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Multivalued Solutions of the Cauchy Problem for One Quasilinear System

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Abstract: The article proposes a generalization of the method of characteristics to quasilinear systems of evolutionary equations with one spatial variable. Such systems are a special case of hydrodynamic type systems. It is shown that in some cases self-intersection of solution graphs is possible. An example of a system of two equations is considered in detail.

Keywords: Hydrodynamic type systems, multivalued solutions, jet spaces, caustics, self-intersection, Cauchy problem

1. INTRODUCTION

Consider the following first order quasilinear system of m equations

$$\begin{cases} \dfrac{\partial u^1}{\partial t} + A(u^1, \ldots, u^m)\dfrac{\partial u^1}{\partial x} = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dfrac{\partial u^m}{\partial t} + A(u^1, \ldots, u^m)\dfrac{\partial u^m}{\partial x} = 0. \end{cases} \tag{1.1}$$

Here t is time, x is a spatial coordinate, the function $A \in C^\infty(\mathbb{R}^m)$. This is a special case of a hydrodynamic type systems and a generalization of the scalar Euler–Hopf equation

$$u_t + uu_x = 0. \tag{1.2}$$

It is well known that solutions of equation (1.2) become discontinuous at some moment of time even for a smooth initial condition. A similar phenomenon occurs for the considered system. In addition, we will show that for such systems the effect of self-intersection of the graphs of their solutions is possible.

Consider the following Cauchy problem for equation (1.1)

$$u^i|_{t=0} = U^i(x), \quad i = 1, \ldots, m. \tag{1.3}$$

Assume that the initial functions U^i belong to the class C^∞.

2. MULTIVALUED SOLUTIONS

Let $J^1 = J^1(2, m)$ be the 1-jet space with canonical coordinates (see [1,2])

$$t, x, u^1_{0,0}, u^1_{1,0}, u^1_{0,1}, \ldots, u^m_{0,0}, u^m_{1,0}, u^m_{0,1}; \quad i = 1, \ldots, m.$$

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The Cartan distribution $\mathcal{C}$ is generated by the differential 1-forms

$$\varkappa_i = du^i_{0,0} - u^i_{1,0}dt - u^i_{0,1}dx,$$

i.e.

$$\mathcal{C}(\theta) = \bigcap_{i=1}^{m} \ker \varkappa_i \,|_\theta \subset T_\theta J^1.$$

System (1.1) generates a $2(m+1)$-dimensional submanifold

$$\mathcal{E} = \left\{ u^i_{1,0} + A u^i_{0,1} = 0 \;\middle|\; i = 1, \ldots, m \right\} \subset J^1.$$

A two-dimensional integral manifold of the Cartan distribution lying in $\mathcal{E}$ is called a *multivalued solution* of system (1.1).

Let us show a method for constructing multivalued solutions to Cauchy problem (1.3). Define the following vector field on the 0-jet space $J^0(2, m)$:

$$X = -\frac{\partial}{\partial t} - A(u^1_{0,0}, \ldots, u^m_{0,0})\frac{\partial}{\partial x}.$$

Its prolongation to $J^1(2, m)$ is

$$X^{(1)} = X + \sum_{j=1}^{m}\left(\sum_{i=1}^{m} \frac{\partial A}{\partial u^i_{0,0}} u^i_{1,0}\right) u^j_{0,1}\frac{\partial}{\partial u^j_{1,0}} + \sum_{j=1}^{m}\left(\sum_{i=1}^{m} \frac{\partial A}{\partial u^i_{0,0}} u^i_{0,1}\right) u^j_{0,1}\frac{\partial}{\partial u^j_{0,1}}.$$

Lemma 2.1:
The vector field $X^{(1)}$ is tangent to $\mathcal{E}$.

