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Stability Criteria for Periodic Selector-Linear Difference Inclusions

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Abstract: The paper considers periodic selector-linear difference inclusions. Lyapunov functions from the parametric class of homogeneous forms of even degree are constructed. These functions establish necessary and sufficient conditions of asymptotic stability and can be used in the development of numerical methods for investigating the stability of systems equivalent to the considered difference inclusions. Using piecewise linear Lyapunov functions, an algebraic criterion of asymptotic stability is obtained. An example of a mechanical system leading to periodic selector-linear difference inclusion is considered.

Keywords: periodic selector-linear difference inclusions, asymptotic stability, Lyapunov functions, algebraic criterion of asymptotic stability

1. INTRODUCTION. STATEMENT OF THE PROBLEM

The study of discrete control systems in a number of cases leads to difference inclusions. In [9] a brief review of publications on this topic is given, see for example [1-3,5]. In [9] for periodic difference inclusions a necessary and sufficient condition of uniform asymptotic stability in the form of some limit relation is obtained on the basis of the variational approach. In [10], for periodic selector-linear difference inclusions, a class of time-periodic Lyapunov functions of quasi-quadratic form, as well as parametric classes of piecewise quadratic and piecewise linear Lyapunov functions were distinguished. With the help of these functions the necessary and sufficient conditions for asymptotic stability were obtained. This paper is a continuation of [9,10] and is devoted to obtaining new asymptotic stability criteria for periodic selector-linear difference inclusions based on the method of Lyapunov functions.

Consider periodic selector-linear difference inclusion

$$x(s+1) \in F(s,x), \;\; s = 0,1,..., \;\; x \in R^n, \tag{1.1}$$

where the set-valued map $F : R^{n+1} \to R^n$ has the form

$$F(s,x) = \{y : y = B(s)x, \; B(s) \in \Omega(s)\},$$

here $\Omega(s)$, $\Omega(s+N) = \Omega(s)$ ($s = 0,1,...,$ N is a natural number) is a convex, compact set of real $(n \times n)$ - matrices B. The sequence of vectors $\{x(s)\}$, satisfying for all $s = 0,1,...$ inclusion (1.1), is the solution of inclusion (1.1). The definitions of asymptotic stability, uniform asymptotic stability and uniform exponential stability of inclusion (1.1) are given in [9]. The equivalence of these properties for inclusion (1.1) is proved there. Keeping this in mind, further we will speak about asymptotic stability of inclusion (1.1).

The problem is to construct stability criteria for inclusion (1.1) using the discrete analogue of the direct Lyapunov method [4].

2. RESULTS

In [10] parametric classes of piecewise quadratic

$$V_M(s,x)=\max_{1\le j\le M}\left\langle l^j(s),x\right\rangle^2, \quad l^j(s+N)=l^j(s) \tag{2.1}$$

and piecewise linear

$$V_M(s,x)=\max_{1\le j\le M}\left|\left\langle l^j(s),x\right\rangle\right|, \quad l^j(s+N)=l^j(s) \tag{2.2}$$

Lyapunov functions were considered. In (2.1) and further we denote by $\langle\cdot,\cdot\rangle$ a scalar product of vectors. In [10] it was proved that for asymptotic stability of inclusion (1.1) it is necessary and sufficient that for some integer $M\ge n$ there exists periodic on s (period N) Lyapunov function (2.1) or (2.2) satisfying the condition

$$rank L(s)=n\le M,\ \ L(s)=(l^1(s),...,l^M(s)) \tag{2.3}$$

and the inequality

$$\max_{y\in F(s,x)} v_M(s+1,y)\le\theta v_M(s,x) \tag{2.4}$$

for all $x\in R^n$, $s\ge 0$ and some $\theta\,(0<\theta<1)$.

Consider periodic on s Lyapunov functions from the class of homogeneous forms of degree $2r$ (hereafter r is natural)

$$V_{M,r}(s,x)=\sum_{j=1}^{M}\left\langle l^j(s),x\right\rangle^{2r}, \quad l^j(s+N)=l^j(s), \quad s=0,1,... \tag{2.5}$$

The condition of positive definiteness of function (2.5), as well as for functions (2.1), (2.2), is condition (2.3). If this condition is fulfilled, the function $V_{M,r}(s,x)$ will be strictly convex on $x\in R^n$ for every $s\ge 0$.

Theorem 2.1:

Inclusion (1.1) is asymptotically stable iff there exists Lyapunov function (2.5) periodic in s *(of period* N *), its vectors* $l^j(s)$ $(l^j(s+N)=l^j(s))$, $j=\overline{1,M}$ *satisfy for all* $s\ge 0$ *condition (2.3), for the function inequality (2.4) is satisfied for some* $r\ge 1$.

