in \{m+1, \ldots, m+q\}$ for which the following inequalities hold:

$$(w_i, v_\mu) \leqslant 0 \text{ for all } i \in \big(J \setminus J(\mu)\big) \cup \{s(\mu)\}. \tag{3.5}$$

It follows from condition (3.4) that for any $\Lambda \in \Omega_q^0(J_1)$ there is a number $i_\Lambda \in \Lambda$ such that $\lambda(\omega_{i_\Lambda}, v_\mu) = 0$. By virtue of the properties of the function λ [4] we find that $\|v_\mu\| = 1$. Therefore, the condition $\lambda(\omega_{i_\Lambda}, v_\mu) = 0$ is equivalent to the condition $(\omega_{i_\Lambda}, v_\mu) \leqslant 0$.

Let $\Lambda_1 = \{1, \ldots, q\}$. Then there is a number $i_1 \in \Lambda_1$ such that $(w_{i_1}, v_\mu) \leqslant 0$. Take $\Lambda_2 = \Lambda_1 \setminus \{i_1\} \cup \{q+1\}$. Then there is a number $i_2 \in \Lambda_2$ such that $(\omega_{i_2}, v_\mu) \leqslant 0$. Continuing this process further, we find that for the set $\Lambda_{m-q+1} = \Lambda_{m-q} \setminus \{i_{m-q}\} \cup \{m\}$ there is a number $i_{m-q+1} \in \Lambda_{m-q+1}$ for which $(\omega_{i_{m-q+1}}, v_\mu) \leqslant 0$. In addition, for the set $\Lambda = \{(m+1, \ldots, m+q)\}$ there is a number $s(\mu)$ such that $(\omega_{s(\mu)}, v_\mu) \leqslant 0$. Consider a set $J(\mu) = J \setminus \{i_1, \ldots, i_{m-q+1}\}$. Then $|J(\mu)| = q-1$ and $(\omega_s, v_\mu) \leqslant 0$ for all $s \in J \setminus J(\mu)$. This proves (3.5).

We now show that there exists a set $J_0 \subset J$, $|J_0| = q-1$, such that

$$0 \notin \operatorname{Intco}\{a_s, s \in J \setminus J_0, c_1, p_1\}. \tag{3.6}$$

Let $\widehat{\mu} = 1$. Then there is $\mu_1 > \hat{\mu}$ for which there are a vector $v_1 \in V$, $\|v_1\| = 1$, a set $J(\mu_1) \subset J$, $|J(\mu_1)| = q-1$, and an index $s(\mu_1) \in \{m+1, \ldots, m+q\}$ such that

$$(a_i, v_1) \leqslant 0 \text{ for all } i \in J \setminus J(\mu_1),$$
$$(b^0_{s(\mu_1)-m} + \mu_1 c_1, v_1) \leqslant 0,\ (p_1, v_1) \leqslant 0.$$

Take $\widehat{\mu} = \mu_1 + 1$. Then there is $\mu_2 > \widehat{\mu}$ for which there are a vector $v_2 \in V$, $\|v_2\| = 1$, a set $J(\mu_2) \subset J$, $|J(\mu_2)| = q-1$, and an index $s(\mu_2) \in \{m+1, \ldots, m+q\}$ such that

$$(a_i, v_2) \leqslant 0 \text{ for all } i \in J \setminus J(\mu_2),$$
$$(b^0_{s(\mu_2)-m} + \mu_2 c_1, v_1) \leqslant 0,\ (p_1, v_1) \leqslant 0.$$

Continuing this process further, we find that there exist a sequence of real numbers $\{\mu_l\}_{l=1}^\infty$, $\lim\limits_{l \to +\infty} \mu_l = +\infty$, a sequence of vectors $\{v_l\}_{l=1}^\infty$, $\|v_l\| = 1$, a sequence of sets $\{J(\mu_l)\}_{l=1}^\infty$, $J(\mu_l) \subset J$, $|J(\mu_l)| = q-1$, and a sequence of natural numbers $\{s(\mu_l)\}_{l=1}^\infty$, $s(\mu_l) \in \{m+1, \ldots, m+q\}$ for which the following inequalities hold:

$$(a_i, v_l) \leqslant 0 \text{ for all } i \in J \setminus J(\mu_l), \quad (b^0_{s(\mu_l)-m} + \mu_l c_1, v_1) \leqslant 0, \quad (p_1, v_1) \leqslant 0. \tag{3.7}$$