Proof
The Lie derivative of the function $F_k = u^k_{1,0} + A u^k_{0,1}$ is

$$X^{(1)}(F_k) = \left(\sum_{i=1}^{m} \frac{\partial A}{\partial u^i_{0,0}} u^i_{1,0}\right) u^k_{0,1} + \left(\sum_{i=1}^{m} \frac{\partial A}{\partial u^i_{0,0}} u^i_{0,1}\right) A u^k_{0,1}.$$

Therefore, the restriction

$$X^{(1)}(F_k)|_{\mathcal{E}} = 0$$

for $k = 1, \ldots, m$. □

3. CAUCHY DATA

The initial data (1.3) define a curve in the 1-jet space:

$$\mathcal{K} : \left\{ t = 0, x = \chi, u^i_{0,0} = U^i(\chi), u^i_{0,1} = \frac{dU^i}{d\chi}, u^i_{1,0} = -\widetilde{A}\frac{dU^i}{d\chi} \;\middle|\; i = 1, \ldots, m \right\},$$

where $\widetilde{A} = A(U^1(\chi)), \ldots, U^m(\chi))$. This curve is called *Cauchy curve*.

Let Φ_τ be a flow of $X^{(1)}$. Since the vector field $X^{(1)}$ is transversal to the hyperplane $t = 0$, we see that $X^{(1)}_a \notin T_a\mathcal{K}$ for any point , i.e. trajectories of the vector field intersect the Cauchy curve. So, for sufficiently small τ, the union of $\Phi_\tau(\mathcal{K})$ forms a surface L:

$$L = \bigcup_{\tau} \Phi_\tau(\mathcal{K}). \tag{3.4}$$

Theorem 3.1:
Surface (3.4) *is a multivalued solution of equation* (1.1).

Proof
Due to Lemma 2.1, the vector field $X^{(1)}$ is tangent to $\mathcal{E}$. Then $L \subset \mathcal{E}$. It remains to prove that L is an integral manifold of the Cartan distribution.

The Cauchy curve is an integral curve of the Cartan distribution. Since the vector field $X^{(1)}$ is contact, the curve $\Phi_\tau(\mathcal{K})$ is an integral for the Cartan distribution, too. That is $T_a\Phi_\tau(\mathcal{K}) \subset \mathcal{C}(a)$ for any point $a \in \mathcal{K}$.

The value of the 1-form $\varkappa_i$ on the vector field $X^{(1)}$ is $\varkappa_i(X^{(1)}) = u^i_{1,0} + Au^i_{0,1}$ for $i = 1, \dots, m$. Therefore, the restriction $\varkappa_i(X^{(1)}|_{\mathcal{E}}) = \varkappa_i(X^{(1)})|_{\mathcal{E}} = 0$ and the integral curves of the vector field $X^{(1)}|_{\mathcal{E}}$ are integral for the Cartan distribution as well.

Let $a \in L$ be an arbitrary point. The tangent space T_aL is the direct sum

$$T_aL = T_a\Phi_\tau(\mathcal{K}) \oplus \mathbb{R}X^{(1)}_a \subset \mathcal{C}(a).$$

Therefore, L is a 2-dimensional integral manifold of the Cartan distribution. □

4. EXAMPLE

Consider the system

$$\begin{cases} u_t + uu_x = 0, \\ v_t + uv_x = 0 \end{cases} \tag{4.5}$$

with initial data

$$u|_{t=0} = \frac{1}{1+x^2}, \qquad v|_{t=0} = \frac{1}{1+(x-1)^2}.$$

Then the vector field

$$X = -\frac{\partial}{\partial t} - u_{0,0}\frac{\partial}{\partial x}$$

and its prolongation is

$$X^{(1)} = -\frac{\partial}{\partial t} - u_{0,0}\frac{\partial}{\partial x} + u_{1,0}u_{0,1}\frac{\partial}{\partial u_{1,0}} + u_{0,1}^2\frac{\partial}{\partial u_{0,1}} + u_{1,0}v_{0,1}\frac{\partial}{\partial v_{1,0}} + u_{0,1}v_{0,1}\frac{\partial}{\partial v_{0,1}}.$$

