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Grading Structure for Derivations of Group Algebras

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Abstract: In this paper we give a way of equipping the derivation algebra of a group algebra with the structure of a graded algebra. The derived group is used as the grading group. For the proof, the identification of the derivation with the characters of the adjoint action groupoid is used. These results also allow us to obtain the analogous structure of a graded algebra for outer derivations. A non-trivial graduation is obtained for all groups that are not perfect.

Keywords: derivations, bimodule, outer derivations, graded algebras

Calculation of derivations in a group algebra is a well-known problem. Present work elaborates results from articles [1–3] focused on studying derivations in terms of characters of adjoint action gruppoid. An important result of this research are handy formulas for quick calculation of derivations. These articles explore derivations' link to combinatorial properties of the group.
Among applications note use in coding theory (see [4, 5]), Novikov algebras (see recent work [6]) and more general constructions, like $(\sigma, \tau)-$derivations (see [7]).

1. INTRODUTION

Aim of the present work is grading the derivation algebra by identifying derivations and characters on a certain groupoid (all the necessary definitions are given in **Section 2**). The main result of the paper follows.
Let N be a fixed normal subgroup in G such that G/N is abelian.

Theorem 1.1. *If* $|G/N| > 1$, Der *is graded with* G/N*, that is*

$$\mathrm{Der} = \bigoplus_{k \in G/N} \mathrm{Der}_k,$$
$$\forall k, l \in G/N : [\mathrm{Der}_k, \mathrm{Der}_l] \subset \mathrm{Der}_{kl}.$$

Here Der_k is a subalgebra of derivation whose characters' support is localised entirely in one coset $aN = k$.
The structure of the work follows. **Section 2** provides main definitions and propositions. **Section 3** describes the construction of grading and contains the main result and its proof. **Section 4** provides an example of grading for G equal to discrete Heisenberg group along with an example of localising central derivations for such groups G that G is not a stem group.
Fix an infinite finitely-generated group G for the rest of the text.

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2. PRELIMINARIES

Recall that **Group algebra** $\mathbb{C}[G]$ is an algebra of formal finite sums of type $(a_1, \dots, a_n \in \mathbb{C}, g_1, \dots, g_n \in G)$

$$a_1 g_1 + \dots + a_n g_n$$

We define **derivation** as a linear operator d that satisfies the Leibniz rule (for all $a, b \in \mathbb{C}[G]$)

$$d(ab) = d(a) \cdot b + a \cdot d(b)$$

Derivations over a group algebra form a Lie algebra with respect to commutator.
We will denote this algebra as Der *or* $\mathrm{Der}(\mathbb{C}[G])$.

2.1. Characters

We use the technique of characters following [1–3].

Definition 2.1. For a given group G consider a **small groupoid** Γ:

1. **objects** (Obj) — elements of G,
2. **arrows** (Hom) — pairs of elements of G. For an arrow (u, v) its source $\mathrm{S}(u, v)$ is given by $v^{-1}u$, and its target $\mathrm{T}(u, v)$ — by uv^{-1} ($Hom(a, b)$ denotes a set of all arrows for which the source is a and target is b),
3. Consider two arrows $\varphi = (u_2, v_2) \in Hom(b, c), \psi = (u_1, v_1) \in Hom(a, b)$ (we will call a(n ordered) pair of arrows φ, ψ such that $\mathrm{S}(\varphi) = \mathrm{T}(\psi)$ **composable**). The **composition** for these two arrows is given by:

$$(u_2, v_2) \circ (u_1, v_1) := (v_2 u_1, v_2 v_1)$$

The formula for composition does not comprise u_2 since if we have a pair of composable arrows $(u_2, v_2), (u_1, v_1)$, u_2 can be expressed in terms of u_1, v_1, v_2. The reader may consider this as an exercise. Γ is the groupoid of group's inner action on itself.
Fix an element a of G. Define following symbols:

Definition 2.2.
- $[a] = \{xax^{-1} : x \in G\}$ is a**'s conjugacy class** in G,
- $$G^G := \{[g] : g \in G\}$$
- $\Gamma_{[a]}$ is Γ's subgroupoid, *informally, a connected component in* Γ, given by:

$$Obj(\Gamma_{[a]}) := [a] = \{x \in Obj : x \in [a]\}$$
$$Hom(\Gamma_{[a]}) := \{(u, v) \in Hom : u, v \in [a]\}$$

Definition 2.3. A **character** on Γ is a function $\chi : Hom \to \mathbb{C}$, such that:

- ***(Composition)*** for each pair of composable arrows φ, ψ:

$$\chi(\varphi \circ \psi) = \chi(\varphi) + \chi(\psi)$$

- ***(Locally finite)*** $\forall \boldsymbol{y} \in G$ there is a **finite** set of $\boldsymbol{x} \in G$, such that $\chi(x, y) \neq 0$.

Holds the following decomposition:

Lemma 2.1. *(Decomposition)* $\Gamma = \bigsqcup_{[a] \in G^G} \Gamma_{[a]}$

Remark. Although characters being locally finite may seem as an alien and a bit too technical detail, it is deliberately placed in the definition to stress that we will **not** consider *"non-locally finite characters"*. The reasons will become clear, among the rest, in **Theorem 2.1**.

We will need the following statement:

Statement 2.1. *Let* $(u, v) = \varphi \in Hom,\quad a \in G$. *Then the following statements are equivalent* $\varphi \in Hom(\Gamma_{[a]}),\ \ \mathrm{S}(\varphi) \in [a],\ \ \mathrm{T}(\varphi) \in [a]$.

It is proved by direct calculation.

2.2. Connection between Characters and Derivations

The following theorem motivates to consider **(locally finite)** characters when studying derivations. *Informally,* ***Theorem 2.1*** *shows that characters may be seen as a generalization of linear operator's matrix.*

Theorem 2.1 (Derivation formula and derivation character, [1, section 2])**.** *For each derivation* d *there exists a unique character* χ *such that for each* $x \in G$ *holds*

$$d(x) = \sum_{k \in G} \chi(k, x)k \tag{2.1}$$

Consider d, χ from **Theorem 2.1**. We will say that character χ **gives** derivation d (derivation d **is given by character** χ; we will omit words "derivation" and "character"). For a derivation d let χ^d be a character such that χ^d gives d.
Theorem 2.1 implies:

Corollary 2.1. *Let* d, ∂ *be derivations given by characters* χ^d, χ^∂ *correspondingly, Then* $d + \partial$ *be given by* $\chi^d + \chi^\partial$.

Definition 2.4. Let d be given by α, ∂ be given by β. Then $\{\alpha, \beta\}$ is the character that gives $[d, \partial]$.

Statement 2.2 (*"Matrix" product*, [2, Proposition 2.4])**.** *Let* α, β *be characters. Then* $\{\alpha, \beta\}$ *satisfies* $(a, b \in G)$

$$\{\alpha, \beta\}(a, b) = \sum_{k \in G} \alpha(a, k)\beta(k, b) - \beta(a, k)\alpha(k, b)$$

Two examples of derivations follow. **Example 2.1** will be needed to prove **Theorem 1.1**.
Let $a \in G$. Recall that derivation d_a is called **inner** if for any $x \in \mathbb{C}[G]$

$$d_a(x) = [x, a] = xa - ax$$

Example 2.1 (Character of inner derivation [3, Proposition 3]). Let $a \in G$. Then character χ_a given by formula

$$\chi_a(\varphi) = \begin{cases} 1, & a = \mathrm{S}(\varphi), \\ -1, & a = \mathrm{T}(\varphi), \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

gives $d_a(x) = [x, a]$.
Example 2.2. Another possible example of derivations are *central derivations*. We will call derivation d central if there exists such central element $z \in Z(G)$ and homomorphism $\tau : G \to (\mathbb{C}, +)$ such that for all *basis elements* $g \in G$:

$$d(g) = \tau(g)gz$$

Such an operator is indeed a derivation, see [3, Proposition 4]. [3, Proposition 5] shows that non-trivial central derivations are not inner. Moreover, [3, Proposition 6] shows that central derivations form a Lie subalgebra in $\mathrm{Der}(G)$.