Proof. Sufficiency. The sufficiency of the conditions of Theorem 2.1 is proved by using inequality (2.4) and estimates

$$\eta_1\|x\|^{2r}\le v_{M,r}(s,x)\le\eta_2\|x\|^{2r}, \ \ \eta_2\ge\eta_1>0$$

for the positively homogeneous strictly convex function (2.5), which were obtained in Lemma [8].

Necessity. To prove the necessity, we will use the statement of Theorem 3.2 in [10], according to which for asymptotically stable inclusion (1.1) there exists a piecewise quadratic Lyapunov function (2.1) satisfying (2.3) and the inequality

$$\max_{y\in F(s,x)} V_M(s+1,y)\le\theta_1 V_M(s,x) \tag{2.6}$$

for some $\theta_1\ (0<\theta_1<1)$.

We construct the function $V_{M,r}(s,x)$ by choosing as vectors $l^j(s)$, $j=\overline{1,M}$ in (2.5) the vectors defining the Lyapunov function $V_M(s,x)$.

Since the inequalities are true

$$\max_{1\le j\le M}\left\langle l^j(s),x\right\rangle^{2r}\le\sum_{j=1}^{M}\left\langle l^j(s),x\right\rangle^{2r}\le M\max_{1\le j\le M}\left\langle l^j(s),x\right\rangle^{2r},$$

then the estimates

$$V_M^r(s,x)\le V_{M,r}(s,x)\le MV_M^r(s,x), \quad x\in R^n,\ \ s=0,1,... \tag{2.7}$$

The inequality follows from (2.6) and (2.7)

$$\max_{y\in F(s,x)} V_{M,r}(s+1,y) \le M\theta_1^r V_{M,r}(s,x).$$

Choosing a positive integer $r \ge r_0$, where $r_0 = [\ln M/(-\ln\theta_1)]+1$, we obtain that the inequality $\max\limits_{y\in F(s,x)} v_{M,r}(s+1,y) \le \theta_2 v_{M,r}(s,x)$ is satisfied for the constructed function $v_{M,r}(s,x)$, since for $r \ge r_0$ the number $\theta_2 = M\theta_1^2$ satisfies the condition $0<\theta_2<1$. The periodicity of the function $v_M(s,x)$ implies the fulfillment of the equality $V_{M,r}(s+N,x) = V_{M,r}(s,x)$. Theorem 2.1 is proved. □

Lyapunov function (2.5) can be represented in the form

$$V_{M,r}(s,x) = \sum_{i=1}^{N_r} \alpha_i(s)\psi_i(x), \quad \alpha_i(s+N) = \alpha_i(s), \quad s = 0,1,..., \quad i = \overline{1,N_r}, \tag{2.8}$$

where $\psi_i(x),\ i=\overline{1,N_r}$ – all possible elementary monomials of degree $2r$ $(\psi_i(x) = x_1^{m_{1i}}...x_n^{m_{ni}},\ \sum\limits_{j=1}^{n} m_{ji} = 2r)$ $N_r = C_{n+2r-1}^{2r}$ – the total number of such monomials.

In accordance with the discrete analog of the direct Lyapunov method [4], the construction of Lyapunov function (2.8) for difference inclusion (1.1) is reduced to the search of the parameter vector $\alpha = (\alpha_i(x)),\ i=\overline{1,N_r}$, defining the solution of the set of inequalities

$$\delta_B(\alpha,s,x) = V_{M,r}(s+1,B(s)x) - V_{M,r}(s,x) < 0, \quad x \ne 0, \quad B(s)\in\Omega(s). \tag{2.9}$$

Equation (2.9) specifies the conditions of strict monotonic decreasing of the function $V_{M,r}(s,x)$ on the solutions of inclusion (1.1).

Considering (2.8) and (2.9) we obtain the inequalities

$$\delta_B(\alpha,s,x) = \sum_{i=1}^{N_p} (\alpha_i(s+1)\psi_i(B(s)x) - \alpha_i(s)\psi_i(x)) < 0, \quad x\ne 0, \quad B(s)\in\Omega(s). \tag{2.10}$$

Let $\|x\|_\infty = \max\limits_{1\le i\le n}|x_i|$ be cubic norm of vector x. Consider the problem of mathematical programming

$$\beta = \min_{\alpha\in G}\ \max_{0\le s\le N-1}\ \max_{\|x\|_\infty=1}\ \max_{B(s)\in\Omega(s)} \Delta_B(\alpha,s,x), \qquad G = \left\{\alpha : \sum_{i=1}^{N_r}\alpha_i^2 \le 1\right\}. \tag{2.11}$$

Using the scheme of the proof of Theorem 1 in [11], it can be shown that for existence of Lyapunov function (2.8), satisfying condition (2.10), it is necessary and sufficient that the solution of the problem (2.11) satisfies the inequality $\beta<0$. To solve the minimax problem (2.11), known methods of numerical solution can be used. The periodic components of the vector α, $\alpha_i(x)$, $i=\overline{1,N_r}$, can be searched, for example, in the form of Fourier series segments [11].

