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# A Cauchy Problem for Functional Inclusions with Causal Operators in Banach Spaces

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**Abstract:** In this paper, we consider the Cauchy problem for functional inclusions containing the sum of n causal single-valued operators and a multivalued causal operator in Banach spaces. The peculiarity of single-valued operators is that, starting from the second term, each operator is represented by an integral with a variable upper limit of the previous term. Such functional inclusions generalize the Cauchy problem for semilinear differential equations and inclusions of arbitrary order n, as well as a Cauchy-type problem in the case of inclusions and equations of fractional order not exceeding n. To solve the problem, we will apply the theory of topological degree for multivalued condensing mappings. To prove the existence of a solution, we will construct a resolving multivalued operator in the space of continuous functions corresponding to the problem. Based on the properties of the resolving operator, we will prove a theorem on the existence of solutions to this problem.

***Keywords:*** causal multioperator, functional inclusion, Cauchy type problem, differential inclusion, fractional derivative, measure of non-compactness, fixed point, topological degree, condensing multioperator

## 1. INTRODUCTION

The study of boundary value problems for differential equations and inclusions is a complex and extremely important section of modern mathematical science, which has numerous applications and currently attracts the attention of many scientists. Many physical, economic, biological and engineering problems, primarily related to the flow of processes in dynamical systems, lead to the need for new research in the field of differential equations and inclusions in abstract spaces. In turn, the development of the theory of differential inclusions is due to the fact that they are a convenient and natural tool for describing control systems of various classes, systems with discontinuous characteristics studied in various sections of optimal control theory, mathematical physics, radio physics, acoustics, etc. One of the best tools for studying this kind of problem, provide multivalued and nonlinear analysis methods that stand out as very powerful, efficient and useful.

Recently, the attention of many researchers (see [1], [2], [3], [4] and references therein) has been attracted to generalizations of differential equations and inclusions, namely to the class of functional equations and inclusions with causal operators. The term of a causal or Volterra operator in the sense of A.N. Tikhonov (see [5]), was used in mathematical physics to solve problems of differential equations, integro-differential equations, functional-differential equations with a finite or infinite delay, integral equations of Volterra type, functional equations of a neutral type, etc. (see, for example, [6]). The papers [7], [8], [9], [10], [11] among others are devoted to the study of equations and inclusions with causal operators of

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various types, theorems on the existence of solutions, description of qualitative properties of solutions and various applications.

In this paper we study a Cauchy type problem in Banach spaces for various classes of functional inclusions with causal operators. Basing on the topological degree theory for condensing multimaps we prove a global theorem on the existence of trajectories for systems governed by functional inclusions. As an application, we obtain generalizations of existence theorems for solutions of a Cauchy type problem for semilinear differential inclusions an arbitrary order $n$ and semilinear differential inclusions of a fractional order $n-1<q<n$.

## 2. PRELIMINARIES

### *2.1. Measures of Noncompactness*

We denote a Banach space by $\mathcal{E}$ and introduce the following notation:

- $P(\mathcal{E})=\{A\subseteq\mathcal{E}:A\neq\varnothing\}$ is the collection of all non-empty subsets of $\mathcal{E}$;
- $Pb(\mathcal{E})=\{A\in P(\mathcal{E}):A$ is bounded$\}$ ;
- $Pv(\mathcal{E})=\{A\in P(\mathcal{E}):A$ is convex$\}$ ;
- $C(\mathcal{E})=\{A\in P(\mathcal{E}):A$ is closed$\}$ ;
- $Cv(\mathcal{E})=Pv(\mathcal{E})\cap C(\mathcal{E})$;
- $K(\mathcal{E})=\{A\in P(\mathcal{E}):A$ is compact$\}$ ;
- $Kv(\mathcal{E})=Pv(\mathcal{E})\cap K(\mathcal{E})$.

**Definition 2.1:**
*(See [12]). Let $(\mathcal{A},\geq)$ be a partially ordered set. A function $\beta:Pb(\mathcal{E})\to\mathcal{A}$ is called the measure of noncompactness (MNC) in $\mathcal{E}$ if for each $\Omega\in Pb(\mathcal{E})$ we have:*

$$\beta(\overline{\mathrm{co}}\,\Omega)=\beta(\Omega),$$

*where $\overline{\mathrm{co}}\,\Omega$ denotes the closure of the convex hull of $\Omega$.*

A measure of noncompactness $\beta$ is called:

1) *monotone*, if for each $\Omega_0,\Omega_1\in Pb(\mathcal{E})$, from $\Omega_0\subseteq\Omega_1$ follows $\beta(\Omega_0)\leq\beta(\Omega_1)$;
2) *nonsingular*, if for each $a\in\mathcal{E}$ and each $\Omega\in Pb(\mathcal{E})$ we have $\beta(\{a\}\cup\Omega)=\beta(\Omega)$.

If $\mathcal{A}$ is a cone in a Banach space, the MNC $\beta$ is called:

3) *regular*, if $\beta(\Omega)=0$ is equivalent to the relative compactness of $\Omega\in Pb(\mathcal{E})$;
4) *real*, if $\mathcal{A}$ is the set of all real numbers $\mathbb{R}$ with the natural ordering.

As the example of a real MNC obeying all above properties, we can consider the Hausdorff MNC $\chi(\Omega)$:

$$\chi(\Omega)=\inf\{\varepsilon>0,\text{for which }\Omega\text{ has a finite }\varepsilon\text{-net in }\mathcal{E}\ \}.$$

As other examples, consider the measures of noncompactness defined in the space of continuous functions $C([a,b];\mathcal{E})$ with values in the Banach space $\mathcal{E}$:

(1) *the modulus of fiber noncompactness:*

$$\varphi(\Omega)=\sup_{t\in[a,b]}\chi_{\mathcal{E}}(\Omega(t)),$$

where $\chi_{\mathcal{E}}$ is the Hausdorff MNC in $\mathcal{E}$ and $\Omega(t)=\{y(t):y\in\Omega\}$;

(2) *the fading modulus of fiber noncompactness:*

$$\gamma(\Omega)=\sup_{t\in[a,b]}e^{-Lt}\chi_{\mathcal{E}}(\Omega(t)),$$

where $L>0$ is a given number;

(3) *the modulus of equicontinuity:*

$$\mathrm{mod}_C(\Omega) = \lim_{\delta\to 0} \sup_{y\in\Omega} \max_{|t_1-t_2|\le\delta} \|y(t_1) - y(t_2)\|.$$

These measures of noncompactness satisfy all the above properties, except for the regularity.