The corresponding flow is

$$\Phi_\tau : \begin{cases} t \mapsto t - \tau, \\ x \mapsto x - u_{0,0}\tau, \\ u_{0,0} \mapsto u_{0,0}, \\ v_{0,0} \mapsto v_{0,0}, \\ u_{1,0} \mapsto \dfrac{u_{1,0}}{1-\tau u_{0,1}} \\ u_{0,1} \mapsto \dfrac{u_{0,1}}{1-\tau u_{0,1}} \\ v_{1,0} \mapsto \dfrac{u_{1,0}v_{0,1}\tau - v_{1,0}u_{0,1}\tau + v_{1,0}}{1-\tau u_{0,1}}, \\ v_{0,1} \mapsto \dfrac{v_{0,1}}{1-\tau u_{0,1}}. \end{cases}$$

The Cauchy curve is

$$\mathcal{K}:\begin{cases} t=0,\\ x-\chi=0,\\ u_{0,0}-\dfrac{1}{1+\chi^2}=0,\\ v_{0,0}-\dfrac{1}{1+(\chi-1)^2}=0,\\ u_{1,0}-\dfrac{2\chi}{(1+\chi^2)^3}=0,\\ v_{1,0}-\dfrac{2\chi-2}{(1+\chi^2)(1+(\chi-1)^2)^2}=0,\\ u_{0,1}+\dfrac{2\chi}{(1+\chi^2)^2}=0,\\ v_{0,1}+\dfrac{2\chi-2}{(1+(\chi-1)^2)^2}=0. \end{cases}$$

Shifting the curve $\mathcal{K}$ along the vector field $X^{(1)}$, we obtain the multivalued solution to the Cauchy problem:

$$L_{\mathcal{K}}:\begin{cases} t=\tau,\\ x=\dfrac{\tau+\chi+\chi^3}{1+\chi^2},\\ u_{0,0}=\dfrac{1}{1+\chi^2},\\ v_{0,0}=\dfrac{1}{2+\chi^2-2\chi},\\ u_{1,0}=\dfrac{2\chi}{(1+\chi^2)(1+2\chi^2+\chi^4-2\tau\chi)},\\ u_{0,1}=-\dfrac{2\chi}{1+2\chi^2+\chi^4-2\tau\chi},\\ v_{1,0}=\dfrac{2\chi^3-2\chi^2+2\chi-2}{(2+\chi^2-2\chi)^2(1+2\chi^2+\chi^4-2\tau\chi)},\\ v_{0,1}=-\dfrac{2(\chi-1))(1+\chi^2)^2}{(2+\chi^2-2\chi)^2(1+2\chi^2+\chi^4-2\tau\chi)}. \end{cases}$$

The projections of this solution to 3-dimensional spaces t,x,u and t,x,v are shown in Fig. 4.1.

Let us construct caustics. For this goal, we introduce two planes Π and Γ with coordinates t,x and τ,χ respectively and construct the mapping $\pi:\Gamma\longrightarrow\Pi$, where

