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Goal Inversion in Multi-Criteria Decision Making Problems

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Abstract: Within the framework of multi-criteria decision making (MCDM) problems, a comprehensive analysis of procedures for the consistent normalization of benefit and cost attributes was conducted. Rank-based methods for solving multi-criteria decision making (MCDM) problems boil down to criteria convolution, which predetermines the transformation of all criteria to either the maximum or minimum value of the objective function using an inversion procedure. Criteria convolution is performed only for normalized data, so the inversion procedure must be consistent with the normalization procedure. It is shown that thousands of studies in MCDM using nonlinear inversion of cost attributes based on the $1/x$ transformation should be recognized as inaccurate. The argument is quite simple: distortion of the original data. This study explains what the error is. Nonlinear data inversion leads to a violation of mutual distances in the original data, the measurement scales of various attributes are not consistent and there is a shift in the areas of normalized values. Also, nonlinear inversion does not have a reasonable interpretation of values. The solution to the above problems is achieved by using the reverse sorting (ReS) algorithm. The ReS algorithm is a linear transformation and preserves the original information about the object: the location of attribute values, preserves the mutual location of the domains of various attributes and can be applied to both original and normalized data sets. The ReS algorithm is recommended for use in the inversion of values when coordinating the optimization goals of a multi-criteria problem.

Keywords: MCDM; normalization of multivariate data; non-linear inversion; coordination of scale; Reverse Sorting algorithm (ReS)

1. INTRODUCTION

The standard formalization of the MCDM problem includes defining alternatives A_i, and criteria $C_j, (i = 1, \dots, m, j = 1, \dots, n)$ and specifying the decision matrix: $D = (a_{ij})$ $[m \times n]$. Next, a decision model is formed, which includes a method for assessing the significance of criteria w_j $[1 \times n]$, a method for normalizing the decision matrix $x_{ij} = Norm(a_{ij})$ and a method for aggregating private features of alternatives into an efficiency indicator $Q_i = F(x_{ij}, w_j, par)$, where par are additional parameters of the model.

Since there are no criteria for selecting the trio of methods (F, w, x), the choice is made based on some general principles [1].

In this article, we will focus on the analysis of the procedure for consistent normalization of benefit and cost criteria $x_{ij} = Norm(a_{ij})$ and indicate why nonlinear inversion should not be used. In MCDM problems, there are three types of criteria: benefit criteria for which the goal is Larger-The-Better (LTB), cost criteria for which the goal is Smaller-The-Better (STB), and criteria with a Target Value The Better (TVTB) for which the optimal value is within the data interval. If the methodology for solving a multi-criteria problem is reduced to the convolution of criteria, then this predetermines the transformation of all criteria to the goal of either "max" or "min". To align the goals of the criteria, data inversion is used [1, 2]. With a common LTB goal, cost criteria are inverted, and with STB, benefit criteria are inverted. For

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criteria with a set target value, with a common LTB problem goal, inversion is performed for all values greater than the target, and with an STB goal, inversion is performed for all values less than the target [3]. Data inversion is an integral part of normalization. Normalization of multidimensional data is not only bringing data to a dimensionless form, but also aligning the normalization scales. This is a necessary element of the transformation of multidimensional data. The requirement is that no data set receives priority over the others at the normalization stage. In fact, this is impossible [1]. Since the attributes of objects and the ranges of their values differ greatly from each other, each of the features is assigned its own scale. In this case, the normalizations are not "isotropic", i.e. they compress the data cloud in some directions (for some attributes) more, in others – less. However, despite some violation of the data structure (mutual distances), this approach is considered generally accepted. Additionally, for some normalization methods, there is a shift in the domains of normalized values, which also determines the priorities of individual attributes.

Thus, since the convolution of multidimensional data criteria is performed only for normalized data, the inversion procedure must be consistent with the normalization procedure.

A significant part of MCDM models use a nonlinear inversion of the decision matrix of the form $1/a$. This inversion is often used as a component of such models as, for example, Weighted Sum Model (WSM), Weighted Product Model (WPM), Weighted Aggregated Sum Product Assessment (WASPAS) [4], COmplex PRoportional ASsessment (COPRAS) [5], Additive Ratio Assessment (ARAS) [6], COmbinative Distance-based ASsessment (CODAS) [7], Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) [8], etc.