Definition 2.5. For a given character χ we define support of χ as following

$$\mathrm{supp}\,\chi = \{\varphi \in Hom : \chi(\varphi) \neq 0\}$$

For the given subset $M \subset G$ denote by $\mathbf{Der}_{\boldsymbol{M}}$ the set of derivations d such that for character χ that gives d: $\mathrm{supp}\,\chi \subset M$.

Example 2.3. Recall character χ_a from Example 2.1 (where $a \in G$.) Its support is easily calculated

$$\mathrm{supp}\,\chi_a = \{\varphi : \mathrm{S}(\varphi) = a\} \cup \{\psi : \mathrm{T}(\psi) = a\}$$

By **Statement 2.1**, we can localize $\mathrm{supp}\,\chi_a$ in a single conjugacy class a (we will need such a localisation later in **Theorem 1.1**)

$$\mathrm{supp}\,\chi_a \subset \Gamma_{[a]}$$

2.3. Applying Decomposition

Lemma 2.1 establishes decomposition of groupoid Γ. The current section presents *decompositions for **(locally finite)** characters and derivations.*
The following two lemmas are equivalent. We prove the first one.

Lemma 2.2. *Let χ be a character. Then there exists finitely many $a_1, \ldots, a_N \in G$ such that*

$$\mathrm{supp}\,\chi \leq \bigcup_{k=1}^{N} \Gamma_{[a_k]}$$

Lemma 2.3 (Derivation decomposition)**.** *Then for each $u \in G$ such that χ_u is a character, holds decomposition $d = \sum_{[u] \in G^G} d_u$, and the set $\{[u] \in G^G : \exists x \in G : d_u(x) \neq 0\}$ is finite.*

*Proof for **Lemma 2.2***
Let $G = \langle X \mid R \rangle$, where $X =: \{x_1, \ldots, x_k\}$ is finite (G is assumed to be finitely-generated throughout the text).
Consider $(u, v) \in \Gamma$. Let $n = n(v)$ be minimal nonnegative integer such that

$$\exists y_0, \ldots, y_n \in X \cup X^{-1} : v = y_0 y_1 \ldots y_n$$

1. Show that
$$\exists z_0, \ldots, z_n \in G : (z_0, y_0) \circ \cdots \circ (z_n, y_n) = (u, v) \tag{2.3}$$

   Subproof
   Induction by $n = n(v)$. **Base:** $n = 0$ — $z_0 = u$. **Step:** Consider $z_0 = uv^{-1}y_0$. Then $(z, y_0), (y_0^{-1}u, y_0^{-1}v)$ are composable since

$$\mathrm{S}(z_0, y_0) = y_0^{-1} z_0 = y_0^{-1} u v^{-1} y_0 = \\ = y_0^{-1} u (y_0^{-1} v)^{-1} = \mathrm{T}(y_0^{-1} u, y_0^{-1} v)$$

   Moreover,
$$(z_0, y_0) \circ (y_0^{-1} u, y_0^{-1} v) = (u, v)$$

   Notice that $y_0^{-1} v = y_1 \ldots y_n$, thus $n(y_0^{-1} v) < n = n(v)$. Therefore, applying the step of induction, get eq. (2.3). □