$$\pi:\quad t=\tau,\qquad x=\frac{\chi^3+\chi+\tau}{1+\chi^2}.$$

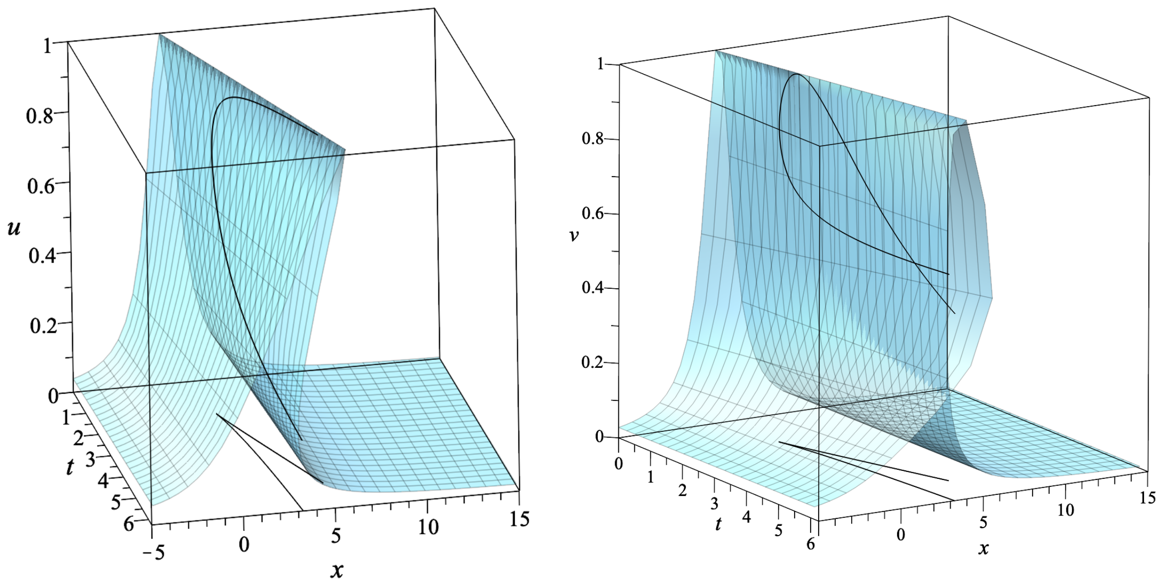

Fig. 4.1. Projections $L^u_{\mathcal{K}}$ and $L^v_{\mathcal{K}}$ of the multivalued solution $L_{\mathcal{K}}$ to the spaces t, x, u and t, x, v respectively.

The Jacobi matrix of π is

$$J_\pi = \begin{pmatrix} 1 & 0 \\ \dfrac{1}{1+\chi^2} & \dfrac{1+2\chi^2+\chi^4-2\chi\tau}{(1+\chi^2)^2} \end{pmatrix}.$$

A caustic Σ is defined by the critical points of this mapping, that is, the points at which the determinant of the Jacobi matrix vanishes (see Fig. 4.2):

$$\Sigma = \{1 + 2\chi^2 + \chi^4 - 2\chi\tau = 0\} \subset \Gamma.$$

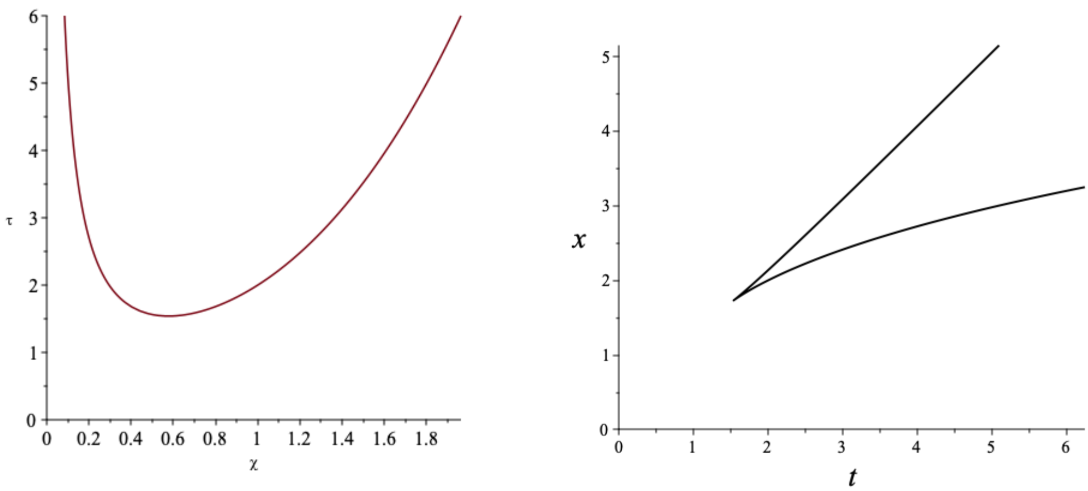

Fig. 4.2. The caustics Σ on the plane Γ (left) and its projection to the plane Π (right).