### 2.2. A Family of Cosine Operator Functions

**Definition 2.2:**
*(See [13]). A family of bounded operators $\{C(t)\}_{t\in\mathbb{R}}$ in a Banach space $E$ is called a strongly continuous family of cosine operator functions if:*

*(1)* $C(0) = I$;
*(2)* $C(s+t) + C(s-t) = 2C(s)C(t)$ *for all* $t, s \in \mathbb{R}$;
*(3)* $t \to C(t)x$ *is continuous for all* $x \in E$.

The family of strongly continuous sine operator functions associated with the family of cosine operator functions $\{C(t)\}_{t\in\mathbb{R}}$ is the family of operators $\{S(t)\}_{t\in\mathbb{R}}$ such that

$$S(t)x = \int_0^t C(s)x ds,\ x \in E, t \in \mathbb{R}.$$

The operator $A$ is a generator a family of cosine operator functions $\{C(t)\}_{t\in\mathbb{R}}$ if:

$$Ax = \frac{d^2}{dt^2} C(t)x|_{t=0},$$

for all $x \in D(A)$ for which the last expression is well defined.

### 2.3. Multivalued Maps

Let $X$ be a metric space and $Y$ be a normed space. Let us recall some notations (see, for example, [14], [15]).

**Definition 2.3:**
*A multivalued map (multimap) $\mathcal{F} : X \to P(Y)$ is said to be upper semicontinuous (u.s.c.) at a point $x \in X$, if for every open set $V \subset Y$ such that $\mathcal{F}(x) \subset V$, there exists a neighborhood $U(x)$ of $x$ such that $\mathcal{F}(U(x)) \subset V$.*

**Definition 2.4:**
*A multivalued map $\mathcal{F} : X \to P(Y)$ is called closed if its graph $G_{\mathcal{F}} = \{(x, y) : x \in X, y \in \mathcal{F}(x)\}$ is a closed subset of $X \times Y$.*

**Definition 2.5:**
*A multivalued map $\mathcal{F} : X \to P(Y)$ is called quasicompact if its restriction to each compact subset $A \subset X$ is compact.*

**Definition 2.6:**
*For a given $p \ge 1$, a multifunction $G : [0, T] \to K(Y)$ is called:*

- *$L^p$–integrable* if it admits an $L^p$–Bochner integrable selection, i.e., there exists a function $g \in L^p([0, T]; Y)$ such that $g(t) \in G(t)$ for a.e. $t \in [0, T]$;
- *$L^p$–integrably bounded* if there exists a function $\xi \in L^p([0, T])$ such that

$$\|G(t)\|_Y \le \xi(t)$$

for a.e. $t \in [0, T]$.

 

**Definition 2.7:**
*A multimap $\mathcal{F} : X \subseteq \mathcal{E} \to K(\mathcal{E})$ is called condensing with respect to a MNC $\beta$ (or $\beta$–condensing) if for each bounded set $\Omega \subseteq X$ which is not relatively compact, we have:*

$$\beta(\mathcal{F}(\Omega)) \not\geq \beta(\Omega).$$

Let $\mathcal{D} \subset \mathcal{E}$ be a non-empty closed convex subset, $V$ be a non-empty bounded open subset of $\mathcal{D}$, $\beta$ is a monotone nonsingular MNC in $\mathcal{E}$ and $\mathcal{F} : \overline{V} \to Kv\,(\mathcal{D})$ be a u.s.c. $\beta$-condensing map such that $x \notin \mathcal{F}\,(x)$ for all $x \in \partial V$, where $\overline{V}$ and $\partial V$ denote the closure and the boundary of the set $V$ in the relative topology of $\mathcal{D}$.

In such a setting, the (relative) topological degree

$$\deg_{\mathcal{D}}\left(i - \mathcal{F}, \overline{V}\right)$$

of the corresponding vector field $i - \mathcal{F}$, satisfying the standard properties is well defined (see, for example, [14], [15]). In particular, the condition

$$\deg_{\mathcal{D}}\left(i - \mathcal{F}, \overline{V}\right) \neq 0$$

implies that the fixed points set $\operatorname{Fix}\mathcal{F} = \{x : x \in \mathcal{F}(x)\}$ is a nonempty subset of $V$.

Application of topological degree theory leads to the following fixed point principles, which will be used in the that follows.

**Theorem 2.1:**
*(See [14], Corollary 3.3.1). Let $\mathcal{M}$ be a convex closed bounded subset of $\mathcal{E}$ and $\mathcal{F} : \mathcal{M} \to Kv(\mathcal{M})$ a $\beta$–condensing multimap, where $\beta$ is a monotone nonsingular MNC in $\mathcal{E}$. Then the fixed point set $\operatorname{Fix}\mathcal{F}$ is non-empty.*

**Theorem 2.2:**
*(See [14], Theorem 3.3.4). Let $V \subset \mathcal{D}$ be a bounded open neighborhood of a point $a \in V$ and $\mathcal{F} \colon \overline{V} \to Kv(\mathcal{D})$ a u.s.c. $\beta$-condensing multimap, where $\beta$ is a monotone nonsingular MNC in $\mathcal{E}$, satisfying the boundary condition*

$$x - a \notin \lambda(\mathcal{F}(x) - a)$$

*for all $x \in \partial V$ and $0 < \lambda \leq 1$. Then $\operatorname{Fix}\mathcal{F} \neq \emptyset$ is a non-empty compact set.*

### *2.4. Causal Multioperators*

Let $E$ be a separable Banach space. By $L^p\left([0,T];E\right)$, $1 \leq p \leq \infty$, we denote the Banach space of all Bochner summable functions $f : [0,T] \to E$ with the usual norm.

For each subset $\mathcal{N} \subset L^p\left([0,T];E\right)$ and $\tau \in (0,T)$ we define restriction $\mathcal{N}$ on $[0,\tau]$ as

$$\mathcal{N}\mid_{[0,\tau]} = \{f\mid_{[0,\tau]} : f \in \mathcal{N}\}.$$

**Definition 2.8:**
*A multivalued map $\mathcal{Q} : C\left([0,T];E\right) \multimap L^p\left([0,T];E\right)$ is said to be a causal multioperator, if for each $\tau \in (0,T]$ and for every $u, v \in C\left([0,T];E\right)$ the condition $u\mid_{[0,\tau]} = v\mid_{[0,\tau]}$ implies that $\mathcal{Q}(u)\mid_{[0,\tau]} = \mathcal{Q}(v)\mid_{[0,\tau]}$.*

Let us give examples of causal multioperators.

**Example 2.1:**
*We assume that the multimap $F : [0,T] \times E \to Kv\,(E)$ satisfies the following conditions:*

$(F1)$ *for each $\phi \in E$ the multifunction $F\,(\cdot,\phi) : [0,T] \to Kv\,(E)$ admits a measurable selection;*

$(F2)$ *for a.e.* $t \in [0,T]$ *the multifunction* $F(t,\cdot) : E \to Kv(E)$ *is u.s.c.;*
$(F3)$ *there exists a function* $\alpha \in L^p_+[0,T],\ 1 \le p \le \infty$, *such that*

$$\|F(t,\phi)\|_E := \sup\{\|z\|_E : z \in F(t,\phi)\} \le \alpha(t)(1 + \|\phi\|_E)$$

*for a.e.* $t \in [0,T]$ *and* $\phi \in E$.