Through $\|D(s)\|_\infty = \max\limits_{1\le i\le M}\sum\limits_{j=1}^{M}|d_{ij}|$ denote the cubic norm of the square matrix $D = (d_{ij})_{i,j=1}^{M}$ of order M. With Lyapunov functions (2.2) we obtain the criterion of asymptotic stability of inclusion (1) in algebraic form.

Theorem 2.2:

For asymptotic stability of inclusion (1.1) it is necessary and sufficient that the following conditions are satisfied:

1. For some $p\ge 1$ there exist periodic $(n\times n)$- matrices $B_i(s)$ ($B_i(s+N) = B_i(s),\ i=\overline{1,p}$), satisfying the condition $\Omega(s)\subset co(B_1(s),...,B_p(s)),\ s=0,1,...$

2. There exist a number $M \ge n$, *periodic* $(n \times M)$*- matrix* $L(s)$ $(L(s+N)=L(s))$ *of rank* n, *and periodic* $(M \times M)$*-matrices* $D_i(s)$ $(D_i(s+N)=D_i(s))$, $i=\overline{1,p}$, *satisfying the conditions* $\|D_i(s)\|_\infty < 1$ *(*$i=\overline{1,p}$, $s=0,1,\ldots$*), such that the matrix relations are satisfied*

$$B_i'(s)L(s+1)=L(s)D_i'(s), \quad i=\overline{1,p}, \quad s=0,1,\ldots \tag{2.12}$$

In (2.12) and further the dash denotes the transpose operation.

Proof. Necessity. By Theorem 3.3 [10], for asymptotically stable inclusion (1.1), there exists piecewise linear Lyapunov function (2.2) satisfying condition (2.3) and inequality (2.4) for some θ $(0<\theta<1)$. It follows from Theorem 20.4 [12] that any non-empty closed bounded set can be approximated by a polyhedral convex set. Thus, for the set $\Omega(s)$ there exists a natural number $p \ge 1$ and such periodic $(n \times n)$- matrices $B_i(s)$ ($B_i(s+N)=B_i(s)$, $i=\overline{1,p}$), that $\Omega(s) \subset co(B_1(s),\ldots,B_p(s))$, and the sets $\Omega(s)$ and $co(B_1(s),\ldots,B_p(s))$ will differ from each other as little as possible. Also, the vectors $y_i(s)=B_i(s)x$, $i=\overline{1,p}$, $s \ge 0$ will be as close as desired to the elements of the set $F(s,x)$ in (1.1). Therefore, from (2.2) and (2.4) follow the inequalities

$$\max_{1\le f\le M}\left|\left\langle l^f(s+1),x\right\rangle,B_i(s)x\right| \le \theta \max_{1\le j\le M}\left|\left\langle l^j(s),x\right\rangle\right|, \quad i=\overline{1,p}, \quad s=0,1,\ldots, \quad x\in R^n,$$

hence the inequalities

$$\left|\left\langle B_i'(s)l^f(s+1),x\right\rangle,B_j(s)x\right| \le \max_{1\le j\le M}\left|\left\langle ql^j(s),x\right\rangle\right|, \ \ x\in R^n, \tag{2.13}$$

for all f,i and s $(f=\overline{1,M},\ i=\overline{1,p},\ s=0,1,\ldots)$. Applying Lemma [7] to (2.13) we obtain that for each f and i $(f=\overline{1,M},\ \ i=\overline{1,p})$ there exist periodic (period N) functions $\varphi_{fj}^i(s)$, $j=\overline{1,M}$, $s=0,1,\ldots,$ such that

$$B_i'(s)l^f(s+1)=\sum_{j=1}^{M} q\varphi_{fj}^i(s)l^j(s), \qquad \sum_{j=1}^{M}\left|\varphi_{fj}^i(s)\right| \le 1.$$

Introducing the periodic functions $d_{fj}^i(s)=q\varphi_{fj}^i(s)$, we come to the following equations

$$B_i'(s)l^f(s+1)=\sum_{j=1}^{M} d_{fj}^i(s)l^j(s),\ f=\overline{1,M},\ i=\overline{1,p},\ s=0,1,\ldots \tag{2.14}$$

Since $\sum_{j=1}^{M}\left|d_{fj}^i(s)\right| \le q$ for any f,i и s $(f=\overline{1,M},\ i=\overline{1,p},\ s=0,1,\ldots)$, then each of the $(M\times M)$-matrices $D_i(s)=(d_{fj}^i(s))_{f,j}^M$, $i=\overline{1,p}$, satisfies the condition $\|D_i(s)\|_\infty \le q<1$ for all $s=0,1,\ldots$ In matrix notation, vector equations (2.14) are equivalent to (2.12). The periodicity of the matrices $L(s)$ and $D_i(s)$ in (2.12) follows from the periodicity of the vectors $l^f(s)$, $f=\overline{1,M}$ in (2.2). The necessity is proved.