It follows from (3.7) that there exist a subsequence $\{v_{l_p}\}_{p=1}^\infty$, $\|v_{l_p}\| = 1$, a set $J_0 \subset J$, $|J_0| = q-1$, and a natural number $s \in \{m+1, \ldots, m+q\}$ for which for all $p = 1, 2, \ldots$ the following inequalities hold:

$$(a_i, v_{l_p}) \leqslant 0 \text{ for all } i \in J \setminus J_0, \quad (b^0_{s-m} + \mu_{l_p} c_1, v_{l_p}) \leqslant 0, \quad (p_1, v_{l_p}) \leqslant 0. \tag{3.8}$$

From the sequence $\{v_{l_p}\}$ one can choose a subsequence converging to v_0, $\|v_0\| = 1$. Assume that the sequence $\{v_{l_p}\}$ itself converges. Passing in (3.8) to the limit as $p \to +\infty$, we find that the following inequalities hold:

$$(a_i, v_0) \leqslant 0 \text{ for all } i \in J \setminus J_0, \quad (c_1, v_0) \leqslant 0, \quad (p_1, v_0) \leqslant 0.$$

By Theorem 3.1, this implies that

$$0 \notin \text{Intco}\{a_i, i \in J \setminus J_0, c_1, p_1\}.$$

This proves (3.6). We have obtained a contradiction with the condition of the lemma. The lemma is proved. □

Lemma 3.2:
Let $r = 1$, $\alpha = 0$, $m \geqslant q$, $a_1, \dots, a_m, c_1, p_1 \in \mathbb{R}^k$ such that for any $J_0 \subset J = \{1, \dots, m\}$, $|J_0| = q - 1$, one has

$$0 \in \text{Intco}\{a_l, l \in J \setminus J_0, c_1, p_1\}.$$

Then for any $b_1, \dots, b_q \in \mathbb{R}^k$ there is $\widehat{\mu} > 0$ such that for each $\mu > \widehat{\mu}$ there is a time instant $T(\mu) > 0$ such that for any admissible control $v(\cdot)$ of evaders E_1 and E_2 there is a set $\Lambda^* \in \Omega_q^0(J_1)$ such that for all $l \in \Lambda^*$ the following inequalities hold:

$$\int_0^{T(\mu)} \lambda(\omega_l, v(s))\, ds \geqslant 1,$$

where

$$J_1 = J \cup \{m+1, \dots, m+q\}, \quad \Omega_q^0(J_1) = \Omega_q(J) \cup \{(m+1, \dots, m+q)\},$$

$$\omega_i = \begin{cases} a_i & \text{if } i \in J, \\ b_{i-m} + \mu c_1 & \text{if } i = m+1, \dots, m+q. \end{cases}$$

Proof
Take arbitrary vectors $b_1, \dots, b_q \in \mathbb{R}^k$. By Lemma 3.1, there exists $\widehat{\mu} > 0$ such that for all $\mu > \widehat{\mu}$ the inequality $\delta(\mu) > 0$ is satisfied, where

$$\delta(\mu) = \min_{v \in V} \max\big\{ \max_{\Lambda \in \Omega_q^0(J_1)} \min_{i \in \Lambda} \lambda(\omega_i, v), (p_1, v) \big\}.$$

Prove that the statement of the lemma is valid for all $\mu > \widehat{\mu}$. Let $v(\cdot)$ be an arbitrary admissible control of the evaders. For each $t > 0$, $\mu > \widehat{\mu}$ we define the sets

$$T_1(t, \mu) = \{\tau \mid \tau \in [0, t],\ (p_1, v(\tau)) \geqslant \delta(\mu)\}, \quad T_2(t, \mu) = \{\tau \mid \tau \in [0, t],\ (p_1, v(\tau)) < \delta(\mu)\}.$$