Exactly the same approach is used in objective methods for assessing the weight of criteria based on the information contained in the decision-making matrix: Entropy weight method – EWM [9], MEthod based on the Removal Effects of Criteria (MEREC) [10], Simultaneous Evaluation of Criteria and Alternatives (SECA) [11], method of Criterion Impact LOSs (CILOS) [12], etc.

It is clear that the monotonic strictly the decreasing function $1/a$ will preserve the order of the values in the list and invert the data, so that larger values become smaller and vice versa.

In this study, based on the analysis and simple examples, we will show that such an inversion is not correct. The nonlinear transformation changes the data, or rather changes the distances between the values. This means that the normalized data is fundamentally distorted. As an alternative, a linear method, Reverse Sorting (ReS), is proposed, an algorithm that eliminates all the problems of nonlinear transformation.

Although the basic publication of the linear inversion of ReS in the journal IJITDM [2] was in 2020, very few studies use this method. Meanwhile, this is the simplest and most universal inversion for both univariate and multivariate data. The author aims to increase the visibility of the ReS method, which is not available to many researchers due to paid access.

2. NONLINEAR INVERSION OF THE FORM 1/*A* AND LINEAR INVERSION RES

2.1. Nonlinear Inversion of the Form 1/a

When choosing a general LTB goal:

1) Max normalization method:

for benefit criteria:

$$x_{ij} = \frac{a_{ij}}{k_j}, k_j = a_j^{\max} \tag{1}$$

for cost criteria, iMax inverse normalization:

$$\bar{x}_{ij} = \frac{a_j^{\min}}{a_{ij}} = \frac{1/a_{ij}}{t_j}, t_j = \frac{1}{a_j^{\min}}, \tag{1'}$$

2) Sum normalization method:
for benefit criteria:

$$x_{ij} = \frac{a_{ij}}{k_j}, k_j = \sum_{j=1}^{m} a_{ij}\,, \tag{2}$$

for cost criteria, iSum inverse normalization:

$$\bar{x}_{ij} = \frac{1/a_{ij}}{t_j}, t_j = \sum_{i=1}^{m} \left(\frac{1}{a_{ij}}\right) \tag{2'}$$

(see, for example, ARAS method [6]).

3) Vec normalization method:
for benefit criteria:

$$x_{ij} = \frac{a_{ij}}{k_j}, k_j = \left(\sum_{i=1}^{m} a_{ij}^2\right)^{0.5} \tag{3}$$

for cost criteria, iVec inverse normalization:

$$\bar{x}_{ij} = \frac{1/a_{ij}}{t_j}, t_j = \left(\sum_{i=1}^{m} \left(\frac{1}{a_{ij}}\right)^2\right)^{0.5}. \tag{3'}$$

Note 1: that the normalization coefficients t_j are constants for fixed j (similarly, k_j), and, therefore, all three inverse normalizations of the same type use the $1/a$ transformation.

Note 2: in MCDM problems, objective function maximization (LTB) is typically used, so transformations (1)–(3) are applied. If objective function minimization (STB) is used in multi-criteria decision-making problems, the formulas for the benefit and cost criteria should be reversed.

The selection of normalizations should be such that the normalized values are dimensionless (for example, fractions of the largest) and the ranges of normalized values x_{ij} and $\bar{x}_{ij}$ are comparable. The normalization coefficient t_j in the inverse normalization is selected so that the inversion produces data from the same range of values as the corresponding normalization of the benefit criteria.

The first requirement is met, but the interpretation of the $\bar{x}_{ij}$ values is difficult. To interpret the normalized values, the connecting term “fraction of” is used, for example, the fraction of the largest value. Then, for example, how to interpret the value $1/a_j^{\min}$. It is even more difficult to do this for t_j, defined by formulas (2') and (3'). And the value $1/a_{ij}$ itself in the numerator of these formulas does not make sense?

The second requirement is met only for the iMax normalization, but not for the iSum and iVec methods, although the shift is small. That is, the value ranges for the benefit and cost criteria are shifted, and one of the sets will have priority when integrating indicators to determine the rating of alternatives, for example, when summing.