Sufficiency. According to the conditions of the theorem $\Omega(s)\subset co(B_1(s),\ldots,B_p(s))$, therefore, any vector $y\in F(s,x)$ in (1.1) can be represented as

$$y=\sum_{i=1}^{p}\lambda_i(s)B_i(s)x, \quad \lambda_i(s)\ge 0, \quad i=\overline{1,p}, \quad \sum_{i=1}^{p}\lambda_i(s)=1.$$

Determining by the matrix $L(s)$ in (2.12), satisfying condition (2.3), Lyapunov function (2.2), we obtain

$$\max_{y\in F(s,x)} v_M(s+1,y)=\max_{1\le i\le p} V_M(s+1,B_i(s)x(s))= \tag{2.15}$$

$$= \max_{1 \le i \le p} \max_{1 \le f \le M} \left| \left\langle B_i'(s) l^f(s+1), x(s) \right\rangle \right|.$$

Since matrix equalities (2.12) are equivalent to vector equalities (2.14), for any f, i, s ($f = \overline{1,M}$, $i = \overline{1,p}$, $s = 0,1,\ldots$) the following relations are true

$$\left| \left\langle B_i'(s) l^f(s+1), x \right\rangle \right| = \left| \sum_{j=1}^{M} d_{fj}^i(s) \left\langle l^j(s), x \right\rangle \right| \le \sum_{j=1}^{M} \left| d_{fj}^i(s) \right| \left| \left\langle l^j(s), x \right\rangle \right| \le V_M(s,x) \sum_{j=1}^{M} \left| d_{fj}^i(s) \right|.$$

Hence

$$\max_{1 \le f \le M} \left| \left\langle B_i'(s) l^f(s+1), x \right\rangle \right| \le \left\| D_i(s) \right\|_\infty V_M(s,x), \quad i = \overline{1,p}.$$

Therefore, the equality is valid

$$\max_{1 \le i \le p} \max_{1 \le f \le M} \left| \left\langle B_i'(s) l^f(s+1), x \right\rangle \right| \le \theta V_M(s,x), \tag{2.16}$$

where $\theta = \max_{0 \le s \le N-1} \max_{1 \le i \le p} \left\| D_i(s) \right\|_\infty < 1$.

It follows from (2.15) and (2.16) that the function $V_M(s,x)$ satisfies inequality (2.4). Therefore, by Theorem 3.3 [9] inclusion (1.1) will be asymptotically stable. Theorem 2.2 is proved. □

4. EXAMPLE

Consider a pendulum of length l, whose suspension axis makes vertical harmonic oscillations with small amplitude α and frequency ω. The differential equation of motion of the pendulum is [6]

$$\frac{d^2 z}{dt^2} = -\frac{g}{l\omega^2} \left(1 + \frac{\alpha\omega^2}{g} \sin t \right) z = 0, \tag{3.1}$$

where g is acceleration of free fall. As for a pendulum with a fixed pendulum axis, the vertical is the equilibrium position. But in contrast to the case of a pendulum with a fixed axis, this equilibrium position can be either stable or unstable, depending on the value of ω.

Equation (3.1) is a special case of an equation of the more general form [6]

$$\frac{d^2 z}{dt^2} + bf(t) z = 0, \tag{3.2}$$

where $f(t)$ is a periodic function of time (with period $T > 0$), and $b \in I$, $I = [b_1, b_2]$ is a certain parameter. Just as in the examples in [9,10], it can be shown that equation (3.1) can be represented as a second-order system of differential equations with periodic coefficients, depending on the parameter $b \in I$. The discrete analog of this system is equivalent to inclusion (1.1).

4. CONCLUSION

For periodic selector-linear difference inclusion (1.1) the asymptotic stability criterion is obtained using Lyapunov functions (2.5). Such functions can be used in the development of numerical methods for investigating the stability of systems equivalent to difference inclusion (1.1). It is shown that the construction of Lyapunov functions (2.5) reduces to the solution of the minimax problem (2.11). The algebraic criterion of asymptotic stability is established using piecewise linear Lyapunov functions. The example of a mechanical problem leading to the consideration of a periodic difference inclusion is given.

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