Since $v(\cdot)$ is the admissible control of the evaders, the following inequalities hold for all $t \geqslant 0$:

$$(p_1, y_1(t)) \leqslant \beta_1, \quad (p_1, y_2(t)) \leqslant \beta_1.$$

Therefore, for all $t \geqslant 0$ the following inequality holds:

$$\int_0^t (p_1, v(s))\, ds \leqslant \beta_0, \text{ where } \beta_0 = \min\{\beta_1 - (p_1, y_1^0), \beta_1 - (p_1, y_2^0)\}.$$

Then one has

$$\beta_0 \geqslant \int_0^t (p_1, v(s))\, ds \geqslant \delta(\mu) \int_{T_1(t,\mu)} ds - \int_{T_2(t,\mu)} ds,$$

$$t = \int_{T_1(t,\mu)} ds + \int_{T_2(t,\mu)} ds.$$

This implies that the following inequality holds:

$$\int_{T_2(t,\mu)} ds \geqslant \frac{\delta(\mu)t - \beta_0}{1+\delta(\mu)}.$$

Next, we have

$$\max_{\Lambda\in\Omega_1^0(J_1)} \min_{i\in\Lambda} \int_0^t \lambda(\omega_i, v(s))\, ds \geqslant \max_{\Lambda\in\Omega_1^0(J_1)} \int_0^t \min_{i\in\Lambda} \lambda(\omega_i, v(s))\, ds.$$

For any nonnegative numbers $\gamma_\Lambda \big(\Lambda \in \Omega_1^0(J_1)\big)$ the following inequality holds:

$$\max_{\Lambda\in\Omega_1^0(J_1)} \gamma_\Lambda \geqslant \frac{1}{M} \sum_{\Lambda\in\Omega_1^0(J_1)} \gamma_\Lambda, \text{ where } M = \frac{m!}{q!(m-q)!} + 1.$$

Therefore,

$$\max_{\Lambda\in\Omega_1^0(J_1)} \int_0^t \min_{i\in\Lambda} \lambda(\omega_i, v(s))\, ds. \geqslant \frac{1}{M}\int_0^t \sum_{\Lambda\in\Omega_q^0(J_1)} \min_{i\in\Lambda} \lambda(\omega_i, v(s))\, ds \geqslant$$

$$\geqslant \frac{1}{M}\int_0^t \max_{\Lambda\in\Omega_1^0(J_1)} \min_{i\in\Lambda} \lambda(\omega_i, v(s))\, ds \geqslant \frac{1}{M}\int_{T_2(t,\mu)} \max_{\Lambda\in\Omega_1^0(J_1)} \min_{i\in\Lambda} \lambda(\omega_i, v(s))\, ds \geqslant$$

$$\geqslant \frac{\delta(\mu)}{M}\int_{T_2(t,\mu)} ds \geqslant \frac{\delta(\mu)}{M}\Big(\frac{\delta(\mu)t - \beta_0}{1+\delta(\mu)}\Big).$$

Hence,

$$\max_{\Lambda\in\Omega_1^0(J_1)} \min_{i\in\Lambda} \int_0^t \lambda(\omega_i, v(s))\, ds \geqslant \frac{\delta(\mu)}{M}\Big(\frac{\delta(\mu)t - \beta_0}{1+\delta(\mu)}\Big).$$

The required inequality follows from the last inequality. This proves the lemma. □

Lemma 3.3:
Let $r = 1,\ \alpha < 0,\ \beta_1 = 0,\ m \geqslant q,\ a_1, \ldots, a_m, c_1, p_1 \in \mathbb{R}^k$ *such that for any* $J_0 \subset J = \{1, \ldots, m\}$, $|J_0| = q-1$, *one has*

$$0 \in \operatorname{Intco}\{a_l, l \in J \setminus J_0, c_1, p_1\}.$$

Then for any $b_1, \ldots, b_q \in \mathbb{R}^k$ *there is* $\widehat{\mu} > 0$ *such that for each* $\mu > \widehat{\mu}$ *there is a time instant* $T(\mu) > 0$ *such that for any admissible control of evaders* E_1 *and* E_2 *there is a set* $\Lambda^* \in \Omega_q^0(J_1)$ *such that for all* $l \in \Lambda^*$ *the following inequalities hold:*

