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On the External Estimation of the Limit Reachable Set for the Linear Discrete-Time System Based on Support Hyperplanes

Anastasiya Simkina*

Moscow Aviation Institute, Moscow, Russia

Abstract: A linear discrete-time system with geometric constraints on control is considered. For the given system, the problem of constructing the limit reachable set is solved. It is proven that it is a cylinder oriented along certain elements of the real Jordan basis. Statements about the structure of the supporting hyperplane to the convex section of this cylinder are formulated and proven. Necessary and sufficient conditions for the boundedness of the limit reachable set are determined.

Keywords: discrete-time system, limit reachable set, support hyperplane, convex set, polyhedral approximation

1. INTRODUCTION

The classical controllability condition for linear dynamic systems assumes unbounded control actions. However, when considering practical problems, constraints often arise that are determined by various technical characteristics of the control system under consideration. As a result, even with an infinite number of steps, it is not always possible to transfer the system from a given initial state to a certain fixed final state. This fact makes it relevant to study not only the issues of reachability and controllability of various dynamic systems but also to develop methods for constructing and evaluating the limit reachable and controllable sets for an arbitrary control system. On the other hand, these sets can be applied to solving synthesis problems for discrete-time systems [7].

Currently, two main directions can be distinguished in this area: the study of individual states for controllability [3–5, 12] and geometric methods for constructing reachable and controllable sets [**?**, 1, 15, 16]. Thus, when studying nonlinear systems, it is possible to obtain only general properties of controllable sets [3] or their estimates [15, 16]. For the case of linear dynamic equations, it is possible to construct more constructive results for various classes of systems: periodic [11], switchable [4], with positive control [1]. The most rigorous results are formulated for the case of compact and convex constraints on control values [**?**, 7], even allowing the description of limit reachable and controllable sets [2, 5, 12]. Analogous problems were also studied in the works of Kurzhanskiy A.B. [13–15]. In [8], for linear discrete-time systems with scalar control, which is subject to a first-order summary constraint, it is shown that in the case of stable systems, it is possible to explicitly find the limit reachable set, which is a convex polyhedron symmetric about zero. For higher-order constraints, the description of the limit reachable and null-controllable sets is obtained using supporting half-spaces [9].

The structure of the article is as follows. Section 2 is devoted to the problem statement. Section 3 describes the procedure for decomposing the original system into subsystems of smaller dimensions, allowing the problem to be simplified. Section 4 develops a method

*Corresponding author: abv1998@yandex.ru

for the external evaluation of limit reachable sets. The resulting estimates are cylinders or polyhedra, whose borders are determined by supporting hyperplanes and half-spaces. Based on the developed methods, Section 5 constructs the limit reachable set for the system of stabilizing the glucose level in blood plasma.

2. PROBLEM STATEMENT

Consider linear stationary system with discrete time and geometric constraints on the control $(A,\mathcal{U})$:

$$\begin{aligned} &x(k+1) = Ax(k) + u(k), \\ &x(0) = 0,\ u(k) \in \mathcal{U},\ k \in \mathbb{N} \cup \{0\}, \end{aligned} \tag{2.1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^n$ is the control action at step $k, \mathcal{U} \subset \mathbb{R}^n$ denotes the set of admissible control values defining the geometric constraints, and $A \in \mathbb{R}^{n\times n}$ is the system matrix. It is assumed that $\mathcal{U}$ is a convex compact set and $0 \in \operatorname{int}\mathcal{U}$.

We define the limit reachable set $\mathcal{Y}_\infty$ of the system (2.1) as the set of states to which the system $(A,\mathcal{U})$ can be transferred from the origin in any finite number of steps:

$$\mathcal{Y}_\infty = \left\{ x \in \mathbb{R}^n \colon x = \sum_{k=0}^{N-1} A^{N-k-1}u(k),\ u(0),\dots,u(N-1) \in \mathcal{U},\ N \in \mathbb{N} \right\}. \tag{2.2}$$

It is required to investigate the properties and construct estimates of the set (2.2).

As an internal estimate of the limit reachable set, the reachable set in a finite number of steps $\mathcal{Y}(N)$ can be considered, where $\mathcal{Y}(N)$ is the set of terminal states to which the system $(A,\mathcal{U})$ can be transferred from 0 in $N \in \mathbb{N}$ steps. A description of $\mathcal{Y}(N)$ is known [7, Lemma 1]. From (2.2) the following representation for $\mathcal{Y}_\infty$ follows:

$$\mathcal{Y}_\infty = \bigcup_{N=0}^{\infty} \mathcal{Y}(N).$$

It is known that the set (2.2) is convex [18, Theorem 1]. This determines the range of problems addressed in this article. It is necessary to determine the necessary and sufficient conditions for the boundedness of the limit reachable set $\mathcal{Y}_\infty$, to investigate its closedness property, and, in the case of boundedness, to construct a description in terms of supporting hyperplanes, which is guaranteed for any convex set [17, Theorem 18.8].

In many respects, the reachable sets share properties with the null-controllable sets, which were studied in [2]. For this reason, some of their properties will be analogous. However, there are also fundamental differences, which will be revealed in the subsequent sections.

3. THE GENERAL STRUCTURE OF REACHABLE SETS

The article [19] demonstrates that the limit reachable and null-controllable sets for systems with scalar control are cylinders. Their orientation is determined by the elements of the real Jordan basis and the eigenvectors of the matrix A. A similar result was obtained in [2, Lemmas 2–4] for 0-controllable sets for systems with vector control. We will use a similar approach to describe the limit reachable sets (2.2) for the system (2.1).