*From above conditions* $(F1)-(F3)$ *it follows that the multimap* $\mathcal{P}_F : C([0;T];E) \to P(L^p([0,T];E))$, *given in the following way*

$$\mathcal{P}_F(x) = \{f \in L^p([0,T];E) : f(t) \in F(t,x(t)) \text{ for a.e. } t \in [0,T]\}$$

*is well defined (see, for example, [14], [15]). It is clear that the multioperator* $\mathcal{P}_F$ *is causal.*

**Example 2.2:**
*Let* $F : [0,T] \times E \to Kv(E)$ *be a multimap satisfying conditions* $(F1)-(F3)$ *from Example 2.1. Suppose that* $\{K(t,s) : 0 \le s \le t \le T\}$ *is a continuous (with respect to the corresponding norm) family of bounded linear operators in* $E$ *and* $m \in L^1([0,T];E)$ *is a given function. Consider the Volterra integral multioperator* $\mathcal{V} : C([0,T];E) \multimap L^1([0,T];E)$ *defined as*

$$\mathcal{V}(u)(t) = m(t) + \int_0^t K(t,s)F(s,u)ds,$$

*i.e.,*

$$\mathcal{V}(u) = \left\{ y \in L^1([0,T];E) : y(t) = m(t) + \int_0^t K(t,s)f(s)ds : f \in \mathcal{P}_F(u) \right\}.$$

*It is also clear that the multioperator* $\mathcal{V}$ *is causal.*

## 3. CAUCHY TYPE PROBLEM FOR FUNCTIONAL INCLUSIONS WITH THE CAUSAL OPERATORS

We will assume that the causal operator $\mathcal{Q} : C([0,T];E) \to C(L^p([0,T];E))$ satisfies the following conditions:

$(\mathcal{Q}1)$ the operator $\mathcal{Q}$ is weakly closed in the following sense: conditions $\{u_n\}_{n=1}^{\infty} \subset C([0,T];E)$, $\{f_n\}_{n=1}^{\infty} \subset L^p([0,T];E)$, $1 \le p \le \infty$, $f_n \in \mathcal{Q}(u_n)$, $n \ge 1$, $u_n \to u_0$, $f_n \xrightarrow{L^1} f_0$ implies $f_0 \in \mathcal{Q}(u_0)$;
$(\mathcal{Q}2)$ there exists a function $\alpha \in L^{\infty}_+([0,T])$ such that

$$\|\mathcal{Q}(u)(t)\|_E \le \alpha(t)(1 + \|u(t)\|_E), \quad \text{for a.e. } t \in [0,T],$$

for all $u \in C([0,T];E)$;
$(\mathcal{Q}3)$ there exists a function $\omega : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that
- $(\omega 1)$ for all $x \in \mathbb{R}_+ : \omega(\cdot,x) \in L^p_+([0,T])$, $1 \le p \le \infty$;
- $(\omega 2)$ for a.e. $t \in [0,T]$ a function $\omega(t,\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, nondecreasing and quasihomogeneous in the sense that $\omega(t,\lambda x) \le \lambda\omega(t,x)$ for all $x \in \mathbb{R}_+$ and $\lambda \ge 0$;
- $(\omega 3)$ for each bounded set $\Delta \subset C([0,T];E)$ we have

$$\chi(\mathcal{Q}(\Delta)(t)) \le \omega\left(t, \sup_{s\in[0,t]} \varphi(\Delta(s))\right) \text{ for a.e. } t \in [0,T],$$

where the set $\Delta(s) = \{y(s) : y \in \Delta\} \subset E$ and $\varphi$ is the modulus of fiber noncompactness in $C([0,T];E)$.

 

Note that the condition $(\omega 2)$ means that $\omega(t,0)=0$ for a.e. $t\in[0,T]$ and as an example of such a function we can consider $\omega(t,x)=k(t)\cdot x$, where $k(\cdot)\in L^p_+([0,T])$.

Consider a linear operator $\mathcal{S}: L^p([0,T];E)\to C([0,T];E)$, which is causal in the sense that for every $\tau\in(0,T]$ and $f,g\in L^p([0,T];E)$ condition $f(t)=g(t)$ for a.e. $t\in[0,\tau]$ implies $(\mathcal{S}f)(t)=(\mathcal{S}g)(t)$ for all $t\in[0,\tau]$. Following [14], we impose the next conditions on operator $\mathcal{S}$ :

$(\mathcal{S}1)$ for $1\le p<\infty$ there exist $D\ge 0$ such that

$$\|\mathcal{S}f(t)-\mathcal{S}g(t)\|_E^p\le D\int_0^t\|f(s)-g(s)\|_E^p ds$$

for all $f,g\in L^p([0,T];E)$ and $0\le t\le T$;
if $p=\infty$ there exist $D_1\ge 0$ such that

$$\|\mathcal{S}f(t)-\mathcal{S}g(t)\|_E\le D_1\int_0^t\|f(s)-g(s)\|_E ds$$

for all $f,g\in L^\infty([0,T];E)$ and $0\le t\le T$.

$(\mathcal{S}2)$ for an arbitrary compact set $K\subset E$ and a sequence $\{f_n\}_{n=1}^\infty\subset L^p([0,T];E)$, $1\le p\le\infty$, such that $\{f_n(t)\}_{n=1}^\infty\subset K$ for all $t\in[0,T]$ the weak convergence $f_n\overset{L^1}{\rightharpoonup}f_0$ implies $\mathcal{S}f_n\to\mathcal{S}f_0$ in $C([0,T];E)$.

Also we suppose that $\mathcal{S}$ satisfies the relation:

$(\mathcal{S}3)$ $\quad(\mathcal{S}f)(0)=0$ for each function $f\in L^p([0,T];E)$.

Notice, that condition $(\mathcal{S}1)$ implies that the operator $\mathcal{S}$ satisfies the Lipschitz condition:

$(\mathcal{S}1')$ $\|\mathcal{S}f-\mathcal{S}g\|_C\le D\|f-g\|_{L^1}$.

Consider following important examples of special cases of the operator $\mathcal{S}$.

(i) The operator $\mathcal{M}_1: L^1([0,T];E)\to C([0,T];E)$ defined as

$$\mathcal{M}_1 f(t)=\int_0^t e^{A(t-s)}f(s)ds,$$

where a closed linear operator $A: D(A)\subset E\to E$ is the infinitesimal generator of a uniformly bounded $C_0$-semigroup $\{e^{At}\}_{t\ge 0}$.