$$\int_0^{T(\mu)} e^{-\alpha s} \lambda(\omega_l, v(s))\, ds \geqslant 1,$$

where

$$J_1 = J \cup \{m+1, \ldots, m+q\}, \quad \Omega_q^0(J_1) = \Omega_q(J) \cup \{(m+1, \ldots, m+q)\},$$

$$\omega_i = \begin{cases} a_i & \text{if } i \in J, \\ b_{i-m} + \mu c_1 & \text{if } i = m+1, \ldots, m+q. \end{cases}$$

Proof
Let $v(\cdot)$ be the admissible control of the evaders. Then for all $t \geqslant 0$ the following inequalities hold: $(p_1, y_1(t)) \leqslant 0$, $(p_1, y_2(t)) \leqslant 0$. Therefore, for all $t \geqslant 0$ the following inequality is valid:

$$\int_0^t e^{-as}(p_1, v(s))\, ds \leqslant \beta_0, \quad \text{where} \quad \beta_0 = \min\{-(p_1, y_1^0), -(p_2, y_2^0)\}.$$

Further reasoning is similar to that used in the proof of Lemma 3.2. □

4. SUFFICIENT CONDITIONS FOR CAPTURE IN THE CASE $\alpha = 0$

Theorem 4.1:
Let $\alpha = 0$, $r = 1$, $n \geqslant 2q - 1$ and suppose that there exists a set $I_0 \subset I$, $|I_0| = n - q$, such that for any $J_0 \subset I_0$, $|J_0| = q - 1$, one has

$$0 \in \operatorname{Intco}\{x_l^0 - y_1^0, x_l^0 - y_2^0, l \in I_0 \setminus J_0, p_1\}. \tag{4.9}$$

Then a q-fold capture occurs in the game $\Gamma(n + 2)$.

Proof
Assume that $I_0 = \{1, \ldots, n - q\}$. Denote $c = y_1^0 - y_2^0$. Since the equation

$$x_l^0 - y_2^0 = x_l^0 - y_1^0 + c$$

holds for all $l \in I$, it follows from (4.9) that for any $J_0 \subset I_0$, $|J_0| = q - 1$, we have

$$0 \in \operatorname{Intco}\{z_{l1}^0, l \in I_0 \setminus J_0, c, p_1\}.$$

It follows from Lemma 3.1 that there exists $\mu > 0$ such that

$$\delta(\mu) = \min_{v \in V} \max\Big\{ \max_{\Lambda \in \Omega_q^0(I)} \min_{i \in \Lambda} \lambda(\omega_i, v), (p_1, v)\Big\} > 0,$$

where $\Omega_q^0(I) = \Omega_q(I_0) \cup \{(n - q + 1, \ldots, n)\}$,

$$\omega_i = \begin{cases} z_{i1}^0 & \text{if } i \in I_0, \\ z_{i2}^0 + \mu c & \text{if } i = n - q + 1, \ldots, n. \end{cases}$$

It follows from Lemma 3.2 that

$$T_0 = \min\{t \geqslant 0 \mid \inf_{v(\cdot)} \max_{\Lambda \in \Omega_q^0(I)} \min_{i \in \Lambda} \int_0^t \lambda(\omega_i, v(s))\, ds \geqslant 1\}$$

is finite. Let $v(\cdot)$ be the admissible control of the evaders. Define the functions

$$h_i(t) = 1 - \int_0^t \lambda(\omega_i, v(s))\, ds.$$

Let pursuer P_i, $i \in I$, construct a control as follows. If the inequality $h_i(t) \geqslant 0$ holds at time t, then we assume

$$u_i(t) = v(t) - \lambda(\omega_i, v(t))\omega_i.$$

If τ is the first time instant for which $h_i(\tau) = 0$, then we assume that $\lambda(\omega_i, v(t)) = 0$ for all $t \geqslant \tau$.