Another requirement for normalization is to preserve the proportions between the values (preserve the dispositions) [13]. The dispositions are determined for a fixed j by the formula:

$$\Delta x_{ij} = \frac{|x_{i+1,j} - x_{ij}|}{x_j^{\max} - x_j^{\min}}, i = 1, \dots, m-1\ \forall j \tag{4}$$

It is obvious that dispositions are invariant with respect to data scaling. In particular, dispositions will not depend on the normalization coefficients k_j and t_j in formulas (1)–(3), (1')–(3'). Dispositions show the relative remoteness of values. Preservation of dispositions is scaling of the object, i.e., when transforming data, proportions are preserved. Only linear transformations have this property. Note that the normalizations Max, Sum, Vec use linear transformations, i.e., preserve data dispositions, and the inverse normalizations iMax, iSum, iVec are nonlinear, and, therefore, the data and the object are distorted.

Additional comment for the Sum and Vec normalization methods. Few people attach importance to the fact that for multi-criteria problems the Sum normalization method produces the same arithmetic means ($m_j = \mathrm{mean}_i(x_{ij}) = m_0, \forall j$), and the Vec normalization method produces the same root mean squares ($r_j = \mathrm{rms}_i(x_{ij}) = r_0, \forall j$) for all normalized attribute values. This is the target task of these methods – to equalize the normalized values of all attributes by the mean in order to eliminate the priority of the contributions of individual attributes during subsequent integration of values. In terms of content, the Sum and Vec methods have nothing in common with their abbreviation. These are normalization variants based on the arithmetic mean and root mean square. When using nonlinear inversion (2') and (3') for individual criteria, the property of equality of means is not fulfilled, which means that the goal: to equalize all means for all attributes will not be achieved.

The acceptability of formulas (1')–(3') is due only to the fact that these distortions are not so great, and in some cases do not affect the final result.

2.2 Linear Inversion Reverse Sorting (ReS)

The basic publication of the linear inversion ReS was in 2020 in the journal IJITDM [2]. As shown in this study, it is the simplest and most versatile inversion for both univariate and multivariate data.

The ReS-algorithm is universal, preserves value dispositions, preserves domain positions, and can be applied to both the original (5) and normalized (6) data set for jth criteria [1, 2]:

$$\mathrm{ReS}(a_{ij}) = -a_{ij} + a_j^{\min} + a_j^{\max} \text{ for } j \in C^- - \text{cost criteria}, \quad (5)$$

$$ReS(x_{ij}) = -x_{ij} + x_j^{\min} + x_j^{\max}, \text{for } j \in C^- - \text{cost criteria}. \quad (6)$$

In accordance with (5) and (6), the following are possible for the normalization option. First, using (5), an inversion is performed for the cost criteria for the original decision matrix, and then one of the linear normalization methods is applied. Another option is to first perform one of the linear normalization methods for the entire decision matrix, and then, using (6), perform an inversion of the normalized values of the cost criteria.

The study [1, 2] showed that $\mathrm{ReS}(\mathrm{Max}(a)) = \mathrm{Max}(\mathrm{ReS}(a))$, but $\mathrm{ReS}(\mathrm{Sum}(a)) \neq \mathrm{Sum}(\mathrm{ReS}(a))$ and $\mathrm{ReS}(\mathrm{Vec}(a)) \neq \mathrm{Vec}(\mathrm{ReS}(a))$. The dispositions of the values are preserved, but there is a shift in the domains for the case of $\mathrm{ReS}(\mathrm{Sum}(a))$ and $\mathrm{ReS}(\mathrm{Vec}(a))$ from the corresponding domains when normalizing $\mathrm{Sum}(a)$ and $\mathrm{Vec}(a)$. Therefore, it is recommended to use the sequence: inversion/normalization, i.e., first perform the inversion using formula (5) and then normalize the data. In this case, the goal of the Sum and Vec normalization methods are achieved: to equalize all means for all attributes (see Section 2.1). Note also that this is due to the asymmetry of the data. The asymmetry coefficient changes sign under linear inversion.

The ReS transformation inverts both cost and benefit criteria equally. For example, you can invert the cost attribute values and then solve the LTB problem, or invert the benefit attribute values and then solve the STB problem.

As is known, for optimization problems, the LTB goal is changed to the STB goal by simply changing the sign of the objective function. Known inversions using this simple technique, which were previously used but did not gain popularity in MCDM are:

$$\bar{x}_{ij} = -x_{ij},$$

$$\bar{x}_{ij} = 1 - xx_{ij}.$$

In the first case, the value domain becomes negative, and some people simply do not like negative numbers. In the second case, there is a strong data shift – antiphase. Obviously, ReS is just a modification of the above methods. But in formulas (5) and (6), an exact shift in the value domain is performed, which allows you to preserve the value domain after the inversion. This is exactly what eliminates the priority of the attribute contribution to the integral rating.