Let $h_1,\dots,h_n \in \mathbb{R}^n$ denote the real Jordan basis of the matrix A, which is a set of linearly independent eigenvectors and generalized eigenvectors, in which basis A is described by its real Jordan form, that is the following decomposition [6] holds for the matrix $S =$

$(h_1, \ldots, h_n) \in \mathbb{R}^{n\times n}$:

$$A = S \begin{pmatrix} J_1 & \ldots & O \\ \vdots & \ddots & \vdots \\ O & \ldots & J_m \end{pmatrix} S^{-1},\ J_i \in \mathbb{R}^{n_i \times n_i},\ i = \overline{1, m},\ n_1 + \ldots + n_m = n, \tag{3.3}$$

where the Jordan blocks $J_1, \ldots, J_m$ have one of the following two forms:

$$J = \begin{pmatrix} \lambda & 1 & \cdots & 0 \\ 0 & \lambda & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & \lambda \end{pmatrix} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}, \tag{3.4}$$

$$J = \begin{pmatrix} rA_\varphi & I & \cdots & O \\ O & rA_\varphi & \ddots & O \\ \vdots & \vdots & \ddots & I \\ O & O & \cdots & rA_\varphi \end{pmatrix} \in \mathbb{R}^{2\tilde{n}\times 2\tilde{n}},\ A_\varphi = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix},\ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.5}$$

Let $P \in \mathbb{R}^{\tilde{n}\times n}$ denote the projection matrix onto an $\tilde{n}$-dimensional subspace:

$$P = \underbrace{\begin{pmatrix} 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix}}_{\tilde{n}\times n}.$$

We describe the structure of the limit reachable set in the following theorem.

Theorem 3.1:
Let the order of $h_1, \ldots, h_n$ in the decomposition (3.3) *be chosen such that the representation holds*

$$A = S \begin{pmatrix} A_{\geqslant 1} & O \\ O & A_{<1} \end{pmatrix} S^{-1},$$

where $A_{\geqslant 1} \in \mathbb{R}^{(n-\tilde{n})\times(n-\tilde{n})}$ consists of Jordan blocks of the form (3.4) *and* (3.5) *corresponding to eigenvalues with modulus not less than 1, and $A_{<1} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ consists of Jordan blocks corresponding to eigenvalues with modulus less than 1. Let $\tilde{\mathcal{Y}}_\infty \subset \mathbb{R}^{\tilde{n}}$ be the limit reachable set* (2.2) *for the system $(A_{<1}, PS^{-1}\mathcal{U})$.*

Then the relation holds

$$\mathcal{Y}_\infty = S\left(\mathbb{R}^{n-\tilde{n}} \times \tilde{\mathcal{Y}}_\infty\right).$$

Proof
Let us denote $\tilde{\mathcal{U}} = PS^{-1}\mathcal{U}$. First, we will demonstrate that the theorem holds for the case when the matrix A coincides with its real Jordan form, i.e. S is the identity matrix.

Since $\mathcal{U}$ is bounded, there exists a convex set $\mathcal{V} \in \mathbb{R}^{(n-\tilde{n})}$ such that

$$\mathcal{U} \subset \mathcal{V} \times \tilde{\mathcal{U}}. \tag{3.6}$$

Consider the dynamic equations (2.1) for the system $(A, \mathcal{V} \times \tilde{\mathcal{U}})$ and an arbitrary control sequence $\{u(k)\}_{k=0}^{N-1} \subset \mathcal{V} \times \tilde{\mathcal{U}}$. For all $N \in \mathbb{N}$ the following representation holds:

$$x(N) = A^{N-1}u(0) + A^{N-2}u(1) + \ldots + u(N-1) =$$

$$= \begin{pmatrix} \sum\limits_{k=0}^{N-1} A_{\geqslant 1}^{N-k-1} v(k) \\ \sum\limits_{k=0}^{N-1} A_{<1}^{N-k-1} \tilde{u}(k) \end{pmatrix} = \begin{pmatrix} y(N) \\ \tilde{x}(N) \end{pmatrix},$$

where $v(0), \ldots, v(N-1) \in \mathcal{V}$, $\tilde{u}(0), \ldots, \tilde{u}(N-1) \in \tilde{\mathcal{U}}$, $y(N) \in \mathbb{R}^{n-\tilde{n}}$, $\tilde{x}(N) \in \mathbb{R}^{\tilde{n}}$. Therefore, if we denote by $\hat{\mathcal{Y}}_\infty \subset \mathbb{R}^n$ the limit reachable set for the system $(A, \mathcal{V} \times \tilde{\mathcal{U}})$, then due to the representation (2.2), for any $x(N) \in \hat{\mathcal{Y}}_\infty$ the inclusion $\tilde{x}(N) \in \tilde{\mathcal{Y}}_\infty$ must also hold. We obtain

$$\hat{\mathcal{Y}}_\infty \subset \mathbb{R}^{n-\tilde{n}} \times \tilde{\mathcal{Y}}_\infty.$$

On the other hand, from (3.6) it follows that $\mathcal{Y}_\infty \subset \hat{\mathcal{Y}}_\infty$. Finally, we obtain:

$$\mathcal{Y}_\infty \subset \mathbb{R}^{n-\tilde{n}} \times \tilde{\mathcal{Y}}_\infty. \tag{3.7}$$

Let's prove the reverse inclusion. To do this, we will choose an arbitrary $x = \left(y^{\mathrm{T}},\ \tilde{x}^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^{n-\tilde{n}} \times \tilde{\mathcal{Y}}_\infty$. From the inclusion $0 \in \operatorname{int}\mathcal{U}$ it follows that there exists $v' > 0$ such that

$$[-v'; v']^{n-\tilde{n}} \times \{0\} \subset \mathcal{U}.$$

Therefore, for all $N \in \mathbb{N}$ the following control will satisfy the constraints of the system (2.1):

$$u(k) = \begin{pmatrix} v(k) \\ 0 \end{pmatrix},\ v(k) \in [-v'; v']^{n-\tilde{n}},\ k = \overline{0, N-1}.$$

since $\tilde{x} \in \tilde{\mathcal{Y}}_\infty$, by definition (2.2), there exist $M \in \mathbb{N}$ and $\tilde{u}(N), \ldots, \tilde{u}(N+M-1) \in \tilde{\mathcal{U}}$ such that

$$\tilde{x} = \sum_{k=N}^{N+M-1} A_{<1}^{N+M-k-1} \tilde{u}(k).$$

Additionally, considering the definition of $\tilde{\mathcal{U}}$, it is possible to choose $v(N), \ldots, v(N+M-1) \in \mathbb{R}^{n-\tilde{n}}$ such that

$$u(k) = \begin{pmatrix} v(k) \\ \tilde{u}(k) \end{pmatrix} \in \mathcal{U},\ k = \overline{0, M-1}.$$