**Remark 3.1:**
*The Laplace transform for a real-valued function $f$*

$$\mathcal{L}[f](\lambda)=\int_0^\infty e^{-\lambda s}f(s)ds,\ \lambda\in\mathbb{R},\lambda>0,$$

*is a special cases of the operator $\mathcal{S}$. Notice, that the inverse Laplace transform is a causal operator also and satisfy conditions $(\mathcal{S}1)-(\mathcal{S}3)$.*

(ii) The operator $\mathcal{M}_2: L^1([0,T];E)\to C([0,T];E)$ defined as

$$\mathcal{M}_2 f(t)=\int_0^t(t-s)sh(At)\ f(s)ds,$$

where a closed linear operator $A: D(A)\subset E\to E$ is a generator of a uniformly bounded family of strongly continuous cosine operator functions $\{ch(At)\}_{t\ge 0}$.

(iii) The operator $L_1 : L^p([0,T];E) \to C([0,T];E)$, $p > 1/q$, defined as

$$L_1 f(t) = \int_0^t (t-s)^{q-1} \mathcal{T}_1(t-s) f(s) ds, \quad 0 < q < 1,$$

where

$$\mathcal{T}_1(t) = q \int_0^\infty \theta \xi_q(\theta) U(t^q \theta) d\theta, \qquad \xi_q(\theta) = \frac{1}{q} \theta^{-1-\frac{1}{q}} \Psi_q(\theta^{-1/q}),$$

$$\Psi_q(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-qn-1} \frac{\Gamma(nq+1)}{n!} \sin(n\pi q), \theta \in \mathbb{R}^+,$$

and $A : D(A) \to E$ is a closed linear operator in $E$ generating a uniformly bounded $C_0$-semigroup $\{U(t)\}_{t \ge 0}$.

(iv) The operator $L_2 : L^p([0,T];E) \to C([0,T];E)$, $p \ge 1$, defined as

$$L_2 f(t) = \int_0^t (t-s)^{q-1} \mathcal{T}_2(t-s) f(s) ds, \quad 0 < q \le 1,$$

where

$$\mathcal{T}_2(t) = q \int_0^\infty \theta \xi_q(\theta) S(t^q \theta) d\theta, \quad \xi_q(\theta) = \frac{1}{q} \theta^{-1-\frac{1}{q}} \Psi_q(\theta^{-1/q}),$$

$$\Psi_q(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-qn-1} \frac{\Gamma(nq+1)}{n!} \sin(n\pi q), \theta \in \mathbb{R}^+.$$

and $A : D(A) \subset E \to E$ is a closed linear operator in $E$ generating a uniformly bounded family of strongly continuous cosine operator functions $\{C(t)\}_{t \ge 0}$.

Let a closed linear operator $A : D(A) \subset E \to E$ be a generator of a uniformly bounded family operator functions $\{\mathcal{G}_0(t)\}_{t \ge 0}$. We suppose the following conditions:

($\mathcal{G}1$) For each $t \in [0,T]$, $\mathcal{G}_0(\cdot)$ are linear bounded operators, more precisely, for each $x \in E$

$$\|\mathcal{G}_0(t)x\|_E \le M \|x\|_E$$

where $M = \sup\{\|\mathcal{G}_0(t)\|; t \in [0, +\infty)\}$.

($\mathcal{G}2$) The operator function $\mathcal{G}_0(\cdot)$ are strongly continuous, i.e., functions $t \in [0,T] \to \mathcal{G}_0(t)x$ are continuous for each $x \in E$.

Consider the operator functions $\mathcal{G}_i$, $i = 1, 2, ..., n-1$ defined as

$$\mathcal{G}_i(t) = \int_0^t \mathcal{G}_{i-1}(s) ds.$$

**Lemma 3.1:**
*Under conditions ($\mathcal{G}1$), ($\mathcal{G}2$) the operator functions $\mathcal{G}_i$, $(i = 1, \cdots, n-1)$ possess the following properties:*

1) *for each $t \in [0,T]$, $\mathcal{G}_i$ are linear bounded operators, more precisely, for each $x \in E$ we have*

$$\|\mathcal{G}_i(t)x\|_E \le M \|x\|_E \frac{t^i}{i!};$$

 

2) *the operator functions* $t^i\mathcal{G}_i(\cdot)$ *are strongly continuous, i.e., functions* $t \in [0,T] \to t^i\mathcal{G}_i(t)x$ *are continuous for each* $x \in E$.

*Proof*
1) For every fixed $t \geq 0$ and for each $x \in E$, we prove the statement using the principle of mathematical induction. Show it is true for $i = 1$. By using condition $(\mathcal{G}1)$ we have

$$\|\mathcal{G}_1(t)x\|_E \leq \int_0^t \|\mathcal{G}_0(s)x\|_E \, ds \leq M\|x\|_E \, t.$$

Let us assume that

$$\|\mathcal{G}_k(t)x\|_E \leq M \, \|x\|_E \, \frac{t^k}{k!}$$

is true. Now, we prove that it is true for $i = k+1$.

$$\|\mathcal{G}_{k+1}(t)x\|_E \leq \int_0^t \|\mathcal{G}_k(s)x\|_E \, ds \leq M\|x\|_E \, \frac{1}{k!}\int_0^t s^k \, ds = M \, \|x\|_E \, \frac{t^{k+1}}{(k+1)!}.$$

2) For any $x \in E, \; i = 1, \cdots, n-1$, and any $0 \leq t_1 \leq t_2$ we have

$$\|\mathcal{G}_i(t_1)x - \mathcal{G}_i(t_2)x\|_E \leq \int_{t_1}^{t_2} \|\mathcal{G}_{i-1}(s)x\|_E ds \leq$$
$$\int_{t_1}^{t_2} M \, \|x\|_E \, \frac{s^{i-1}}{(i-1)!} ds \leq M \, \|x\|_E \, \frac{(t_2-t_1)^i}{i!}.$$

Then if $t_2 \to t_1$ we have $\|\mathcal{G}_i(t_1)x - \mathcal{G}_i(t_2)x\|_E \to 0$ for every $x \in E$.

□

By using Lemma 3.1 it is easy to see that the operator functions $\mathcal{G}_i, \; i = 0, 1, \cdots, n$, are satisfy conditions $(\mathcal{S}1) - (\mathcal{S}3)$.