It is easy to prove that WSM with inversion ReS is equivalent to WSM with inversion $\bar{x}_{ij} = -x_{ij}$, and is also equivalent to WSM with inversion $\bar{x}_{ij} = 1 - x_{ij}$, where x_{ij} is obtained using any linear normalization method, i.e. the bias in the WSM method can be omitted. In contrast, the authors of [14] singled out the approach with inversion $\bar{x}_{ij} = -x_{ij}$, as a separate component (the Ratio System approach):

$$Q_i = \sum_{j=1}^{g} w_j \cdot x_{ij} - \sum_{j=g+1}^{n} w_j \cdot x_{ij}.$$

What is the "zest" of the proof? Some people do not like not only negative numbers, but also decimal fractions, preferring natural numbers, such as ranks instead of ratings, although ratings contain more information. The ratings of the Q_i alternatives for different methods of aggregating partial attribute values are defined on different scales. Two rating lists are the same if they are transformed into each other using linear transformations, since the fine structure of the relationships between the data is preserved [15]. To do this, it is sufficient to transform both lists to the form of Relative Performance Indicator (RPI) dQ [15]:

$$dQ_p = \frac{Q_{p+1} - Q_p}{Q_{\max} - Q_{\min}}, p = 1, \dots, m-1, \tag{7}$$

meaning relative (given in the measurement scale) distances between alternatives. If formula (7) is applied to a sorted (ranked) list of alternatives, then dQ_i determine the distance in scale fractions (possibly in percent). In more detail, the usefulness of transformation (7) can be found in [15]. Note that formula (7) is similar to formula (4) and has the same meaning.

Another well-known variant of checking the equivalence of two lists is when the Pearson pair correlation coefficient is equal to 1. This coefficient measures the magnitude of the linear relationship between two variables and is invariant under linear transformations. However, this is a statistical parameter and it is required that the studied variables should be normally distributed. This creates some discomfort, since in the MCDM problem statistics are not used and the normality of features is not checked. A simpler and more visual check of the equivalence of ratings obtained by different methods is based on formula (7) for an ordered (in accordance with the purpose of the problem) list. An example of the equivalence of rating lists is given below in section 3, example 1.

Now it is easy to prove that WSM with inversion ReS is equivalent to WSM with inversion $\bar{x}_{ij} = -x_{ij}$, and is also equivalent to WSM with inversion $\bar{x}_{ij} = 1 - x_{ij}$, since dQ_i are the same in all three cases.

Note also that the ReS transformation is equivalent to the generally accepted inversion for the Max-Min normalization method:

$$x_j^{\max} + x_j^{\min} - x_{ij} = 1 - 0 - \frac{a_j^{\max} - a_{ij}}{a_j^{\max} - a_j^{\min}} = \frac{a_{ij} - a_j^{\min}}{a_j^{\max} - a_j^{\min}}.$$

2.3. Illustrations for Comparison of Nonlinear Inversion $1/a$ and Linear Inversion ReS

The final illustration in Figures 1, 2 and 3 compares the nonlinear inversion $1/a$ and linear ReS for the Max, Sum and Vec normalizations for the following vector of one of the attributes for 8 alternatives: $\bar{a} = (1056\ 2680\ 1230\ 1480\ 1350\ 2065\ 1650\ 1750)$. The ReS transformation is performed for pre-normalized data using formula (6).

Figure 1 illustrates the linearity property for ReS and the nonlinearity of the $1/a$ transformation for the Max, Sum and Vec normalization.

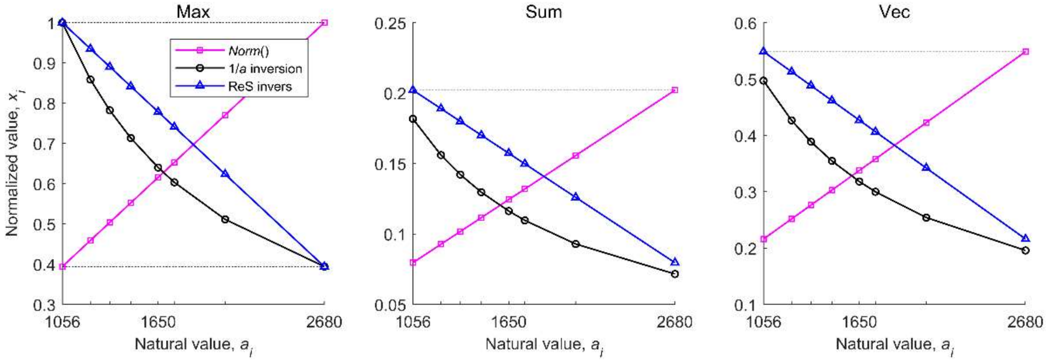

Fig. 1. Normalized values using $1/a$ and ReS inverse techniques

The regions of normalized values x_{ij} and $\bar{x}_{ij}$ are equal only for the iMax normalization, but not for the iSum and iVec methods, although the shift is small.