From the definition of controls and the system (2.3) it follows that for all $t \geqslant 0$ the following equations hold:

$$z_{i1}(t) = z_{i1}^0 h_i(t),\ i \in I_0, \quad z_{i2}(t) = z_{i2}^0 h_i(t) - \mu c(1 - h_i(t)),\ i \in I \setminus I_0. \tag{4.10}$$

It follows from Lemma 3.2 that there exists $\Lambda^* \in \Omega_q^0(T)$ for which $h_i(T_0) = 0$ for all $i \in \Lambda^*$.

Two cases are possible.

1. $\Lambda^* \in \Omega_q(I_0)$. In this case, $z_{i1}(T_0) = 0$ for all $i \in \Lambda^*$. Hence, a q-fold capture of evader E_1 occurs.

2. $\Lambda^* = \{(n - q + 1, \ldots, n)\}$. Then $z_{i2}(T_0) = -\mu c$ for all $i \in I \setminus I_0$. We show that for any $J_0 \subset I$, $|J_0| = q - 1$, we have

$$0 \in \operatorname{Intco}\{z_{i2}(T_0), i \in I \setminus J_0, p_1\}. \tag{4.11}$$

Assume that this is not the case. Then there exists $J_0 \subset I$, $|J_0| = q - 1$, for which

$$0 \notin \operatorname{Intco}\{z_{i2}(T_0), i \in I \setminus J_0, p_1\}.$$

Hence, there exists a vector $v_0 \in V$, $\|v_0\| = 1$ such that

$$(p_1, v_0) \leqslant 0,\ (z_{i2}(T_0), v_0) \leqslant 0 \text{ for all } i \in I \setminus J_0.$$

It follows from the condition $|J_0| = q - 1$ that there exists a number $l \in I \setminus I_0$ such that $l \in \Lambda^*$. Then from the equation $z_{l2}(T_0) = -\mu c$ we find that $(c, v_0) \geqslant 0$.

Since the equation $z_{i1}(T_0) = z_{i2}(T_0) - c$ holds for all $i \in I_0$, the inequality $(z_{i1}(T_0), v_0) \leqslant 0$ holds for all $i \in I_0 \setminus J_0$. Therefore, it follows from (4.10) that $(z_{i1}^0, v_0) \leqslant 0$ for all $i \in I_0 \setminus J_0$. In addition, from the equation ($i \in I_0$)

$$z_{i2}(T_0) = z_{i1}(T_0) + c = z_{i1}^0 h_i(T_0) + z_{i2}^0 - z_{i1}^0$$

it follows that

$$z_{i2}^0 = z_{i2}(T_0) + z_{i1}^0(1 - h_i(T_0))$$

and hence $(z_{i2}^0, v_0) \leqslant 0$ for all $i \in I_0 \setminus J_0$. Thus, the following inequalities hold:

$$(p_1, v_0) \leqslant 0,\ (z_{i1}^0, v_0) \leqslant 0,\ (z_{i2}^0, v_0) \leqslant 0 \text{ for all } i \in I_0 \setminus J_0.$$

Let $J_2 \subset I_0 \setminus J_0$ be a set such that for the set $J_1 = (I_0 \cap J_0) \cup J_2$ the equation $|J_1| = q - 1$ is satisfied. Then, by Theorem 3.1,

$$0 \notin \operatorname{Intco}\{z_{i1}^0, z_{i2}^0, i \in I_0 \setminus J_1, p_1\},$$

which contradicts the condition of the theorem. Thus, (4.11) is proved. Taking T_0 as the initial time instant and using Theorem [12], we find that a q-fold capture of evader E_2 occurs in the game $\Gamma(n + 2)$. This proves the theorem. □

Theorem 4.2:
Let $\alpha = 0$, $D = \mathbb{R}^k$, $n \geqslant 2q - 1$ and suppose that there exists a set $I_0 \subset I$, $|I_0| = n - q$, such that for any $J_0 \subset I_0$, $|J_0| = q - 1$, we have

$$\operatorname{Intco}\{x_l^0, l \in I_0 \setminus J_0\} \cap \operatorname{co}\{y_1^0, y_2^0\} \neq \varnothing. \tag{4.12}$$

Then a q-fold capture occurs in the game $\Gamma(n + 2)$.