Thus, the control sequence $\{u(k)\}_{k=0}^{N+M-1}$ is admissible for the system $(A, \mathcal{U})$ and, with a zero initial state, leads to the following relations:

$$x(N+M) = \begin{pmatrix} \sum\limits_{k=0}^{M+N-1} A_{\geqslant 1}^{M+N-k-1} v(k) \\ \sum\limits_{k=0}^{M+N-1} A_{<1}^{M+N-k-1} \tilde{u}(k) \end{pmatrix} =$$

$$= \begin{pmatrix} \sum\limits_{k=0}^{N-1} A_{\geqslant 1}^{N+M-k-1} v(k) + \sum\limits_{k=N}^{N+M-1} A_{\geqslant 1}^{N+M-k-1} v(k) \\ \sum\limits_{k=0}^{N-1} A_{<1}^{N+M-k-1} \tilde{u}(k) + \sum\limits_{k=N}^{N+M-1} A_{<1}^{N+M-k-1} \tilde{u}(k) \end{pmatrix} = \begin{pmatrix} A_{\geqslant 1}^{M} \sum\limits_{k=0}^{N-1} A_{\geqslant 1}^{N-k-1} v(k) + y' \\ \tilde{x} \end{pmatrix},$$

where

$$y' = \sum_{k=N}^{N+M-1} A_{\geqslant 1}^{N+M-k-1} v(k).$$

Therefore, the equality $x(N+M)=x$ can be obtained by demonstrating that for some $N \in \mathbb{N}$, it is possible to choose $v(0),\ldots,v(N-1) \in [-v';v']^{n-\tilde{n}}$ that satisfy the following relations:

$$y = A_{\geqslant 1}^{M} \sum_{k=0}^{N-1} A_{\geqslant 1}^{N-k-1} v(k) + y',$$

$$A_{\geqslant 1}^{-M}(y-y') = \sum_{k=0}^{N-1} A_{\geqslant 1}^{N-k-1} v(k).$$

Considering (2.2), is equivalent to the inclusion $A_{\geqslant 1}^{-M}(y-y') \in \mathcal{Y}'_\infty$, where $\mathcal{Y}'_\infty \subset \mathbb{R}^{n-\tilde{n}}$ is the limit reachable set for the system $\left(A_{\geqslant 1}, [-v';v']^{n-\tilde{n}}\right)$.

For further reasoning, we will use known analytical representations of reachable and 0-controllable sets for discrete-time linear systems. It is known [20, Lemma 1] that for any $N \in \mathbb{N}$ the 0-controllable set of the system $\left(A_{\geqslant 1}^{-1}, A_{\geqslant 1}^{-1}[-v';v']^{n-\tilde{n}}\right)$ in N steps has the form:

$$\mathcal{X}'(N) = -\sum_{k=1}^{N} \left(A_{\geqslant 1}^{-1}\right)^{-k} \left(A_{\geqslant 1}^{-1}[-v';v']^{n-\tilde{n}}\right) = -\sum_{k=1}^{N} A_{\geqslant 1}^{k-1}[-v';v']^{n-\tilde{n}}.$$

On the other hand, according to [7, Lemma 1], the reachable set in N steps for the system $(A_{\geqslant 1}, [-v';v']^{n_1})$ has the form:

$$\mathcal{Y}'(N) = \sum_{k=0}^{N-1} A_{\geqslant 1}^{k}[-v';v']^{n-\tilde{n}}.$$

Thus, the 0-controllable set of the system $\left(A_{\geqslant 1}^{-1}, A_{\geqslant 1}^{-1}[-v';v']^{n-\tilde{n}}\right)$ and the reachable set of the system $(A_{\geqslant 1}, [-v';v']^{n_1})$ over the same number of steps coincide, and consequently, their limit set analogs $\mathcal{X}'_\infty$ and $\mathcal{Y}'_\infty$ also coincide. However, according to [2, Lemma 3] $\mathcal{X}'_\infty = \mathbb{R}^{n-\tilde{n}}$, since all eigenvalues of $A_{\geqslant 1}^{-1}$ do not exceed 1 in magnitude, being reciprocals of the eigenvalues of the matrix $A_{\geqslant 1}$ [6].

This trivially implies the inclusion:

$$A_{\geqslant 1}^{-M}(y-y') \in \mathcal{Y}'_\infty = \mathbb{R}^{n-\tilde{n}},$$

from which it follows that the equality $x(N+M)=x$ is admissible, and as a consequence, $x \in \mathcal{Y}_\infty$, leading to

$$\mathbb{R}^{n-\tilde{n}} \times \tilde{\mathcal{Y}}_\infty \subset \mathcal{Y}_\infty. \tag{3.8}$$

Together, (3.7) and (3.8) for the case $S=I$ define the identity:

$$\mathcal{Y}_\infty = \mathbb{R}^{n-\tilde{n}} \times \tilde{\mathcal{Y}}_\infty. \tag{3.9}$$

Let us consider the case where $S \in \mathbb{R}^{n\times n}$ is an arbitrary non-singular matrix. From (3.9), it follows that for the system $(S^{-1}AS, S^{-1}\mathcal{U})$, the limit reachable set $\mathcal{Y}^0_\infty$ has the form:

$$\mathcal{Y}^0_\infty = \mathbb{R}^{n-\tilde{n}} \times \tilde{\mathcal{Y}}_\infty.$$

Taking into account the decomposition (3.3) and for $N \in \mathbb{N}$ from the dynamic equations we obtain:

$$x(N) = \sum_{k=0}^{N-1} A^{N-k-1}u(k) = \sum_{k=0}^{N-1} S \begin{pmatrix} A_{\geqslant 1}^{N-k-1} & O \\ O & A_{<1}^{N-k-1} \end{pmatrix} S^{-1}u(k) =$$

$$= S \sum_{k=0}^{N-1} \begin{pmatrix} A_{\geqslant 1}^{N-k-1} & O \\ O & A_{<1}^{N-k-1} \end{pmatrix} S^{-1} u(k).$$

From this, considering (2.2), it follows that $x \in \mathcal{Y}_\infty$ if and only if $S^{-1}x \in \mathcal{Y}_\infty^0$, i.e.

$$\mathcal{Y}_\infty = S\mathcal{Y}_\infty^0.$$

Theorem 3.1 is proven. □

For the limit null-controllable sets, it is also guaranteed that they are open and convex [2, Theorem 1]. However, when considering the limit reachable sets, general statements about their openness or closedness do not seem possible.