Let's sort the components of the vector $\bar{a}$ in ascending order. We get the vector $\bar{a}^* = (1056\ 1230\ 1350\ 1480\ 1650\ 1750\ 2065\ 2680)$. These values correspond to the dispositions of natural values: $\Delta a = (0.11\ 0.07\ 0.08\ 0.10\ 0.06\ 0.19\ 0.38)$. Figure 2 shows the dispositions between the normalized values for ReS transformation and the non-linear transformation $1/a$.

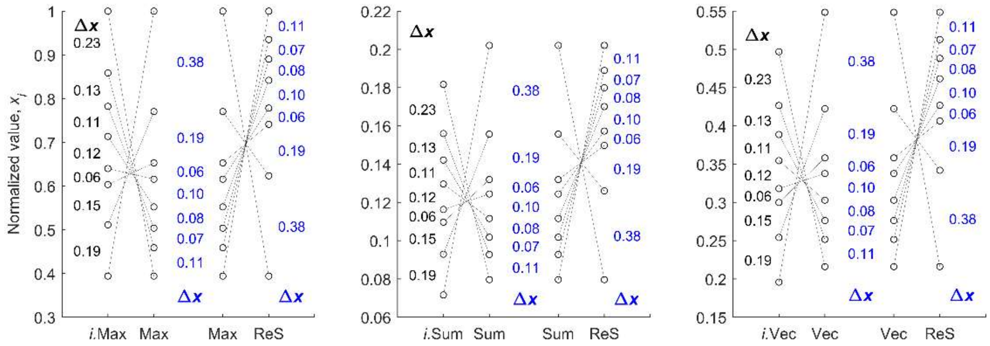

Fig. 2. Disposition distortion when using nonlinear $1/a$ inversion compared to linear ReS inversion

The ReS transformation preserves the dispositions Δx of the original data and has a center of symmetry when displayed. In all three cases of inversions iMax, iSum, iVec, the dispositions do not correspond to the dispositions of the original data, the center of symmetry is absent. The lines illustrating the correspondence intersect at different points, forming a certain polygon. The smaller the size of this polygon, the less the degree of distortion in the dispositions. According to Figure 2, it can be concluded that for some points the dispositions differ by up to 1.7 times. For the case of iSum, iVec, there is a shift in the domain of values towards a decrease, which means a decrease in the contribution of this attribute to the overall integral indicator. For the ReS inversion, the areas of normalized values x_{ij} and $\bar{x}_{ij}$ coincide. Note that, in accordance with formulas (1')–(3'), the dispositions of the transformed data during nonlinear inversion are the same, since the formulas differ only in the compression-expansion coefficients t_j.

Figure 3 shows a polyhedron of attribute dispositions for the original data (Max) and the inversions iMax and ReS.

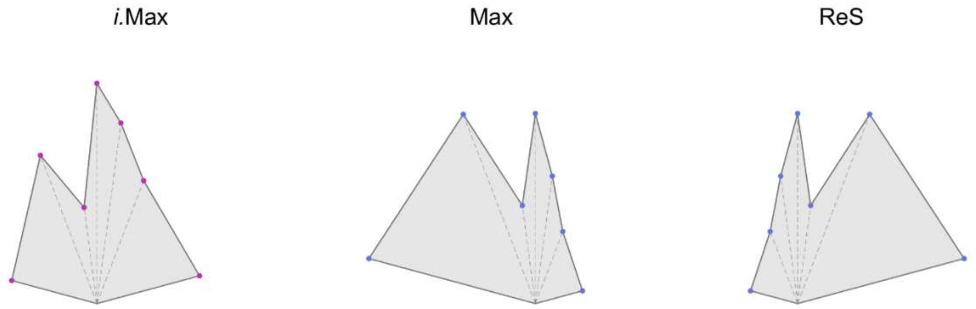

Fig. 3. Polyhedron of attribute dispositions under Max normalization and iMax and ReS inversion

The data polyhedron for Max normalization (which is the same for natural attribute values) coincides with the polyhedron for data obtained by ReS inversion up to mirror symmetry, but differs from the data polyhedron for iMax normalization. Figure 3 demonstrates object scaling during ReS transformation, i.e. the proportions in the data are preserved, unlike the nonlinear transformation based on $1/a$.