Proof
We prove that it follows from condition (4.12) that

$$0 \in \text{Intco}\{x_l^0 - y_1^0, x_l^0 - y_2^0, l \in I_0 \setminus J_0\}. \tag{4.13}$$

Assume that there exists $J_0 \subset I_0$, $|J_0| = q - 1$, for which (4.12) is satisfied, but

$$0 \notin \text{Intco}\{x_l^0 - y_1^0, x_l^0 - y_2^0, l \in I_0 \setminus J_0\}.$$

Hence, $\{0\}$ and $\text{co}\{x_l^0 - y_1^0, x_l^0 - y_2^0, l \in I_0 \setminus J_0\}$ are separable. Therefore, there exists $v_0 \in V$, $\|v_0\| = 1$, such that

$$(x_l^0 - y_1^0, v_0) \leqslant 0, \quad (x_l^0 - y_2^0, v_0) \leqslant 0 \text{ for all } l \in I_0 \setminus J_0.$$

Therefore,

$$(x_l^0, v_0) \leqslant \gamma \leqslant (y_j^0, v_0) \text{ for all } l \in I_0 \setminus J_0, j \in \{1, 2\},$$

where $\gamma = \min\{(y_1^0, v_0), (y_2^0, v_0)\}$. Hence, the sets $\text{co}\{x_l^0, l \in I_0 \setminus J_0\}$ and $\text{co}\{y_1^0, y_2^0\}$ are separable. Therefore,

$$\text{Intco}\{x_l^0, l \in I_0 \setminus J_0\} \cap \text{co}\{y_1^0, y_2^0\} = \varnothing,$$

which contradicts (4.12). Thus, (4.13) is proved. Further reasoning is similar to that used in the proof of Theorem 4.1. This proves the theorem. □

Theorem 4.3:
Let $\alpha = 0$, $n \geqslant 2q - 1$ and there exist a vector $p \in \mathbb{R}^k$, $p \neq 0$, a number $\gamma \in \mathbb{R}^1$ and a set $I_0 \subset I$, $|I_0| = n - q$, such that
1) $D \subset \{x \in \mathbb{R}^k \mid (p, x) \leqslant \gamma\}$;
2) for any set $J_0 \subset I_0$, $|J_0| = q - 1$, we have

$$0 \in \text{Intco}\{x_l^0 - y_1^0, x_l^0 - y_2^0, l \in I_0 \setminus J_0, p\}.$$

Then a q-fold capture occurs in the game $\Gamma(n + 2)$.

The validity of this theorem follows from Theorem 4.1.

Example 1. Let $q = 2$, $\alpha = 0$, $k = 2$, $D = \mathbb{R}^2$, $x_1^0 = (0, 0)$, $x_2^0 = (1, 1)$, $x_3^0 = (-1, 0)$, $x_4^0 = (-1, 1)$, $x_5^0 = (1, -1)$, $x_6^0 = (0, 1)$, $y_1^0 = (0, -2)$, $y_2^0 = (1, 2)$. Taking $I_0 = \{2, 3, 4, 5\}$, we find that the conditions of Theorem 3 are satisfied and hence a two-fold capture occurs in the game $\Gamma(8)$.

Example 2. Let $q = 2$, $\alpha = 0$, $k = 2$, $D = \mathbb{R}^2$, $p_1 = (0, -1)$, $\beta_1 = 3$, $x_1^0 = (-1, 1)$, $x_2^0 = (1, 1)$, $x_3^0 = (-1, 2)$, $x_4^0 = (1, 2)$, $x_5^0 = (0, 1)$, $x_6^0 = (0, 4)$, $y_1^0 = (0, 0)$, $y_2^0 = (0, -1)$. Taking $I_0 = \{2, 3, 4, 5\}$, we find that the conditions of Theorem 4.2 are satisfied and hence a two-fold capture occurs in the game $\Gamma(8)$. Taking $I_0 = \{1, 2, 3, 4\}$, we find that the conditions of Theorem 4.1 are satisfied and hence a two-fold capture occurs in the game $\Gamma(8)$.