Example 3.1:
Consider a two-dimensional system $(A, \mathcal{U})$ with parameters defined as follows:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; \mathcal{U} = [-1; 1] \times [-1; 1].$$

Given that $A^2 = O$, for $N \geqslant 2$ we have

$$x(N) = u(N-1) + Au(N-2).$$

Therefore, by definition (2.2) the inclusions $x \in \mathcal{Y}_\infty$ and $x \in \mathcal{U} + A\mathcal{U}$ are equivalent. We can derive the following representation from this:

$$\mathcal{Y}_\infty = \mathcal{U} + A\mathcal{U} = \operatorname{conv}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\} + \operatorname{conv}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\} =$$

$$= \operatorname{conv}\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \end{pmatrix} \right\}.$$

Finally, $\mathcal{Y}_\infty$ is closed.

Example 3.2:
Consider a two-dimensional system $(A, \mathcal{U})$ with parameters defined as follows:

$$A = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \; \mathcal{U} = [-1; 1] \times [-1; 1].$$

For any $N \in \mathbb{N}$ and control sequence $\{u(k)\}_{k=0}^{N-1} \subset \mathcal{U}$ it holds that

$$x(N) = u(N-1) + Au(N-2) + \ldots + A^{N-1}u(0) = \begin{pmatrix} u_1(N-1) \\ \sum\limits_{k=0}^{N-1} \frac{1}{2^k} u_2(N-k-1) \end{pmatrix},$$

$$|x_1(N)| \leqslant 1, \; |x_2(N)| \leqslant \sum_{k=0}^{N-1} \frac{1}{2^k} < 2.$$

According to (2.2)

$$\mathcal{Y}_\infty \subset [-1; 1] \times (-2; 2).$$

On the other hand, for any $x = (x_1, x_2)^{\mathrm{T}} \in [-1; 1] \times (-2; 2)$ there exists $N \in \mathbb{N}$ such that

$$|x_2| \leqslant 2 - \frac{1}{2^{N-1}}.$$

Consequently, for all $k = \overline{0, N-1}$ the following control actions will be admissible

$$u_2(k) = \frac{x_2}{2 - 2^{-N+1}} \in [-1;1],\ u_1(k) = x_1 \in [-1;1].$$

Then

$$x(N) = \begin{pmatrix} x_1 \\ \sum\limits_{k=0}^{N-1} \frac{1}{2^k} \frac{x_2}{2-2^{-N+1}} \end{pmatrix} = \begin{pmatrix} x_1 \\ \frac{x_2}{2-2^{-N+1}} \sum\limits_{k=0}^{N-1} \frac{1}{2^k} \end{pmatrix} = \begin{pmatrix} x_1 \\ \frac{x_2}{2-2^{-N+1}} \left(2 - 2^{-N+1}\right) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

We have the inclusion

$$[-1;1] \times (-2;2) \subset \mathcal{Y}_\infty.$$

Finally, the representation holds

$$\mathcal{Y}_\infty = [-1;1] \times (-2;2),$$

meaning that the limit reachable set $\mathcal{Y}_\infty$ in this example is neither closed nor open.

Example 3.3:
Let us consider the case where $n = 1$ and in the system $(A, \mathcal{U})$

$$A = \frac{1}{2},\ \mathcal{U} = [-1;1].$$

By repeating the reasoning from Example 3.2 but applying it only to the single coordinate, we obtain:

$$\mathcal{Y}_\infty = (-2;2).$$

Thus, the limit reachable set $\mathcal{Y}_\infty$ in this example is open.

4. CONSTRUCTION OF ESTIMATES OF LIMIT NULL-CONTROLLABLE SETS

Although Theorem 3.1 guarantees that the set (2.2) is a cylinder, it does not determine the structure of the cross-section of this cylinder. Essentially, this statement only makes it possible to reduce the original problem of constructing the limit set to an analogous problem for a system of smaller dimension, but it does not solve it completely. For this reason, in this section, we will consider a method for constructing polyhedral external estimates of the set $\mathcal{Y}_\infty$, based on the apparatus of supporting half-spaces and properties of convex sets. However, Theorem 3.1 allows us to assume, without any loss of generality, that all eigenvalues of the matrix A are strictly less than 1. Indeed, in the opposite case, the original problem is equivalent to the problem of constructing the set $\tilde{\mathcal{Y}}_\infty$ for the subsystem $(A_{<1}, PS^{-1}\mathcal{U})$, which satisfies the eigenvalue constraint. Therefore, we will henceforth assume that the spectral radius of the matrix A is strictly less than 1, i.e., the system (2.1) is stable.

We will use the previously proven property of the convexity of $\mathcal{Y}_\infty$ [18, Theorem 1]. As it is known, every closed and convex set is the intersection of its supporting half-spaces [17, Theorem 18.8]. Although, as demonstrated in Examples 3.1–3.3, the set (2.2) is not necessarily either closed or open. By means of supporting half-spaces, it is possible to obtain an external estimate of its closure, i.e., to reconstruct its structure up to boundary points.

Theorem 4.1:
The set $\mathcal{Y}_\infty$ in the system $(A, \mathcal{U})$ is bounded if and only if the spectral radius of the matrix $A \in \mathbb{R}^{n \times n}$ is less than 1. In this case, when the set is bounded, for each $p \in \mathbb{R}^n \setminus \{0\}$ the following representation holds for the support function:

$$\mathcal{S}(p, \mathcal{Y}_\infty) \triangleq \sup_{x \in \mathcal{Y}_\infty} (p, x) = \sum_{k=0}^{\infty} \max_{u_k \in \mathcal{U}} \left((A^k)^{\mathrm{T}} p, u_k\right).$$

Proof
Let the spectral radius of A be at least 1. Then, in the condition of Theorem 3.1, the inequality $n > \tilde{n}$ holds, and as a consequence, $\mathcal{Y}_\infty$ is not bounded.