3. NUMERIC EXAMPLES

Example 1. The first example shows how the ranking of alternatives A_1 and A_4 for the CODAS/iMax method changed significantly when using the ReS inversion (Table 1 and 2). The original data (Table 1) are from the article promoting the CODAS method [7].

Table 1. Input data

		Criteria, *benefit* (+)/*cost*(–)				
		C_1^+	C_2^+	C_3^-	C_4^+	C_5^-
Alternatives	A_1	60.00	0.40	2540	500	990
	A_2	6.35	0.15	1016	3000	1041
	A_3	6.80	0.10	1727	1500	1676
	A_4	10.00	0.20	1000	2000	965
	A_5	2.50	0.10	560	500	915
	A_6	4.50	0.08	1016	350	508
	A_7	3.00	0.10	1778	1000	920
	w_j	0.036	0.326	0.192	0.326	0.120

As can be seen from Table 2, even a small difference in the inversion of just one attribute changed the ranks, not to mention the ratings. WSM in this example serves as a test method. Columns $dQ^{(4)}$ and $dQ^{(5)}$ of Table 2 illustrate the equivalence of the rating lists obtained using WSM, ReS, and WSM, $\bar{x} = -x$. A similar result holds for the inversion $\bar{x} = 1 - x$.

Table 2. Sorted ranking list Q, alternative number $\#A_i$ and disposition dQ, % for sorted list Q for methods CODAS/iMax, CODAS/ReS, WSM/iMax, WSM/ReS, WSM/ $x = -x$

rank	CODAS, iMax			CODAS, ReS			WSM, iMax			WSM, ReS			WSM, $\bar{x} = -x$		
	$Q^{(1)}$	$\#A_i$	$dQ^{(1)}$	$Q^{(2)}$	$\#A_i$	$dQ^{(2)}$	$Q^{(3)}$	$\#A_i$	$dQ^{(3)}$	$Q^{(4)}$	$\#A_i$	$dQ^{(4)}$	$Q^{(5)}$	$\#A_i$	$dQ^{(5)}$
1	2.193	2	-	2.309	2	-	0.632	2	-	0.684	2	-	0.450	2	-
2	1.214	**1**	22.8	1.258	**4**	26.0	0.563	4	18.1	0.614	4	18.2	0.380	4	18.2
3	1.106	4	2.5	0.864	1	9.7	0.530	1	8.7	0.530	1	22.0	0.295	1	22.0
4	-0.305	3	32.9	-0.332	3	29.5	0.431	3	25.7	0.472	3	14.9	0.238	3	14.9
5	-0.474	5	3.9	-0.782	5	11.1	0.395	5	9.4	0.395	5	20.2	0.161	5	20.2
6	-1.633	7	27.0	-1.577	7	19.6	0.318	7	19.9	0.358	7	9.6	0.123	7	9.6
7	-2.102	6	10.9	-1.740	6	4.0	0.248	6	18.3	0.300	6	15.1	0.066	6	15.1
∑			100			100			100			100			100

The value of dQ for the sorted list of ratings shows how much the alternatives of rank p and $p + 1$ differ. With a small difference, such alternatives can be defined as indistinguishable or as alternatives of the same rank [1, 15].

(*Note:* in the author's version of the solution to this problem [7], the calculations are made with an error. The first three alternatives by rank are 2, 3, 1. There is an obvious miss in Table 2, 3, The attribute "Maximum tip speed" is in the text an attribute of benefits, not costs, and the attribute "Repeatability" is the opposite.)

Example 2. The second example shows how significantly the weight of the criteria for the objective MEREC/iMax method has changed in comparison with MEREC/ReS. The original data (Table 3) are from an article promoting the MEREC method [10].