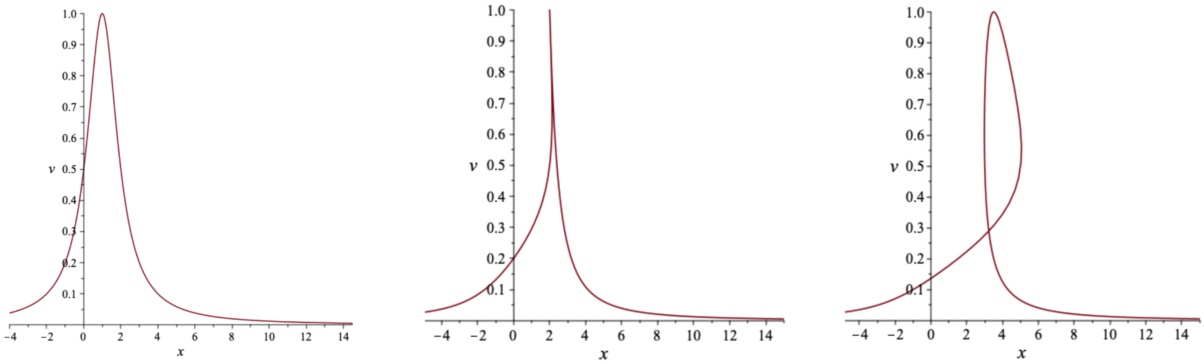

Fig. 4.3. Birth of the self-intersection curve: sections of the surface $L^v_{\mathcal{K}}$ at $t = 0, t = 2, t = 5$ respectively.

As can be seen from Fig. 4.1, the surface

$$L^v_{\mathcal{K}} : \begin{cases} t = \tau, \\ x = \dfrac{\tau + \chi + \chi^3}{1 + \chi^2}, \\ v_{0,0} = \dfrac{1}{2 + \chi^2 - 2\chi}, \end{cases} \tag{4.6}$$

has self-intersection along some curve. Let us find this curve. Coordinates $a = a(\tau)$ and $b = b(\tau)$ of the points of this curve are solutions of the system

$$\begin{cases} \dfrac{\tau + a + a^3}{1 + a^2} = \dfrac{\tau + b + b^3}{1 + b^2}, \\ \dfrac{1}{2 + a^2 - 2a} = \dfrac{1}{2 + b^2 - 2b}. \end{cases}$$

Its real solutions are

$$a = 1 \mp (2\tau - 4)^{1/4}, \quad b = 1 \pm (2\tau - 4)^{1/4}$$

We see that self-intersection occurs only when $\tau \geq 2$. So, we can choose

$$\chi = 1 + (2\tau - 4)^{1/4}.$$

Substituting this value to (4.6), we obtain a parametric representation of the self-intersection curve:

$$\begin{cases} t = \tau, \\ x = \dfrac{\tau + (2\tau - 4)^{1/4} + 1 + ((2\tau - 4)^{1/4} + 1)^3}{1 + ((2\tau - 4)^{1/4} + 1)^2}, \\ v_{0,0} = \dfrac{1}{\sqrt{2\tau - 4} + 1}. \end{cases}$$

5. CONCLUSION

The proposed method can be extended to a wider class of evolutionary systems. Note that another approach to evolutionary systems is proposed in [3–5]. That method is based on the finite-dimensional dynamics.

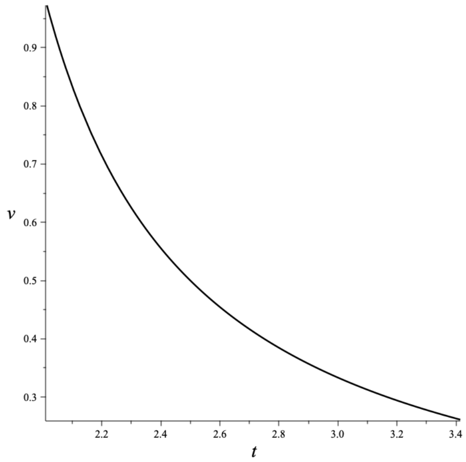

Fig. 4.4. Projection of the self-intersection curve to the plane x, v

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