Now consider the reverse situation. Note that from the condition $0 \in \mathcal{U}$ for all $k \in \mathbb{N} \cup \{0\}$ it follows that:

$$\max_{u_k \in \mathcal{U}} \left((A^k)^{\mathrm{T}} p, u_k \right) \geqslant 0.$$

Then for any $p \in \mathbb{R}^n \setminus \{0\}$ due to (2.2), the following chain of equalities holds:

$$\sup_{x \in \mathcal{Y}_\infty} (p, x) = \sup_{\substack{N \in \mathbb{N} \cup \{0\} \\ u_k \in \mathcal{U} \\ k = \overline{0, N-1}}} \sum_{k=0}^{N} \left(p, A^k u_k \right) = \sup_{N \in \mathbb{N} \cup \{0\}} \sum_{k=0}^{N} \max_{u_k \in \mathcal{U}} \left(p, A^k u_k \right) = \sum_{k=0}^{\infty} \max_{u_k \in \mathcal{U}} \left(p, A^k u_k \right).$$

The convergence of the last series follows from the fact that all eigenvalues of the matrix A are strictly less than 1 in magnitude, boundedness of the set $\mathcal{U}$ and classical inequality: $|(p, A^k v)| \leqslant \|p\| \|A^k\| \|v\|$.

Thus, we conclude that:

$$\mathcal{S}(p, \mathcal{Y}_\infty) = \sum_{k=0}^{\infty} \sup_{u_k \in \mathcal{U}} \left((A^k)^{\mathrm{T}} p, u_k \right) < \infty.$$

The boundedness of $\mathcal{Y}_\infty$ follows from the boundedness of the support function for the basis vectors and their opposites.

Theorem 4.1 is proven. □

The challenge in applying Theorem 4.1 lies in the need to compute the exact value of the series. For an arbitrary choice of the support vector $p \in \mathbb{R}^n \setminus \{0\}$ solving this problem is difficult. However, if p is chosen to be an element of the real Jordan basis of A, it is possible to obtain the exact structure of the supporting half-space.

Corollary 4.1:
Consider the system (2.1) *where all eigenvalues of the matrix A have magnitudes less than 1, and the decomposition* (3.3) *holds. Let $J \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$ be a Jordan block of the form* (3.4)*, located in the rows numbered $n_0 + 1, \ldots, n_0 + \tilde{n}$. Let $p \in \mathbb{R}^n \setminus \{0\}$ be chosen such that for some $i = \overline{1, \tilde{n}}$, the following equality holds:*

$$S^{\mathrm{T}} p = (\underbrace{0, \ldots, 0}_{n_0 + i - 1}, 1, 0, \ldots, 0)^{\mathrm{T}}.$$

Additionally, the following notations are introduced

$$\overline{v_{n_0+j}} = \max_{v \in S^{-1}\mathcal{U}} v_{n_0+j}, \ \underline{v_{n_0+j}} = \min_{v \in S^{-1}\mathcal{U}} v_{n_0+j}, \ j = \overline{1, \tilde{n}}.$$

Then for the set (2.2)*, the following inclusion holds*

$$\mathcal{Y}_\infty \subset \left\{ x \in \mathbb{R}^n \colon \underline{x_{n_0+i}} \leqslant (p, x) \leqslant \overline{x_{n_0+i}} \right\},$$

where:

1. *if $\lambda \geq 0$, then*

$$\underline{x_{n_0+i}} = \sum_{j=0}^{\tilde{n}-i} \frac{\underline{v_{n_0+i+j}}}{(1-\lambda)^{j+1}},$$

$$\overline{x_{n_0+i}} = \sum_{j=0}^{\tilde{n}-i} \frac{\overline{v_{n_0+i+j}}}{(1-\lambda)^{j+1}};$$

2. *if* $\lambda < 0$*, then*

$$\underline{x_{n_0+i}} = \sum_{j=0}^{\tilde{n}-i} \left(\frac{\underline{v_{n_0+i+j}} + \overline{v_{n_0+i+j}}}{2(1+|\lambda|)^{j+1}} - \frac{\overline{v_{n_0+i+j}} - \underline{v_{n_0+i+j}}}{2(1-|\lambda|)^{j+1}} \right),$$

$$\underline{x_{n_0+i}} = \sum_{j=0}^{\tilde{n}-i} \left(\frac{\overline{v_{n_0+i+j}} + \underline{v_{n_0+i+j}}}{2(1+|\lambda|)^{j+1}} + \frac{\overline{v_{n_0+i+j}} - \underline{v_{n_0+i+j}}}{2(1-|\lambda|)^{j+1}} \right).$$

Proof
By the given condition, for the chosen vector p, it holds that $S^{\mathrm{T}}p = (0,\dots,0,1,0,\dots,0)^{\mathrm{T}} \in \mathbb{R}^n$, where 1 corresponds to the (n_0+i)-th coordinate of the vector $S^T p$.

Let $k \geqslant \tilde{n}-1$. Then, as known [6, section 3.2.5]

$$J^k = \begin{pmatrix} \lambda^k & C_k^1\lambda^{k-1} & C_k^2\lambda^{k-2} & \dots & C_k^{\tilde{n}-1}\lambda^{k-\tilde{n}+1} \\ 0 & \lambda^k & C_k^1\lambda^{k-1} & \dots & C_k^{\tilde{n}-2}\lambda^{k-\tilde{n}+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda^k \end{pmatrix},$$

where C_k^j denotes the binomial coefficient:

$$C_k^j = \frac{k!}{(k-j)!j!}.$$

Taking into account (3.3),

$$(A^k)^{\mathrm{T}}p = (S^{-1})^{\mathrm{T}} \operatorname{diag}\left(J_1^k,\dots,J_m^k\right)^{\mathrm{T}} S^{\mathrm{T}}p =$$

$$= (S^{-1})^{\mathrm{T}}(\underbrace{0,\dots,0}_{n_0+i-1},\underbrace{\lambda^k, C_k^1\lambda^{k-1},\dots,C_k^{\tilde{n}-i}\lambda^{k-\tilde{n}+i}}_{\tilde{n}-i+1},\underbrace{0,\dots,0}_{n-n_0-\tilde{n}})^{\mathrm{T}},$$

$$\left((A^k)^{\mathrm{T}}p,u\right) = \lambda^k v_{n_0+i} + C_k^1\lambda^{k-1}v_{n_0+i+1} + C_k^2\lambda^{k-2}v_{n_0+i+2} + \dots + C_k^{\tilde{n}-i}\lambda^{k-\tilde{n}+i}v_{n_0+\tilde{n}} =$$

$$= \sum_{j=0}^{\tilde{n}-i} C_k^j\lambda^{k-j}v_{n_0+j+i},$$

where $S^{-1}u = v$.