Consider a system governed by a functional inclusion with causal operators $\mathcal{Q}$ and $\mathcal{S}$, of the following form:

$$x(t) \in \mathcal{G}_0(t)x_0 + \mathcal{G}_1(t)x_1 + \cdots + \mathcal{G}_{n-1}(t)x_{n-1} + \mathcal{S} \circ \mathcal{Q}(x)(t), \; t \in [0,T] \tag{3.1}$$

$$x(0) = x_0, \quad x'(0) = x_1, \quad \cdots, \quad x^{(n-1)}(0) = x_{n-1}. \tag{3.2}$$

**Definition 3.1:**
*A function* $x \in C([0,T]; E)$ *is called a mild solution to problem* (3.1)-(3.2)*, if it satisfies inclusion* (3.1) *and condition* (3.2).

Consider the multioperator $\Gamma : C([0,T]; E) \multimap C([0,T]; E)$ defined as

$$\Gamma(x) = \{x \in C([0,T]; E) : x(t) = \mathcal{G}_0(t)x_0 + \mathcal{G}_1(t)x_1 + \cdots + \mathcal{G}_{n-1}(t)x_{n-1} + \mathcal{S} \circ \mathcal{Q}(x)\}.$$

It is clear that if the function $x$ is a fixed point of the multioperator $\Gamma$, then $x$ is a solution to the problem (3.1) - (3.2), therefore, our goal is to prove the existence of fixed point of the multioperator $\Gamma$.

 

**Definition 3.2:**
*A sequence of functions* $\{\xi_n\} \subset L^p([0,T];E)$ *is called* $L^p$*–semicompact if it is* $L^p$*–integrably bounded, i.e.,*

$$\|\xi_n(t)\|_E \leq \upsilon(t) \text{ for a.e. } t \in [0,T] \text{ and for all } n = 1, 2, ...,$$

*where* $\upsilon \in L^p([0,T])$*, and the set* $\{\xi_n(t)\}$ *is relatively compact in* $E$ *for a.e.* $t \in [0,T]$.

**Lemma 3.2:**
*(See. [14], Proposition 4.2.1.). Every* $L^p$*–semicompact sequence is weakly compact in* $L^1([0,T];E)$.

We need the following properties of the multioperator $\mathcal{S} \circ \mathcal{Q}$. Since for every $1 < p \leq \infty : L^p([0,T];E) \subset L^1([0,T];E)$, we can formulate a modification of Theorem 5.1.1 from [14] in the following form.

**Lemma 3.3:**
*Let* $\mathcal{S} : L^p([0,T];E) \to C([0,T];E)$ *be an operator satisfying conditions* $(\mathcal{S}1)$ *and* $(\mathcal{S}2)$*. Then for every* $L^p$*-semicompact sequence* $\{f_n\}_{n=1}^{\infty} \subset L^p([0,T];E)$*, the sequence* $\{\mathcal{S}f_n\}_{n=1}^{\infty}$ *is relatively compact in* $C([0,T];E)$ *and, moreover, the weak convergence* $f_n \stackrel{L^1}{\rightharpoonup} f_0$ *implies that* $\mathcal{S}f_n \to \mathcal{S}f_0$ *in* $C([0,T];E)$.

**Theorem 3.1:**
*(See [4]). Let the multioperator* $\mathcal{Q}$ *satisfy conditions* $(\mathcal{Q}1)$*–*$(\mathcal{Q}3)$ *and the operator* $\mathcal{S}$ *satisfy* $(\mathcal{S}1)$, $(\mathcal{S}2)$*. Then the composition* $\mathcal{S} \circ \mathcal{Q} : C([0,T];E) \multimap C([0,T];E)$ *is a u.s.c. multimap with compact values.*

Let us proceed to finding conditions under which the multioperator $\mathcal{S} \circ \mathcal{Q}$ will be condensing with respect to a corresponding MNC. For this we need the following statements.

**Lemma 3.4:**
*(See [4]). Let a sequence of functions* $\{f_n\}_{n=1}^{\infty} \subset L^p([0,T];E)$ *be* $L^p$*-integrally bounded and there exist a function* $\upsilon \in L_+^p([0,T])$ *such that*

$$\chi\left(\{f_n(t)\}_{n=1}^{\infty}\right) \leq \upsilon(t) \text{ for a.e. } t \in [0,T].$$

*If an operator* $\mathcal{S}$ *satisfies conditions* $(\mathcal{S}1)$ *and* $(\mathcal{S}2)$*, then for* $1 \leq p < \infty$ *we have*

$$\chi\left(\{\mathcal{S}f_n(t)\}_{n=1}^{\infty}\right) \leq \left(4^p D \int_0^t \upsilon^p(s)ds\right)^{1/p},$$

*and for* $p = \infty$

$$\chi\left(\{\mathcal{S}f_n(t)\}_{n=1}^{\infty}\right) \leq 2D_1 \int_0^t \upsilon(s)ds,$$

*where* $D, D_1$ *are the constants from condition* $(\mathcal{S}1)$.

Consider the measure of noncompactness $\nu$ in the space $C([0,T];E)$ with values in the cone $\mathbb{R}_+^2$. On a bounded subset of $\Omega \subset C([0,T];E)$ we define the values of $\nu$ as follows:

$$\nu(\Omega) = \left(\gamma\left(\Omega\right), mod_C\left(\Omega\right)\right),$$

where $mod_C$ is the modulus of equicontinuity, $\gamma$ is the fading modulus of fiber noncompactness

$$\gamma(\Omega) = \sup_{t \in [0,T]} e^{-Lt} \chi(\Omega(t)).$$

The constant $L > 0$ is chosen so that

$$\max\{q_1, q_2\} < 1,$$

where

$$q_1 = \sup_{t\in[0,T]} \left( 4D^{1/p} \left( \int_0^t e^{-pL(t-s)} \omega^p (s, 1)\, ds \right)^{1/p} \right),$$

$$q_2 = \sup_{t\in[0,T]} \left( 2D_1 \int_0^t e^{-L(t-s)} \omega (s, 1)\, ds \right),$$

where the constants $D, D_1$ are from condition $(\mathcal{S}1)$, $\omega$ is a function from condition $(\mathcal{Q}3)$.

It is easy to see that the MNC $\nu$ is monotone, nonsingular, and algebraically semi-additive. It follows from the Arzela-Ascoli theorem that it is also regular.

**Theorem 3.2:**
*Let a causal multioperator $\mathcal{Q} : C([0,T]; E) \multimap L^p ([0,T]; E)$ satisfy conditions $(\mathcal{Q}2)$ and $(\mathcal{Q}3)$ and for a causal operator $\mathcal{S} : L^p ([0,T]; E) \to C ([0,T]; E)$ the conditions $(\mathcal{S}1)$–$(\mathcal{S}3)$ be satisfied. Then the multioperator $\Gamma$ is $\nu$-condensing.*

*Proof*
By Lemma 3.1 it is suffices to prove the assertion of the theorem for the multioperator $\mathcal{S} \circ \mathcal{Q}$. Let $\Omega \subset C([0,T]; E)$ be a bounded set such that

$$\nu (\mathcal{S} \circ \mathcal{Q} (\Omega)) \geq \nu (\Omega) . \tag{3.3}$$

Let us show that the set $\Omega$ is relatively compact.