5. SUFFICIENT CONDITIONS FOR CAPTURE IN THE CASE $\alpha < 0$

Theorem 5.1:
Let $\alpha < 0$, $r = 1$, $\beta_1 = 0$, $n \geqslant 2q - 1$ and suppose that there exists a set $I_0 \subset I$, $|I_0| = n - q$, such that for any $J_0 \subset I_0$, $|J_0| = q - 1$, one has

$$0 \in \text{Intco}\{x_l^0 - y_1^0, x_l^0 - y_2^0, l \in I_0 \setminus J_0, p_1\}. \tag{5.14}$$

Then a q-fold capture occurs in the game $\Gamma(n + 2)$.

Proof
Assume that $I_0 = \{1, \ldots, n-q\}$. Denote $c = y_1^0 - y_2^0$. Since the equation

$$x_l^0 - y_2^0 = x_l^0 - y_1^0 + c$$

holds for all $l \in I$, it follows from (5.14) that for any $J_0 \subset I_0$, $|J_0| = q-1$, we have

$$0 \in \mathrm{Intco}\{z_{l1}^0, l \in I_0 \setminus J_0, c, p_1\}.$$

It follows from Lemma 3.1 that there exists $\mu > 0$ such that

$$\delta(\mu) = \min_{v \in V} \max\{\max_{\Lambda \in \Omega_q^0(I)} \min_{i \in \Lambda} \lambda(\omega_i, v), (p_1, v)\} > 0,$$

where $\Omega_q^0(I) = \Omega_q(I_0) \cup \{(n-q+1, \ldots, n)\}$,

$$\omega_i = \begin{cases} z_{i1}^0 & \text{if } i \in I_0, \\ z_{i2}^0 + \mu c & \text{if } i = n-q+1, \ldots, n. \end{cases}$$

It follows from Lemma 3.3 that

$$T_0 = \min\{t \geqslant 0 \mid \inf_{v(\cdot)} \max_{\Lambda \in \Omega_q^0(I)} \min_{i \in \Lambda} \int_0^t e^{-\alpha s} \lambda(\omega_i, v(s))\, ds \geqslant 1\}$$

is finite. Let $v(\cdot)$ be the admissible control of the evaders. Define the functions

$$h_i(t) = 1 - \int_0^t e^{-\alpha s} \lambda(\omega_i, v(s))\, ds.$$

Let pursuer P_i, $i \in I$, construct a control as follows. If the inequality $h_i(t) \geqslant 0$ holds at time t, then we assume

$$u_i(t) = v(t) - \lambda(\omega_i, v(t))\omega_i.$$

If τ is the first time instant for which $h_i(\tau) = 0$, then we assume that $\lambda(\omega_i, v(t)) = 0$ for all $t \geqslant \tau$.

It follows from (2.3) that the solution to the Cauchy problem has the form

$$z_{ij}(t) = e^{\alpha t} \cdot \big(z_{ij}^0 + \int_0^t e^{-\alpha s}(u_i(s) - v(s))\, ds\big). \tag{5.15}$$

From the definition of the controls of the pursuers and (5.15) we find that for all $t \geqslant 0$ the following equations hold:

$$z_{i1}(t)e^{-\alpha t} = z_{i1}^0 h_i(t),\ i \in I_0, \quad z_{i2}(t)e^{-\alpha t} = z_{i2}^0 h_i(t) - \mu c(1 - h_i(t)),\ i \in I \setminus I_0. \tag{5.16}$$

It follows from Lemma 2 that there exists $\Lambda^* \in \Omega_q^0(T)$ for which $h_i(T_0) = 0$ for all $i \in \Lambda^*$.

Two cases are possible.

1. $\Lambda^* \in \Omega_q(I_0)$. In this case we find that $z_{i1}(T_0) = 0$ for all $i \in \Lambda^*$. Hence, a q-fold capture of evader E_1 occurs.