2. Since χ is ***locally finite***, the set $B = (G \times (X \cup X^{-1})) \cap \operatorname{supp} \chi$ is finite. By **??** for each arrow φ there exists a unique element $a \in G$ such that $\varphi \in Hom(\Gamma_{[a]})$; thus, there exists a finite set $A = \{a_1, \ldots, a_N\}$ such that for any $a \notin A$: $B \cap Hom(\Gamma_{[a]}) = \emptyset$. Thus, by item 1 for any $a \notin A$: $\operatorname{supp} \chi \cap Hom(\Gamma_{[a]}) = \emptyset$. Therefore:

$$\operatorname{supp} \chi \leq \bigcup_{k=1}^{N} \Gamma_{[a_k]}$$

■

A very nice alternative proof for **Lemma 2.2** was submitted in an anonymous review.

*Alternative proof for **Lemma 2.2***
Let d be the derivation given by character χ.
Let P be a union of conjugacy classes such that

$$\operatorname{supp} \chi \subset \bigcup_{g \in P} \Gamma_{[g]} =: U$$

The following statements are equivalent:

- $(x, y) \in Hom(U)$,
- $y^{-1}x = \mathrm{S}(x, y) \in P$

Consider an element $y \in G$ such that

$$y^{-1}d(y) \in \langle P \rangle$$

Here $\langle X \rangle$ for $X \subset G$ denotes a set of all finite sums $a_1x_1 + \cdots + a_nx_n$ such that $a_1, \ldots, a_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in X$.
Let's calculate $y^{-1}d(y)$ by **Statement 2.2**.

$$y^{-1}d(y) = y^{-1}\sum_{x \in G} \chi(x, y)x = \sum_{x \in G} \chi(x, y)y^{-1}x = \sum_{x \in G} \chi(x, y)\mathrm{S}(x, y) \in \langle P \rangle$$

Therefore, for each x such that $\chi(x, y) \neq 0 : y^{-1}x \in P$.
Consider a set

$$H = \left\{ y : y^{-1}d(y) \in \langle P \rangle \right\}$$

As a simple exercise, check H's being a subgroup in G.
To summarize, if y is in subgroup $H \leq G$ (that is $y^{-1}d(y) \in \langle P \rangle$) then for each x such that $\chi(x, y) \neq 0 : y^{-1}x \in P$.
To finish the proof let's choose such P that P is a union of a **finite number** of conjugacy classes and $H = G$.
To achieve this, consider a finite generating set S for G and a **finite** subset $M \subset G$ such that for any $s \in S$ there exist complex numbers $a_m, m \in M$ such that

$$s^{-1}d(s) = \sum_{m \in M} a_m m$$

Informally, calculate all $s^{-1}d(s), s \in S$, which are finite sums, and store all elements in G present in at least one of finite sums.

Since M is finite, $P = \bigcup_{m\in M} [m]$ is a union of finite number of conjugacy classes. Moreover, for such P

$$S \subset H$$

Thus, $G = H$.
All in all, there exists a union P of a **finite** number of conjugacy classes such that

$$\text{supp}\, \chi \subset \bigcup_{g\in P} \Gamma_{[g]} =: U$$

■

Let d be the derivation given by character χ, and

$$\chi_u(\varphi) := \begin{cases} \chi(\varphi), & \varphi \in Hom(\Gamma_{[u]}), \\ 0, & otherwise \end{cases}.$$

We will denote the derivation given by χ_u as d_u.

3. CONSTRUCTING GRADED ALGEBRA

Grading with Abelian Quotients

Definition 3.1. Let A be an abelian group, e be a neutral element in A, $\mathfrak{A}$ be a Lie algebra, $\mathfrak{A}$ can be expressed as a direct sum

$$\mathfrak{A} = \bigoplus_{n\in A} \mathfrak{A}_n, \text{ such that}$$
$$\forall n, l \in A : [\mathfrak{A}_n, \mathfrak{A}_l] \subset \mathfrak{A}_{nl}$$

such that $\mathfrak{A}_e \neq \mathfrak{A}$. Then $\mathfrak{A}$ is called **graded (with A)**. The direct sum Def 3.1 is called **$\mathfrak{A}$'s grading with A**.