Inequality (3.3) means that

$$\gamma(\{\mathcal{S} \circ \mathcal{Q} (\Omega)\}) \geq \gamma(\Omega). \tag{3.4}$$

Applying the condition $(\mathcal{Q}3)$ and by using the properties of the function $\omega$, we obtain for a.e. $t \in [0,T]$

$$\chi (\{f(t) : f \in \mathcal{Q} (\Omega)\}) \leq \omega \left( t, \sup_{s\in[0,t]} \varphi (\{y(s) : y \in \Omega\}) \right) = \omega (t, \varphi (\{y : y \in \Omega\})) =$$

$$\omega \left(t, e^{Lt} e^{-Lt} \varphi (\{y : y \in \Omega\})\right) \leq \omega \left(t, e^{Lt} \gamma (\Omega)\right) \leq \omega \left(t, e^{Lt}\right) \cdot \gamma (\Omega) .$$

At first, we consider the case $1 \leq p < \infty$. By Lemma 3.4 we have for each $t \in [0,T]$ :

$$\chi (\{\mathcal{S}f(t) : f \in \mathcal{Q} (\Omega)\}) \leq \left( 4^p D \int_0^t \omega^p \left(s, e^{Ls}\right) ds \cdot \gamma^p (\Omega) \right)^{1/p} \leq$$

$$4D^{1/p} \left( \int_0^t e^{pLs} \omega^p (s, 1)\, ds \right)^{1/p} \cdot \gamma (\Omega) .$$

Inequality (3.4) and the last inequality imply the following estimate

$$\gamma(\Omega) \leq \sup_{t\in[0,T]} \left( 4D^{1/p} \left( \int_0^t e^{-pL(t-s)} \omega^p (s, 1)\, ds \right)^{1/p} \right) \gamma (\Omega) = q_1 \cdot \gamma (\Omega) ,$$

therefore

$$\gamma (\Omega) = 0,$$

 

thus

$$\varphi(\Omega) = 0$$

for all $t \in [0, T]$.

Let us turn to the case $p = \infty$. By Lemma 3.4 we have for each $t \in [0, T]$ :

$$\chi(\{\mathcal{S}(f)(t) : f \in \mathcal{Q}(\Omega)\}) \leq 2D_1 \int_0^t \omega\left(s, e^{Ls}\right) ds \cdot \gamma(\Omega) \leq 2D_1 \int_0^t e^{Ls} \omega(s, 1)\, ds \cdot \gamma(\Omega);$$

Inequality (3.4) and the last inequality imply the following

$$\gamma(\Omega) \leq \sup_{t \in [0,T]} \left(2D_1 \int_0^t e^{-L(t-s)} \omega(s, 1)\, ds\right) \gamma(\Omega) = q_2 \cdot \gamma(\Omega),$$

therefore

$$\gamma(\Omega) = 0,$$

thus

$$\varphi(\Omega) = 0$$

for each $t \in [0, T]$.

Now we will show that the set $\Omega$ is equicontinuous. We take sequences $\{y_n\}_{n=1}^{\infty} \subset \Omega,\ n \geq 1$, and $\{f_n\}_{n=1}^{\infty},\ f_n \in \mathcal{Q}(y_n)$. From conditions $(\mathcal{Q}2)$ and $(\mathcal{Q}3)$ it follows that the sequence $\{f_n\}_{n=1}^{\infty}$ is $L^p$-semicompact, and therefore by Lemma 3.3 the sequence $\{\mathcal{S}f_n\}_{n=1}^{\infty}$ is relatively compact. Hence

$$mod_C(\{\mathcal{S}f_n\}_{n=1}^{\infty}) = 0.$$

Thus

$$\nu(\mathcal{S} \circ \mathcal{Q}(\Omega)) = (0, 0),$$

but then it follows from the inequality (3.3) that

$$\nu(\Omega) = (0, 0),$$

and the last expression yields that the set $\Omega$ is relatively compact. □

To prove the main theorem of the paper, we need the following statements, known as the Gronwall - Bellman Lemma and the generalized Gronwall - Bellman Lemma (see [16]).

**Lemma 3.5:**
*Let $v(t)$ and $f(t)$ be nonnegative continuous functions on the segment $[a, b]$, moreover*

$$v(t) \leq c + \int_a^t f(s) v(s) ds,\ t \in [a, b],$$

*where $c$ is a positive constant. Then for each $t \in [a, b]$ the inequality*

$$v(t) \leq c e^{\int_a^t f(s) ds},$$

*holds.*

**Lemma 3.6:**
*Let $h(t),\ u(t)$ and $v(t)$ be nonnegative functions integrable on $[a, b]$ satisfying the inequality:*

$$v(t) \leq u(t) + \int_a^t h(s) v(s) ds, \quad t \in [a, b].$$

*Then the following inequality holds:*

$$v(t) \leq u(t) + \int_a^t e^{\int_a^t h(\theta) d\theta} h(s) u(s) ds, \quad t \in [a, b].$$

**Theorem 3.3:**
*Let a causal multioperator* $\mathcal{Q} : C([0,T];E) \to Cv(L^p([0,T];E)),\ 1 \le p \le \infty,$ *satisfy conditions* $(\mathcal{Q}1)$–$(\mathcal{Q}3)$ *and a linear causal operator* $\mathcal{S} : L^p([0,T];E) \to C([0,T];E)$ *satisfy conditions* $(\mathcal{S}1)$–$(\mathcal{S}3)$*. Then the set* $\Sigma$ *of all solutions to problem* (3.1)-(3.2) *is a non-empty compact set.*

*Proof*

Let us show that the set of all solutions $x \in C([0,T];E)$ of a one-parameter inclusion

$$x \in \lambda\Gamma(x), \quad \lambda \in [0,1], \tag{3.5}$$

is a priori bounded. We divide the proof into three cases: $p = 1,\ 1 < p < \infty,\ p = \infty.$