2. $\Lambda^* = \{(n-q+1, \ldots, n)\}$. Then $z_{i2}(T_0) = -\mu e^{\alpha T_0} c$ for all $i \in I \setminus I_0$. We show that for any $J_0 \subset I$, $|J_0| = q-1$,

$$0 \in \mathrm{Intco}\{z_{i2}(T_0), i \in I \setminus J_0, p_1\}. \tag{5.17}$$

Assume that this is not the case. Then there exists $J_0 \subset I$, $|J_0| = q - 1$, for which

$$0 \notin \operatorname{Intco}\{z_{i2}(T_0), i \in I \setminus J_0, p_1\}.$$

Consequently, there exists a vector $v_0 \in V$, $\|v_0\| = 1$, such that

$$(p_1, v_0) \leqslant 0,\ (z_{i2}(T_0), v_0) \leqslant 0 \text{ for all } i \in I \setminus J_0.$$

Since $|J_0| = q - 1$, there exists a number $l \in I \setminus I_0$ such that $l \in \Lambda^*$. Then it follows from the equation $z_{l2}(T_0) = -\mu e^{\alpha T_0} c$ that $(c, v_0) \geqslant 0$.

Since the equation $z_{i1}(T_0) = z_{i2}(T_0) - ce^{\alpha T_0}$ holds for all $i \in I_0$, the inequality $(z_{i1}(T_0), v_0) \leqslant 0$ holds for all $i \in I_0 \setminus J_0$. Therefore, it follows from (5.16) that $(z_{i1}^0, v_0) \leqslant 0$ for all $i \in I_0 \setminus J_0$. Also, from the equation $(i \in I_0)$

$$z_{i2}(T_0) = z_{i1}(T_0) + e^{\alpha T_0} c = z_{i1}^0 h_i(T_0) + e^{\alpha T_0}(z_{i2}^0 - z_{i1}^0)$$

it follows that

$$z_{i2}^0 = e^{-\alpha T_0} z_{i2}(T_0) + z_{i1}^0(1 - h_i(T_0))$$

and hence $(z_{i2}^0, v_0) \leqslant 0$ for all $i \in I_0 \setminus J_0$. Thus, the following inequalities hold:

$$(p_1, v_0) \leqslant 0,\ (z_{i1}^0, v_0) \leqslant 0,\ (z_{i2}^0, v_0) \leqslant 0 \text{ for all } i \in I_0 \setminus J_0.$$

Let $J_2 \subset I_0 \setminus J_0$ be a set such that for the set $J_1 = (I_0 \cap J_0) \cup J_2$ the equation $|J_1| = q - 1$ is satisfied. Then, by Theorem 3.1,

$$0 \notin \operatorname{Intco}\{z_{i1}^0, z_{i2}^0, i \in I_0 \setminus J_1, p_1\},$$

which contradicts the condition of the theorem. Thus, (5.17) is proved. Taking T_0 as the initial time instant and using Theorem [12], we find that a q-fold capture of evader E_2 occurs in the game $\Gamma(n + 2)$. This proves the theorem. □

Theorem 5.2:
Let $D = \mathbb{R}^k$, $\alpha < 0$, $n \geqslant 2q - 1$ and suppose that there exists a set $I_0 \subset I$, $|I_0| = n - q$, such that for any $J_0 \subset I_0$, $|J_0| = q - 1$,

$$\operatorname{Intco}\{x_l^0, l \in I_0 \setminus J_0\} \cap \operatorname{co}\{y_1^0, y_2^0\} \neq \varnothing. \tag{5.18}$$

Then a q-fold capture occurs in the game $\Gamma(n + 2)$.

This theorem is proved along the same lines as Theorem 4.1.

Theorem 5.3:
Let $\alpha < 0$, $n \geqslant 2q - 1$ and there exists a vector $p \in \mathbb{R}^k$, $\|p\| = 1$, a set $I_0 \subset I$, $|I_0| = n - q$, such that

1) $D \subset \{x \in \mathbb{R}^k \mid (p, x) \leqslant 0\}$;

2) for any set $J_0 \subset I_0$, $|J_0| = q - 1$, one has

$$0 \in \operatorname{Intco}\{x_l^0 - y_1^0, x_l^0 - y_2^0, l \in I_0 \setminus J_0, p\}.$$

Then a q-fold capture occurs in the game $\Gamma(n + 2)$.

The validity of this theorem follows from Theorem 5.1.

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