Table 3. Input data

		Criteria, *benefit* (+)/*cost*(−)						
		C_1^+	C_2^+	C_3^+	C_4^-	C_5^-	C_6^-	C_7^-
Alternatives	A_1	23	264	2.37	0.05	167	8900	8.71
	A_2	20	220	2.20	0.04	171	9100	8.23
	A_3	17	231	1.98	0.15	192	10800	9.91
	A_4	12	210	1.73	0.20	195	12300	10.21
	A_5	15	243	2.00	0.14	187	12600	9.34
	A_6	14	222	1.89	0.13	180	13200	9.22
	A_7	21	262	2.43	0.06	160	10300	8.93
	A_8	20	256	2.60	0.07	163	11400	8.44
	A_9	19	266	2.10	0.06	157	11200	9.04
	A_{10}	8	218	1.94	0.11	190	13400	10.11

The weights of the criteria are presented in Table 4.

Table 4. Criteria weights when using the MEREC method and the inversion of iMax and ReS

j:	1	2	3	4	5	6	7
$w_j^{(1)}$: MEREC, Max/iMax	0.324	0.055	0.086	0.368	0.044	0.077	0.045
$w^{(2)}$: MEREC, Max/ReS	0.266	0.058	0.082	0.409	0.049	0.085	0.050
Relative change, %	-18	5	-5	11	11	10	11

For the MEREC method, when using the linear inversion of ReS, the weight of the first and third criteria decreased, and the weight of the remaining criteria increased. Weight changes are significant. Table 5 presents the ranking results for different weight values.

Table 5. Ratings and ranks of alternatives in the WSM method when using iMax and ReS inversion method and MEREC weighing method: Sorted ranking list Q, alternative number $\#A_i$ and disposition dQ, %

rank	WSM, iMax, $w^{(1)}$			WSM, iMax, $w^{(2)}$			WSM, ReS, $w^{(1)}$			WSM, ReS, $w^{(2)}$		
	$Q^{(1)}$	$\#A_i$	$dQ^{(1)}$	$Q^{(2)}$	$\#A_i$	$dQ^{(2)}$	$Q^{(3)}$	$\#A_i$	$dQ^{(3)}$	$Q^{(4)}$	$\#A_i$	$dQ^{(4)}$
1	0.930	**2**	-	0.937	**2**	-	0.969	**1**	-	0.967	**1**	-
2	0.913	1	3.6	0.905	1	6.7	0.931	2	7.8	0.938	2	5.8
3	0.828	7	18.6	0.818	7	18.3	0.917	7	2.8	0.917	7	4.2
4	0.785	9	9.5	0.780	9	8.1	0.884	8	6.6	0.883	8	6.6
5	0.778	8	1.3	0.766	8	2.8	0.873	9	2.1	0.878	9	1.0
6	0.589	3	41.5	0.571	3	41.2	0.659	3	43.3	0.649	3	45.6
7	0.565	5	5.2	0.553	5	3.9	0.646	5	2.7	0.643	5	1.4
8	0.550	6	3.2	0.541	6	2.4	0.641	6	1.0	0.642	6	0.1
9	0.481	10	15.2	0.488	10	11.2	0.586	10	11.0	0.606	10	7.2
10	0.472	4	2.0	0.463	4	5.3	0.473	4	22.8	0.465	4	28.0
$\sum$			100			100			100			100

In this example, it is important to know whether the rating would change if the weights were recalculated using ReS instead of $1/a$. The answer is no, but the rating dispositions have changed.

Example 3. The third example shows how the weight of the criteria has changed significantly for the objective SECA/iMax method compared to SECA/ReS. The original data (Table 3) are from the article promoting the SECA method [11].
The weights of the criteria are presented in Table 6.

Table 6. Criteria weights when using the SECA method and the inversion of iMax and ReS

j:	1	2	3	4	5	6	7
$w^{(3)}$: SECA, Max/iMax	0,153	0,151	0,126	0,178	0,118	0,159	0,116
$w^{(4)}$: SECA, Max/ReS	0,156	0,147	0,124	0,173	0,113	0,169	0,118
Relative change, %	2	-3	-2	-3	-4	6	2

For the SECA method, when using the linear inversion of ReS, the weights of the first, sixth and seventh criteria increased, and the weights of the remaining criteria decreased. The change is small, about 5%, which apparently will not affect the ranking. Table 7 presents the ranking results for different weight values.