Let $p = 1,$ if a function $x \in C([0,T];E)$ satisfies condition $(3.5)$, then by Lemma 3.1 for each $t \in [0,T]$, we have the following estimates:

$$\|x(t)\|_E \le \|\mathcal{G}_0(t)x_0\|_E + \|\mathcal{G}_1(t)x_1\|_E + \cdots + \|\mathcal{G}_{n-1}(t)x_{n-1}\|_E + D\int_0^t \|f(s)\|_E ds \le$$

$$M\|x_0\|_E + Mt\|x_1\|_E + \cdots + M\frac{t^{n-1}}{(n-1)!}\|x_{n-1}\|_E + D\int_0^t \|f(s)\|_E ds \le$$

$$M\|x_0\|_E + MT\|x_1\|_E + \cdots + M\frac{T^{n-1}}{(n-1)!}\|x_{n-1}\|_E + D\int_0^t \|f(s)\|_E ds,$$

where $f \in \mathcal{Q}(x)$ and, therefore, by condition $(\mathcal{Q}2)$ :

$$\|f(s)\|_E \le \alpha(s)(1 + \|x(s)\|_E),$$

we have

$$\|x(t)\|_E \le M\|x_0\|_E + MT\|x_1\|_E + \cdots + M\frac{T^{n-1}}{(n-1)!}\|x_{n-1}\|_E +$$

$$D\int_0^t \alpha(s)(1 + \|x(s)\|_E)ds \le M\|x_0\|_E + MT\|x_1\|_E + \cdots + M\frac{T^{n-1}}{(n-1)!}\|x_{n-1}\|_E +$$

$$D\|\alpha\|_{L^\infty}T + D\|\alpha\|_{L^\infty}\int_0^t \|x(s)\|_E ds.$$

The last expression is a non-decreasing function of $t$, so we get

$$\|x(t)\|_E \le M\|x_0\|_E + MT\|x_1\|_E + \cdots + M\frac{T^{n-1}}{(n-1)!}\|x_{n-1}\|_E +$$

$$D\|\alpha\|_{L^\infty}T + \int_0^t D\|\alpha\|_{L^\infty}\|x(s)\|_E ds.$$

This means that the function $v(t) = \|x(t)\|_E$ satisfies the estimate

$$v(t) \le M\|x_0\|_E + MT\|x_1\|_E + \cdots + M\frac{T^{n-1}}{(n-1)!}\|x_{n-1}\|_E +$$

$$D\|\alpha\|_{L^\infty}T + \int_0^t D\|\alpha\|_{L^\infty}v(s)ds.$$

 

Applying Lemma 3.5, we obtain the required a priori boundedness:

$$v(t) = \|x(t)\|_E \le U e^{D\|\alpha\|_{L^\infty}} = \gamma_1,$$

where

$$U = M\|x_0\|_E + MT\|x_1\|_E + \cdots + M\frac{T^{n-1}}{(n-1)!}\|x_{n-1}\|_E + \|x_n\|_E + D\|\alpha\|_{L^\infty}T.$$

Then $\|x\|_C = \sup_{t\in[0,T]} \|x(t)\|_E \le \gamma_1$.

For the case $1 < p < \infty$, we have

$$\|x(t)\|_E \le M\|x_0\|_E + MT\|x_1\|_E + \cdots + M\frac{T^{n-1}}{(n-1)!}\|x_{n-1}\|_E + \Big(D\int_0^t \|f(s)\|_E^p ds\Big)^{\frac{1}{p}} \le$$

$$M\|x_0\|_E + MT\|x_1\|_E + \cdots + M\frac{T^{n-1}}{(n-1)!}\|x_{n-1}\|_E + \Big(D\int_0^t \alpha^p(s)(1+\|x(s)\|_E)^p ds\Big)^{\frac{1}{p}} \le$$

$$M\|x_0\|_E + MT\|x_1\|_E + \cdots + M\frac{T^{n-1}}{(n-1)!}\|x_{n-1}\|_E +$$

$$\Big(D\int_0^t \alpha^p(s)ds + D\int_0^t \alpha^p(s)\|x(s)\|_E^p ds\Big)^{\frac{1}{p}} \le$$

$$M\|x_0\|_E + MT\|x_1\|_E + \cdots + M\frac{T^{n-1}}{(n-1)!}\|x_{n-1}\|_E +$$

$$\Big(D\int_0^t \alpha^p(s)ds\Big)^{1/p} + \Big(D\int_0^t \alpha^p(s)\|x(s)\|_E^p ds\Big)^{\frac{1}{p}}.$$

Let us introduce the following notation:

$$c_0 = M\|x_0\|_E + MT\|x_1\|_E + \cdots + M\frac{T^{n-1}}{(n-1)!}\|x_{n-1}\|_E + D^{1/p}\|\alpha\|_{L^p},$$

$$h(s) = D^{1/p}\alpha(s).$$

Then we get:

$$\|x(t)\|_E \le c_0 + \left(\int_0^t h^p(s)\|x(s)\|_E^p ds\right)^{1/p}.$$

Let $v(t) = \|x(t)\|_E^p$, then from the last inequality we obtain the estimate:

$$v(t) \le 2^p c_0^p + 2^p \int_0^t h^p(s)v(s)ds.$$

Now applying Lemma 3.6 to the last inequality, we get

$$v(t) = \|x(t)\|_E^p \le 2^p c_0^p \left(1 + \int_0^t e^{2^p \int_0^t h^p(\theta)d\theta} h^p(s)ds\right).$$

Then we have the final estimate for $1 < p < \infty$ :

$$\|x(t)\|_E \le 2c_0 \sqrt[p]{1 + \int_0^t e^{2^p \int_0^t h^p(\theta)d\theta} h^p(s)ds} = \gamma_2.$$

Then $\|x\|_C = \sup_{t\in[0,T]} \|x(t)\|_E \le \gamma_2$.

For the case $p=\infty$, in the same way as in the case $p=1$, the following estimate holds:

$$\|x\|_C \le U_1 e^{D_1\|\alpha\|_{L^\infty}} = \gamma_3,$$

where

$$U_1 = M\|x_0\|_E + MT\|x_1\|_E + \cdots + M\frac{T^{n-1}}{(n-1)!}\|x_{n-1}\|_E + D_1\|\alpha\|_{L^\infty}T.$$

Now, if we take $R \ge \max\{\gamma_1,\gamma_2,\gamma_3\}$, we can guarantee that the set $V \subset C([0,T];E)$, given as

$$V = \{x \in C([0,T];E) : \|x\|_C < R\},$$

contains all solutions of inclusion (3.5). Thus, the multioperator $\Gamma$ satisfies on $\partial V$ the condition of Theorem 2.2 with $a=0$, hence the set of its fixed points is nonempty and compact. □

## 4. EXAMPLES

We will need the following notion (see [17, 18]).

**Definition 4.1:**
*The Gerasimov-Caputo fractional derivative of an order $q \ge 0$ of a function $g \in C^n([0,T];E)$ is the function ${}^CD_0^q g$ of the following form:*

$${}^CD_0^q g(t) = \frac{1}{\Gamma(n-q)}\int_0^t (t-s)^{n-q-1} g^{(n)}(s)\,ds, \quad n = [q]+1,$$

*where $\Gamma$ is the Euler's gamma-function*

$$\Gamma(q) = \int_0^\infty x^{q-1}e^{-x}dx.$$

Notice, that the last definition includes the case, when $q = n \in \mathbb{N}$.