Notice that trivial gradings are **excluded**.

Definition 3.2. A commutator subgroup of group G is

$$G' := \{xyx^{-1}y^{-1} : x, y \in G\}$$

Recall a few well-known definitions and statements that will be required below.

Statement 3.1. *For any group G a subgroup G' is normal, $G' \lhd G$*

Let N be a fixed normal subgroup in G such that G/N is abelian. *Conceptually, the latter condition derives from the following **Def 3.1**, however it is also needed for more technical, yet crucial details, like **Lemma 3.1**.*

Statement 3.2. $G' \subset N$.

Lemma 3.1. *Let $a \in N$. Then $[a] \subset aN$.*

Proof
A calculation for any $a, t \in G$ proves lemma:

$$tat^{-1}N = tat^{-1}a^{-1}Na = Na = aN$$

The first and the last equivalences hold since N is normal; the second equivalence holds by **Statement 3.2** since G/N is abelian ($tat^{-1}a^{-1} \in G' \subset N$.) ■

*Note that **Lemma 3.1** would have been false if we did not require G/N to be abelian. Consider $G = S_4, N = V_4, a = (12)$ for counterexample.*
Lemma 3.1 motivates the following symbols:

$$\Gamma_{aN} = \bigcup_{k \in aN} \Gamma_{[k]},$$
$$\mathrm{Der}_{aN} = \left\{ d \in Der : \mathrm{supp}\, \chi^d \subset \Gamma_{aN} \right\}.$$

Lemma 3.2. *Let $a, b \in G$, d is given by character $\alpha : \mathrm{supp}\, \alpha \subset \Gamma_{aN}$, ∂ is given by character $\beta : \mathrm{supp}\, \beta \subset \Gamma_{bN}$. Then $\mathrm{supp}\, \{\alpha, \beta\} \subset \Gamma_{abN}$.*

Proof

Statement 2.2 implies:

$$\{\alpha, \beta\}(h, g) = \sum_{k \in G} \alpha(h, k)\beta(k, g) - \beta(h, k)\alpha(k, g)$$

Consider an arrow $(h, g) : \{\alpha, \beta\}(h, g) \neq 0$. There exists $k \in G$:

$$\left[\begin{array}{l} \alpha(h, k)\beta(k, g) \neq 0 \\ \beta(h, k)\alpha(k, g) \neq 0 \end{array}\right.$$

$$\left[\begin{array}{l} \begin{cases} \alpha(h, k) \neq 0 \\ \beta(k, g) \neq 0 \end{cases} \\ \\ \begin{cases} \beta(h, k) \neq 0 \\ \alpha(k, g) \neq 0 \end{cases} \end{array}\right. \tag{3.4}$$

Expressing eq. (3.4) in terms of supp :

$$\left[\begin{array}{l} \begin{cases} (h, k) \in \mathrm{supp}\, \alpha \subset Hom(\Gamma_{aN}) \\ (k, g) \in \mathrm{supp}\, \beta \subset Hom(\Gamma_{bN}) \end{cases} \\ \\ \begin{cases} (k, g) \in \mathrm{supp}\, \alpha \subset Hom(\Gamma_{aN}) \\ (h, k) \in \mathrm{supp}\, \beta \subset Hom(\Gamma_{bN}) \end{cases} \end{array}\right. \tag{3.5}$$

By **Statement 2.1**, an arrow φ belongs to $\Gamma_{[x]} \subset \Gamma_{xN}$ iff its target $\mathrm{T}(\varphi)$ belongs to $[x] \subset xN$. Thus, eq. (3.5) implies:

$$\left[\begin{array}{l} \begin{cases} hk^{-1} = u \in aN \\ kg^{-1} = v \in bN \end{cases} \\ \\ \begin{cases} kg^{-1} = u \in aN \\ hk^{-1} = v \in bN \end{cases} \end{array}\right. \tag{3.6}$$