Table 7. Ratings and ranks of alternatives in the WSM method when using iMax and ReS inversion method and SECA weighing method: Sorted ranking list Q, alternative number $\#A_i$ and disposition dQ, %

rank	WSM, iMax, $w^{(3)}$			WSM, iMax, $w^{(4)}$			WSM, ReS, $w^{(3)}$			WSM, ReS, $w^{(4)}$		
	$Q^{(1)}$	$\#A_i$	$dQ^{(1)}$	$Q^{(2)}$	$\#A_i$	$dQ^{(2)}$	$Q^{(3)}$	$\#A_i$	$dQ^{(3)}$	$Q^{(4)}$	$\#A_i$	$dQ^{(4)}$
1	0.939	1	-	0.940	1	-	0.967	1	-	0.968	1	-
2	0.921	2	5.5	0.922	2	5.6	0.932	7	10.3	0.931	7	10.7
3	0.884	7	11.8	0.884	7	12.0	0.924	2	2.4	0.925	2	2.0
4	0.856	8	8.8	0.856	8	8.9	0.912	8	3.4	0.911	8	4.1
5	0.847	9	2.9	0.846	9	3.1	0.895	9	5.0	0.893	9	5.1
6	0.711	3	42.9	0.713	3	41.9	0.750	3	42.4	0.752	3	41.5
7	0.698	5	3.9	0.699	5	4.5	0.741	5	2.5	0.741	5	3.1
8	0.678	6	6.3	0.679	6	6.4	0.725	6	4.8	0.724	6	5.0
9	0.632	10	14.5	0.632	10	14.9	0.684	10	11.9	0.682	10	12.3
10	0.621	4	3.5	0.623	4	2.8	0.625	4	17.2	0.627	4	16.2

In this example, what matters is whether the ranking would change if the weights were recalculated using ReS instead of $1/a$. The answer is no, and also the ranking dispositions changed only slightly.

4. CONCLUSIONS

The normalization scale is defined by the scaling factor $k_j = \text{Const}$, which is different for each jth criterion. A linear scale is defined by the requirement that the normalized values be linearly dependent on the natural values. For nonlinear scales, the scaling factor is also Const for each criterion, but the dependence of the normalized values on the natural values is nonlinear, such as $x_{ij} = a_j^{\min} / a_{ij}$ or $x_{ij} = \frac{\log a_{ij}}{\log \Pi_{ii} a_{ij}}$ or $x_{ij} = 1/(1 + \exp(-u_{ij}))$. As noted in the introduction, a strictly monotonically decreasing function is used for inversion. Such a function preserves the ordering of values in the list and inverts the data, so that larger values become smaller and vice versa. A strictly monotonically increasing function is used for normalization. Such a function ensures data ordering.

According to this, the following cases are possible.

1. Linear normalization is applied to all attributes (both cost and benefit criteria).

If a nonlinear inversion of the type $(1/a)$ is applied to some criteria to align the optimization objective, the benefit and cost criteria will have different measurement scales, meaning the values are not aligned across the normalization scales. Furthermore, the relative distances between the normalized values and between the natural values differ, resulting in data distortion.

When applying the linear inversion ReS, which uses a scaling factor of -1, the measurement scales remain unchanged, meaning the normalization and inversion are aligned. The relative distances between the normalized values and between the natural values are the same, meaning a simple scaling of the data is performed.

2. Nonlinear normalization is applied to all attributes (both cost and benefit criteria).

If a nonlinear inversion of the type $(1/a)$ is applied to some criteria to align the optimization objective, the benefit and cost criteria will still have different measurement scales, meaning the normalization scales of the different attributes are not aligned. When using the linear ReS inversion, which uses a scaling factor of -1, the measurement scales remain unchanged, meaning that normalization and inversion are consistent. The relative distances between normalized values and between natural values are the same; the interval of normalized values remains unchanged. Thus, when using the ReS inversion, the normalized data remains consistent during nonlinear normalization.

Note: Nonlinear normalization of multivariate data is useful when it is necessary to enhance or reduce the contribution of individual values to the overall result [1]. The authors are not aware of any other cases or examples where nonlinear inversion of multivariate data could be useful.

Thus, all the above arguments and examples indicate that using a nonlinear inversion of the $1/a$ type does not correctly transform the data. The ReS-algorithm eliminates all the shortcomings and is recommended for use in inverting values when coordinating the optimization goals of a multi-criterial problem. The PRI parameter defined in this paper and earlier in [15] is an effective tool for assessing the independence of the MCDM model, and is recommended for analysing the distinguishability of the ranking of alternatives.

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