Let $k < \tilde{n}-1$. Then

$$J^k = \begin{pmatrix} \lambda^k & C_k^1\lambda^{k-1} & C_k^2\lambda^{k-2} & \dots & C_k^k\lambda^{k-k} & 0 & \dots & 0 \\ 0 & \lambda^k & C_k^1\lambda^{k-1} & \dots & C_k^{k-1}\lambda^{k-(k-1)} & C_k^k\lambda^{k-k} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & \lambda^k \end{pmatrix}.$$

Taking into account (3.3),

$$(A^k)^{\mathrm{T}}p = (S^{-1})^{\mathrm{T}} \operatorname{diag}\left(J_1^k,\dots,J_m^k\right)^{\mathrm{T}} S^{\mathrm{T}}p =$$

$$= (S^{-1})^{\mathrm{T}}(\underbrace{0, \ \dots, \ 0}_{n_0+i-1}, \underbrace{\lambda^k, C_k^1\lambda^{k-1}, \dots, C_k^{k-1}\lambda, 1}_{k+1}, \underbrace{0, \ \dots, \ 0}_{n-n_0-i-k})^{\mathrm{T}},$$

$$\big((A^k)^{\mathrm{T}}p, u\big) = \lambda^k v_{n_0+i} + C_k^1\lambda^{k-1}v_{n_0+i+1} + C_k^2\lambda^{k-2}v_{n_0+i+2} + \ldots + C_k^k\lambda^{k-k}v_{n_0+i+k} =$$

$$= \sum_{j=0}^{k} C_k^j \lambda^{k-j} v_{n_0+j+i}.$$

Thus,

$$\big((A^k)^{\mathrm{T}}p, u\big) = \sum_{j=0}^{\min\{k,\tilde{n}-i\}} C_k^j \lambda^{k-j} v_{n_0+j+i},$$

$$\sup_{u\in\mathcal{U}} \big((A^k)^{\mathrm{T}}p, u\big) = \sup_{v\in S^{-1}\mathcal{U}} \sum_{j=0}^{\min\{k,\tilde{n}-i\}} C_k^j \lambda^{k-j} v_{n_0+j+i} \leqslant$$

$$\leqslant \sum_{j=0}^{\min\{k,\tilde{n}-i\}} C_k^j \sup_{v_{n_0+j+i}\in[\underline{v_{n_0+j+i}};\overline{v_{n_0+j+i}}]} \lambda^{k-j} v_{n_0+j+i} =$$

$$= \begin{cases} \displaystyle\sum_{j=0}^{\min\{k,\tilde{n}-i\}} C_k^j \lambda^{k-j}\overline{v_{n_0+j+i}}, & \lambda \geqslant 0, \\ \displaystyle\sum_{j=0}^{\min\{k,\tilde{n}-i\}} C_k^j |\lambda|^{k-j} \max\left\{(-1)^{k-j}\overline{v_{n_0+j+i}}, (-1)^{k-j}\underline{v_{n_0+j+i}}\right\}, & \lambda < 0. \end{cases}$$

On the other hand,

$$\inf_{u\in\mathcal{U}} \big((A^k)^{\mathrm{T}}p, u\big) = \inf_{v\in S^{-1}\mathcal{U}} \sum_{j=0}^{\min\{k,\tilde{n}-i\}} C_k^j \lambda^{k-j} v_{n_0+j+i} \geqslant$$

$$\geqslant \sum_{j=0}^{\min\{k,\tilde{n}-i\}} C_k^j \inf_{v_{n_0+j+i}\in[\underline{v_{n_0+j+i}};\overline{v_{n_0+j+i}}]} \lambda^{k-j} v_{n_0+j+i} =$$

$$= \begin{cases} \displaystyle\sum_{j=0}^{\min\{k,\tilde{n}-i\}} C_k^j \lambda^{k-j}\underline{v_{n_0+j+i}}, & \lambda \geqslant 0, \\ \displaystyle\sum_{j=0}^{\min\{k,\tilde{n}-i\}} C_k^j |\lambda|^{k-j} \min\left\{(-1)^{k-j}\overline{v_{n_0+j+i}}, (-1)^{k-j}\underline{v_{n_0+j+i}}\right\}, & \lambda < 0, \end{cases}$$

Taking into account Theorem 4.1, for any $x \in \mathcal{Y}_\infty$ we obtain for $\lambda \geqslant 0$

$$(p, x) \leqslant \sum_{k=0}^{\infty} \sum_{j=0}^{\min\{k,\tilde{n}-i\}} C_k^j \lambda^{k-j}\overline{v_{n_0+j+i}} = \sum_{j=0}^{\tilde{n}-i} \overline{v_{n_0+j+i}} \sum_{k=j}^{\infty} C_k^j \lambda^{k-j} = \sum_{j=0}^{\tilde{n}-i} \frac{\overline{v_{n_0+j+i}}}{(1-\lambda)^{j+1}} = \overline{x_{n_0+i}},$$

and we obtain for $\lambda < 0$

$$(p,x) \leqslant \sum_{j=0}^{\min\{k,\tilde{n}-i\}} \sum_{k=0}^{\infty} \max\left\{(-1)^{k-j}\overline{v_{n_0+j+i}}, (-1)^{k-j}\underline{v_{n_0+j+i}}\right\} C_{k+j}^{j}|\lambda|^{k-j} =$$

$$= \sum_{j=0}^{\tilde{n}-i}\sum_{k=0}^{\infty} \max\{(-1)^k\overline{v_{n_0+j+i}}; (-1)^k\underline{v_{n_0+j+i}}\}C_{k+j}^{j}|\lambda|^k =$$

$$= \sum_{j=0}^{\tilde{n}-i}\left(\sum_{k=0}^{\infty}\overline{v_{n_0+j+i}}C_{2k+j}^{j}|\lambda|^{2k} - \sum_{k=0}^{\infty}\underline{v_{n_0+j+i}}C_{2k+j+1}^{j}|\lambda|^{2k+1}\right) =$$

$$= \sum_{j=0}^{\tilde{n}-i}\left(\overline{v_{n_0+j+i}}\sum_{k=0}^{\infty}C_{2k+j}^{j}|\lambda|^{2k} - \underline{v_{n_0+j+i}}\sum_{k=0}^{\infty}C_{2k+j+1}^{j}|\lambda|^{2k+1}\right) =$$

$$= \sum_{j=0}^{\tilde{n}-i}\left(\frac{\overline{v_{n_0+j+i}}}{2}\left(\frac{1}{(|\lambda|+1)^{j+1}} + \frac{1}{(1-|\lambda|)^{j+1}}\right) + \frac{\underline{v_{n_0+j+i}}}{2}\left(\frac{1}{(|\lambda|+1)^{j+1}} - \frac{1}{(1-|\lambda|)^{j+1}}\right)\right) =$$

$$= \sum_{j=0}^{\tilde{n}-i}\left(\frac{\overline{v_{n_0+j+i}} + \underline{v_{n_0+j+i}}}{2(1+|\lambda|)^{j+1}} + \frac{\overline{v_{n_0+j+i}} - \underline{v_{n_0+j+i}}}{2(1-|\lambda|)^{j+1}}\right) = \overline{x_{n_0+i}}.$$