Multiplying the equations in eq. (3.6):

$$\left[\begin{array}{l} hg^{-1} = uv \\ hg^{-1} = vu \end{array}\right. , \textbf{ for } u \in aN, v \in bN$$

By definition, $\mathrm{T}(h,g) = hg^{-1}$. By **Statement 2.1**:

$$\left[\begin{array}{l}(h,g) \in Hom(\Gamma_{[uv]})\\(h,g) \in Hom(\Gamma_{[vu]})\end{array}\right.$$

Since $[uv] = [u \cdot vu \cdot u^{-1}] = [vu]$ and $uv \in abN$, then by **Lemma 3.1**:

$$[uv] \subset uvN = abN$$

Thus:

$$(h,g) \in \Gamma_{[uv]} \subset \Gamma_{abN}, \ \textbf{for}\ u \in aN, v \in bN$$

All in all, $\{\alpha,\beta\}(h,g) \neq 0 \Rightarrow (h,g) \in \Gamma_{abN}$, therefore $\operatorname{supp}\{\alpha,\beta\} \subset \Gamma_{abN}$. ■

Theorem 1.1. *If* $|G/N| > 1$, Der *is graded with* G/N, *that is*

$$\mathrm{Der} = \bigoplus_{k \in G/N} \mathrm{Der}_k,$$
$$\forall k,l \in G/N : [\mathrm{Der}_k, \mathrm{Der}_l] \subset \mathrm{Der}_{kl}.$$

Proof
Consider the sum $\sum_{k \in G/N} \mathrm{Der}_k$.

1. First, show that $\sum_{k \in G/N} \mathrm{Der}_k$ is equal to Der.

   - $\sum_{k \in G/N} \mathrm{Der}_k \subset \mathrm{Der}$ — since Der is closed under finite sums.
   - Now we show an opposite inclusion. Consider an arbitrary $d \in \mathrm{Der}$. **Lemmas 2.3 and 3.1** imply (see **Lemma 2.3** for definition of d_u)

     $$d = \sum_{[u] \subset G} d_u = \sum_{k \in G/N} \left(\sum_{[u] \subset k} d_u \right)$$

     Consider for given $k \in G/N$

     $$s_k := \sum_{[u] \subset k} d_u \in \mathrm{Der}_k$$

     By **Lemma 2.3**, there is only a finite number of such $[u] \subset G$ that d_u is not constant 0. Thus, there exists an integer N and $k_1, \ldots, k_N \in G/N$ such that for any $k \in G/N, k \neq k_1, \ldots, k_N$: s_k is constant 0.
     Thus,

     $$d = \sum_{i=1}^{N} s_{k_i}$$

     Thus,

     $$\mathrm{Der} \subset \sum_{k \in G/N} \mathrm{Der}_k$$

- All in all, $\mathrm{Der} = \sum_{k \in G/N} \mathrm{Der}_k$.

2. $\sum_{k \in G/N} \mathrm{Der}_k$ is direct.

   Subproof
   Let $d_k \in \mathrm{Der}_k$ be given by χ_k. Suppose that

   $$\sum_{k \in G/N} d_k \equiv 0$$

   Since constant 0 is a derivation given by constant character (equal to 0), by **Corollary 2.1** for any $\varphi \in Hom$

   $$\sum_{k \in G/N} \chi_k(\varphi) = 0$$

   Since $\operatorname{supp} \chi_k, \operatorname{supp} \chi_l$ are disjoint (for $k \neq l$), for each $\varphi \in Hom$ exists no more than one $k \in G/N$ such that $\chi_k(\varphi) \neq 0$. Thus, for each $\varphi \in Hom$ and for each $k \in G/N$: $\chi_k(\varphi) = 0$, and for each $k \in G/N : d_k \equiv 0$. Therefore, $\sum_{k \in G/N} \mathrm{Der}_k = \bigoplus_{k \in G/N} \mathrm{Der}_k$ is direct by definition of direct sum. □