Consider the following system governed by a differential inclusion in a separable Banach space $E$ :

$${}^CD_0^q y(t) \in Ay(t) + F(t, y(t)), \ \ t \in [0,T], \ n-1 < q \le n, \tag{4.6}$$

$$y(0) = y_0, \quad y'(0) = y_1, \quad ..., \quad y^{(n-1)}(0) = y_{n-1}. \tag{4.7}$$

Let a multimap $F : [0,T] \times E \to Kv(E)$ is such that:

- $(F1)$ for each $\psi \in C([0,T];E)$ the multifunction $F(\cdot,\psi) : [0,T] \to Kv(E)$ admits a measurable selection;
- $(F2)$ for a.e. $t \in [0,T]$ the multimap $F(t,\cdot) : E \to Kv(E)$ is u.s.c.;
- $(F3)$ there exists a function $\alpha \in L^\infty_+[0,T]$ such that

$$\|F(t,\psi)\|_E := \sup\{\|z\|_E : z \in F(t,\psi)\} \le \alpha(t)(1 + \|\psi\|_E)$$

  for a.e. $t \in [0,T]$ and for all $\psi \in E$;
- $(F4)$ there exists a function $\omega_F : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the conditions $(\omega 1)$-$(\omega 3)$ such that for each bounded set $\Omega \subset C([0,T];E)$ we have

$$\chi(F(t,\Omega)) \le \omega_F(t, \varphi(\Omega)) \ \text{ for a.e. } t \in [0,T].$$

 

The fact that the superposition multioperator $\mathcal{P}_F : C([0,T];E) \multimap L^p([0,T];E)$, $p > 1/q$, satisfies condition $(\mathcal{Q}1)$ can be verified as in paper [19]. Conditions $(\mathcal{Q}2)$ and $(\mathcal{Q}3)$ for $\mathcal{P}_F$ follow from $(F3)$ and $(F4)$, respectively.

If $0 < q \leq 1$ we will suppose that

$(A_I)$ $A : D(A) \subset E \to E$ is a linear closed operator generating a uniformly bounded $C_0$-semigroup $\{U(t)\}_{t\geq 0}$.

Then

$$\mathcal{G}_0(t) = \int_0^\infty \xi_q(\theta) U(t^q\theta) d\theta,$$

and a function $y \in C([0,T];E)$ is a mild solution to problem (4.6)–(4.7), if it can be presented in the form:

$$y(t) = \mathcal{G}_0(t) y_0 + \int_0^t (t-s)^{q-1} \mathcal{T}_1(t-s) f(s) ds,$$

where $f \in \mathcal{P}_F(y)$.

We can consider the relation (4.6) as a special case of functional inclusion (3.1) with $\mathcal{Q} = \mathcal{P}_F$, and the Cauchy type operator $\mathcal{S} = L_1$.

As a direct consequence of Theorem 3.3, we obtain the following result.

**Theorem 4.1:**
*Suppose that conditions* $(A_I)$, $(F1)$–$(F4)$ *hold and* $0 < q \leq 1$. *Then the set of solutions to problem* (4.6)-(4.7) *is a non-empty compact subset of the space* $C([0,T];E)$.

Now, let $1 < q \leq 2$. We will suppose that

$(A_{II})$ $A : D(A) \subset E \to E$ be a closed linear operator in $E$ generating a uniformly bounded family of strongly continuous cosine operator functions $\{C(t)\}_{t\geq 0}$.

Then

$$\mathcal{G}_0(t) = \int_0^\infty \xi_q(\theta) C(t^{q/2}\theta) d\theta, \qquad \mathcal{G}_1(t) = \int_0^t \mathcal{G}_0(s) ds,$$

and a function $y \in C([0,T];E)$ is a mild solution to problem (4.6)–(4.7), if it can be presented in the form:

$$y(t) = \mathcal{G}_0(t) y_0 + \mathcal{G}_1(t) y_1 + \int_0^t (t-s)^{q/2-1} \mathcal{T}_2(t-s) f(s) ds,$$

where $f \in \mathcal{P}_F(y)$.

We can consider the relation (4.6) as a special case of functional inclusion (3.1) with $\mathcal{Q} = \mathcal{P}_F$, and the Cauchy type operator $\mathcal{S} = L_2$.

As a direct consequence of Theorem 3.3, we obtain the following result.

**Theorem 4.2:**
*Suppose that conditions* $(A_{II})$, $(F1)$–$(F4)$ *hold and* $1 < q \leq 2$. *Then the set of solutions to problem* (4.6)-(4.7) *is a non-empty compact subset of the space* $C([0,T];E)$.

Finally, in the general case $n-1 < q < n$, we will suppose that

$(A_{III})$ $A : D(A) \subset E \to E$ generating a uniformly bounded family operator functions $\{\mathcal{G}_0(t)\}_{t\geq 0}$ such that

$(\mathcal{G}_0 1)$ $\mathcal{G}_0(0) = I$;

($\mathcal{G}_0 2$) the operator function $\mathcal{G}_0(\cdot)$ is strongly continuous, i.e., the function $t \to \mathcal{G}_0(t)x$, $t \geq 0$ is continuous for each $x \in E$.

Then, the operator functions $\mathcal{G}_i,\ i = 1, 2, ..., n-1$, will be define as

$$\mathcal{G}_i(t) = \int_0^t \mathcal{G}_{i-1}\ (s) ds.$$

A function $y \in C([0,T]; E)$, is a mild solution to problem (4.6)–(4.7), if it can be presented in the form:

$$y(t) = \mathcal{G}_0(t)y_0 + \mathcal{G}_1(t)y_1 + \cdots + \mathcal{G}_{n-1}(t)y_{n-1} + \mathcal{L}^{-1}\left[(\lambda^q - A)^{-1}\mathcal{L}\left[f\right](\lambda)\right](t),$$

where $f \in \mathcal{P}_F(y)$, $\mathcal{L}$ is a Laplace transform and $\lambda^q \in \rho(A)$.

We can consider the relation (4.6) as a special case of functional inclusion (3.1) with $\mathcal{Q} = \mathcal{P}_F$, and the Cauchy type operator $\mathcal{S} = \mathcal{L}^{-1}\left[(\lambda^q - A)^{-1}\mathcal{L}\left[f\right](\lambda)\right]$.

As a direct consequence of Theorem 3.3, we obtain the following result.

**Theorem 4.3:**
*Suppose that conditions* $(A_{III})$, $(F1)$–$(F4)$ *hold. Then the set of solutions to problem* (4.6)-(4.7) *is a non-empty compact subset of the space* $C([0,T]; E)$.


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