Similarly, from Theorem 4.1, it can be derived since $\max\limits_{u_k\in\mathcal{U}}\big(-(A^k)^{\mathrm{T}}p, u_k\big) = -\min\limits_{u_k\in\mathcal{U}}\big((A^k)^{\mathrm{T}}p, u_k\big)$ that $(p,x) \geqslant \underline{x_{n_0+i}}$.

Corollary 4.1 is fully proven. □

For the case of a Jordan block of size 3.5, it is possible to construct supporting hyperplanes for an infinite number of vectors p. Therefore, the corresponding external estimate of the set $\mathcal{Y}_\infty$ will be a cylinder.

Corollary 4.2:
Consider the system (2.1) *where all eigenvalues of the matrix A have magnitudes less than 1, and the decomposition* (3.3) *holds. Let $J \in \mathbb{R}^{2\tilde{n}\times 2\tilde{n}}$ be a Jordan block of the form* (3.5)*, located in the rows numbered $n_0+1,\dots,n_0+2\tilde{n}$. Let $p \in \mathbb{R}^n \setminus \{0\}$ be chosen such that for some $i = \overline{1,\tilde{n}}$, the following equality holds:*

$$S^{\mathrm{T}}p = (\underbrace{0,\dots,0}_{n_0+2i-2},\tilde{p}_1,\tilde{p}_2,0,\dots,0)^{\mathrm{T}},\ \tilde{p}_1^1 + \tilde{p}_2^2 = 1.$$

Additionally, the following notations are introduced

$$\overline{r_{n_0+j}} = \max_{v\in S^{-1}\mathcal{U}}\sqrt{v_{n_0+2j-1}^2 + v_{n_0+2j}^2},\ j = \overline{1,\tilde{n}}.$$

Then the following inclusion holds for the set (2.2)

$$\mathcal{Y}_\infty \subset \left\{x\in\mathbb{R}^n\colon x_{n_0+2i-1}^2 + x_{n_0+2i}^2 \leqslant \overline{R_{n_0+j}}^2\right\},\quad \overline{R_{n_0+j}} = \sum_{j=0}^{\tilde{n}-i}\frac{\overline{r_{n_0+i+j}}}{(1-r)^{j+1}}.$$

Proof
By the given condition, for the chosen vector p, it holds that $S^{\mathrm{T}}p = (0,\ldots,0,\tilde{p}_1,\tilde{p}_2,0,\ldots,0)^{\mathrm{T}} \in \mathbb{R}^n$, where the two-dimensional subvector $\widetilde{p}$ corresponds to the (n_0+2i-1)-th and (n_0+2i)-th coordinates of the vector p.

Let $k \geqslant \tilde{n}-1$. Then

$$J^k = \begin{pmatrix} r^kA_{k\varphi} & C_k^1 r^{k-1}A_{(k-1)\varphi} & \ldots & C_k^{\tilde{n}-1}r^{k-\tilde{n}+1}A_{(k-\tilde{n}+1)\varphi} \\ 0 & r^kA_{k\varphi} & \ldots & C_k^{\tilde{n}-2}r^{k-\tilde{n}+2}A_{(k-\tilde{n}+2)\varphi} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & r^kA_{k\varphi} \end{pmatrix},$$

Taking into account (3.3),

$$(A^k)^{\mathrm{T}}p = (S^{-1})^{\mathrm{T}}\operatorname{diag}\left(J_1^k,\ldots,J_m^k\right)^{\mathrm{T}}S^{\mathrm{T}}p =$$

$$= (S^{-1})^{\mathrm{T}}\Big(\underbrace{0,\ldots,0}_{n_0+2i-2}, C_k^0 r^{k-0}\left(A^{\mathrm{T}}_{(k-0)\varphi}\widetilde{p}\right)^{\mathrm{T}},\ldots,C_k^{\tilde{n}-i}r^{k-\tilde{n}+i}\left(A^{\mathrm{T}}_{(k-\tilde{n}+i)\varphi}\widetilde{p}\right)^{\mathrm{T}},0,\ldots,0\Big)^{\mathrm{T}}.$$

In case $k < \tilde{n}-1$

$$J^k = \begin{pmatrix} r^kA_{k\varphi} & C_k^1 r^{k-1}A_{(k-1)\varphi} & \ldots & C_k^k r^{k-k}A_{(k-k)\varphi} & 0 & \ldots & 0 \\ 0 & r^kA_{k\varphi} & \ldots & C_k^{k-1}r^{k-(k-1)}A_{(k-(k-1))\varphi} & C_k^k r^{k-k}A_{(k-k)\varphi} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & r^kA_{k\varphi} \end{pmatrix},$$

Taking into account (3.3),

$$(A^k)^{\mathrm{T}}p = (S^{-1})^{\mathrm{T}}\operatorname{diag}\left(J_1^k,\ldots,J_m^k\right)^{\mathrm{T}}S^{\mathrm{T}}p =$$

$$= (S^{-1})^{\mathrm{T}}\Big(\underbrace{0,\ldots,0}_{n_0+2i-2}, C_k^0 r^{k-0}\left(A^{\mathrm{T}}_{(k-0)\varphi}\widetilde{p}\right)^{\mathrm{T}},\ldots,C_k^{k-1}r^{k-(k-1)}\left(A^{\mathrm{T}}_{(k-(k-1))\varphi}\widetilde{p}\right)^{\mathrm{T}},\widetilde{p}^{\mathrm{T}},0,\ldots,0\Big)^{\mathrm{T}}.$$