3. We established that

   $$\mathrm{Der} = \bigoplus_{k \in G/N} \mathrm{Der}_k$$

   Since there exists such $m \in G$ that $mN \neq N$. Thus, by Examples 2.1 and 2.3 the inner derivation $[x, m]$ is given by a character χ_m such that $\operatorname{supp} \chi_m \subset [m] \subset mN$ by **Lemma 3.1**. Thus, $\mathrm{Der} \neq \mathrm{Der}_N$ *(i.e. the grading is not trivial.)*
   Finally, check that $(\forall k, l \in G/N)$

   $$\forall k, l \in G/N : [\mathrm{Der}_k, \mathrm{Der}_l] \subset \mathrm{Der}_{kl}$$

   Proof for item 3
   Let $d_k \in \mathrm{Der}_k, d_l \in \mathrm{Der}_l$. Let character χ_k give d_k, character χ_l give d_l. Thus, by definition of $\mathrm{Der}_k, \mathrm{Der}_l$

   $$\operatorname{supp} \chi_k \leq \Gamma_k, \qquad \operatorname{supp} \chi_l \leq \Gamma_l$$

   Therefore, by **Lemma 3.2**

   $$\operatorname{supp} \{\chi_k, \chi_l\} \leq \Gamma_{kl}$$

   Finally,

   $$[d_k, d_l] \in \mathrm{Der}_{kl}$$

   □

■

Example 3.1. Let G be a perfect group ($|G/G'| = 1$), that is $G' = G$. **Theorem 1.1** yields *a trivial grading*(which we do not regard as a grading in this text) for $\mathrm{Der}(\mathbb{C}[G])$ since $|G/G'| = 1$.
And vice versa:

Corollary 3.1. *If G is NOT a perfect group ($G \neq G'$) then* Der *admits a (non-trivial) grading with G/G'.*

Example 3.2. Let G be a knot group (i.e. let there exist some knot K such that G is the knot group of K). It is well-known that in this case $G/G' = \mathbb{Z}$, therefore $Der(G)$ admits a grading with $\mathbb{Z}$.

4. EXAMPLES

Discrete Heisenberg Group

Consider discrete Heisenberg group (a group of 3×3 upper unitriangular matrices with integer entries). Following [3], we use this group as a handy example since it admits easy calculations.

Definition 4.1. Consider a group of integer unitriangular matrices with respect to matrix multiplication:

$$\boldsymbol{H} = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$$

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} := \begin{pmatrix} 1 & a+x & c+z+ay \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{pmatrix}$$

Since all the matrices in $\boldsymbol{H}$ have determinant 1, the inverse is well-defined and given by

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a & ab-c \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix}$$

Our goal is to grade $\mathrm{Der}(\boldsymbol{H})$.

Definition 4.2. The centre of G is

$$Z(G) = \{z \in G : \forall g \in G : gz = zg\}$$

The follwing statements are well-known and trivial.

Statement 4.1.

$$\boldsymbol{H}' = Z(\boldsymbol{H}) = \left\{ \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : a \in \mathbb{Z} \right\}$$

Statement 4.2.

$$\boldsymbol{H}/\boldsymbol{H}' \simeq \mathbb{Z} \oplus \mathbb{Z}$$

Let $\psi : \mathbb{Z} \oplus \mathbb{Z} \to \boldsymbol{H}/\boldsymbol{H}'$ be an isomorphism. Recall the symbols:

$$\Gamma_{aN} = \bigcup_{k \in aN} \Gamma_{[k]},$$

$$\mathrm{Der}_{aN} = \left\{ d \in Der : \mathrm{supp}\ \chi^d \subset \Gamma_{aN} \right\}.$$

Define:

$$\mathrm{Der}_{(i,j)} := \mathrm{Der}_{\psi(i,j)}$$

Corollary 4.1 (From **Statement 4.2**, **Theorem 1.1**). $\mathrm{Der}(\boldsymbol{H})$ *is graded with* $\mathbb{Z} \oplus \mathbb{Z}$*, that is*

$$\mathrm{Der}(\boldsymbol{H}) = \bigoplus_{(i,j)\in\mathbb{Z}\oplus\mathbb{Z}} \mathrm{Der}_{(i,j)}$$

$$\forall (i,j),(k,l) \in \mathbb{Z}\oplus\mathbb{Z} : [\mathrm{Der}_{(i,j)}, \mathrm{Der}_{(k,l)}] \subset \mathrm{Der}_{(i+k,j+l)}$$

Example 4.1. **Statement 4.1** implies: if d is given by such χ that $\operatorname{supp}\chi \leq \Gamma_{[z]}$ for $z \in Z(\boldsymbol{H})$, then $d \in \mathrm{Der}_{(0,0)}$.

Definition 4.3. G is a stem group if

$$Z(G) \leq G'$$

Example 4.1 can be generalised under the assumption that G is a stem group. Note that $\boldsymbol{H}$ is a stem group since $\boldsymbol{H}' = Z(\boldsymbol{H})$ by **Statement 4.1**. See [8] for more details on (finite, which is not our case) stem groups.

Proposition 4.1. *Let G be a stem group. If d is given by such χ that* $\operatorname{supp}\chi \leq \Gamma_{[z]}$ *for $z \in Z(G)$, then $d \in \mathrm{Der}_{G'}$.*

Central derivations were introduced in [3] as operators given on a group algebra generators $g \in G$ with the homomorphism $\tau : G \to (\mathbb{C}, +)$ and the central element $z \in Z(G)$ by formula

$$d_{\tau,z} : g \mapsto \tau(g)gz.$$

Recall that [3, Proposition 6] shows that central derivations form a Lie subalgebra in $\mathrm{Der}(G)$ (denote as $\mathrm{ZDer}(G)$).

Proposition 4.2. *Let G be NOT a stem group. Then there is an "induced" nontrivial grading of* $\mathrm{ZDer}(G)$ *with G'.*

Note that if G is a stem group, *the grading is trivial*, i.e. not a grading at all in our terms. Another example of an induced grading follows.
Let $a \in G$. Recall that derivation d_a is called **inner** if for any $x \in \mathbb{C}[G]$

$$d_a(x) = [x, a] = xa - ax$$

Let $\mathrm{InnerDer}$ be the set of all derivations of the form ($a_1, \ldots, a_n \in \mathbb{C}; y_1, \ldots, y_n \in G$)

$$G \ni x \mapsto a_1[x, y_1] + \cdots + a_n[x, y_n] = [x, a_1 y_1 + \cdots + a_n y_n]$$

A direct calculations shows that $\mathrm{InnerDer}$ is an ideal in Der (i.e. for any $d \in \mathrm{InnerDer}$ and for any $\partial \in \mathrm{Der} : [d, \partial], [\partial, d] \in \mathrm{InnerDer}$; recall that $[d, \partial](x) = d(\partial(x)) - \partial(d(x))$.) Therefore, there exists a factor-algebra $\mathrm{OuterDer} := \mathrm{Der}/\mathrm{InnerDer}$.

Corollary 4.2. *If G is NOT a stem group, there is an induced (non-trivial) grading of* $\mathrm{OuterDer}$ *with G' of the form*

$$\mathrm{OuterDer} = \bigoplus_{k\in G/G'} \mathrm{Der}_k/\mathrm{InnerDer}_k,$$

where $\mathrm{InnerDer}_k := \mathrm{Der}_k \cap \mathrm{InnerDer}$

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