Let us denote for $v \in S^{-1}\mathcal{U}$ by $v^j \in \mathbb{R}^2$, $j=\overline{1,\tilde{n}}$ its two-dimensional subvectors by: $v^j = (v_{n_0+2j-1}, v_{n_0+2j})^{\mathrm{T}}$. Note that by the assumptions, $\|v^j\| \leqslant \overline{r_{n_0+j}}$. Also, let $u = Sv \in \mathcal{U}$. We obtain for $k \geqslant \tilde{n}-1$

$$\left((A^k)^{\mathrm{T}}p,u\right) = r^k(A_{-k\varphi}\widetilde{p},v^i) + C_k^1r^{k-1}(A_{-(k-1)\varphi}\widetilde{p},v^{i+1}) + C_k^{\tilde{n}-i}r^{k-\tilde{n}+i}(A_{-(k-\tilde{n}+i)\varphi}\widetilde{p},v^{\tilde{n}}) \leqslant$$

$$\leqslant r^k\|A_{-k\varphi}\widetilde{p}\|\|v^i\| + C_k^1r^{k-1}\|A_{-(k-1)\varphi}\widetilde{p}\|\|v^{i+1}\| + \ldots + C_k^{\tilde{n}-i}r^{k-\tilde{n}+i}\|A_{-(k-\tilde{n}+i)\varphi}\widetilde{p}\|\|v^{\tilde{n}}\| \leqslant$$

$$\leqslant r^k\overline{r_{n_0+i}} + C_k^1r^{k-1}\overline{r_{n_0+i+1}} + \ldots + C_k^{\tilde{n}-i}r^{k-\tilde{n}+i}\overline{r_{n_0+\tilde{n}}} = \sum_{j=0}^{\tilde{n}-i} C_k^j r^{k-j}\overline{r_{n_0+i+j}}.$$

For $k < \tilde{n}-1$

$$\left((A^k)^{\mathrm{T}}p,u\right) = r^k(A_{-k\varphi}\widetilde{p},v^i) + C_k^1r^{k-1}(A_{-(k-1)\varphi}\widetilde{p},v^{i+1}) + C_k^kr^{k-k}(A_{-(k-k)\varphi}\widetilde{p},v^{i+k}) \leqslant$$

$$\leqslant r^k\|A_{-k\varphi}\widetilde{p}\|\|v^i\| + C_k^1r^{k-1}\|A_{-(k-1)\varphi}\widetilde{p}\|\|v^{i+1}\| + \ldots + C_k^kr^{k-k}\|A_{-(k-k)\varphi}\widetilde{p}\|\|v^{i+k}\| \leqslant$$

$$r^k\overline{r_{n_0+i}} + C_k^1r^{k-1}\overline{r_{n_0+i+1}} + \ldots + C_k^kr^{k-k}\overline{r_{n_0+i+k}} = \sum_{j=0}^{k} C_k^j r^{k-j}\overline{r_{n_0+i+j}}.$$

Due to the arbitrariness of the choice of $u \in \mathcal{U}$ we obtain

$$\max_{u\in\mathcal{U}} \big((A^k)^{\mathrm{T}} p, u\big) \leqslant \sum_{j=0}^{\min\{k,\tilde{n}-i\}} C_k^j r^{k-j} \overline{r_{n_0+i+j}},$$

According to Theorem 4.1, for any $x \in \mathcal{Y}_\infty$, the following relations hold:

$$(p,x) \leqslant \sum_{k=0}^{\infty} \sum_{j=0}^{\min\{k,\tilde{n}-i\}} C_k^j r^{k-j} \overline{r_{n_0+i+j}} = \sum_{j=0}^{n-i} \sum_{k=0}^{\infty} C_{k+j}^j r^k \overline{r_{n_0+i+j}} =$$

$$= \sum_{j=0}^{n-i} \frac{\overline{r_{n_0+i+j}}}{(1-r)^{j+1}} = \overline{R_{n_0+i}}.$$

Taking into account the choice of p, we obtain the inclusion:

$$\mathcal{Y}_\infty \subset \bigcap_{\tilde{p}_1^2+\tilde{p}_2^2=1} \left\{x \in \mathbb{R}^n \colon (p,x) \leqslant \overline{R_{n_0+i}}\right\} = \left\{x \in \mathbb{R}^n \colon \sqrt{x_{n_0+2i-1}^2 + x_{n_0+2i}^2} \leqslant \overline{R_{n_0+i}}\right\}.$$

Corollary 4.2 is proven. □

Using Corollaries 4.1 and 4.2, it is possible to estimate from above the sought set $\mathcal{Y}_\infty$ in certain specific directions determined by the real Jordan basis. The problem] of constructing supporting hyperplanes is reduced to computing the spectrum of the system matrix. Additionally, the more general Theorem 4.1 allows for refining the computed external estimates if necessary.

5. BLOOD PLASMA GLUCOSE STABILIZATION SYSTEM

We will demonstrate the effectiveness of the developed methods using the example of constructing the limit reachable set for a stabilization glucose and insulin levels in blood plasma system. For clarity and the ability to visualize the computed sets, we will consider the simplest mathematical model by Bergman [10], described by a three-dimensional system:

$$\begin{aligned}
\dot{G}(t) &= -q_1 G(t) - X(t)(G(t) + G_B) + v_1(t),\\
\dot{X}(t) &= -q_2 X(t) + q_3 I(t),\\
\dot{I}(t) &= -m(I(t) + I_B) + \frac{v_2(t)}{V_I},
\end{aligned} \tag{5.10}$$

where $G(t)$ and $I(t)$ represent the deviations of the glucose and free insulin concentrations in blood plasma from their normal values G_B and I_B, respectively. $X(t)$ is a term accounting for the delay in the metabolism processes of free insulin. The coefficient m denotes the partial elimination (removal from the body), V_I is the volume of distribution of insulin in tissues, $v_1(t)$ and $v_2(t)$ are the rates of external glucose and insulin input, respectively, and q_1, q_2, q_3 are auxiliary numerical parameters. The following numerical values of the parameters, obtained experimentally in [10], are used for the calculations,:

$$q_1 = 0.028,\ q_2 = 0.025,\ q_3 = 0.000013,$$

$$V_I = 12\,\mathrm{L},\ m = \frac{5}{54}\ \mathrm{min}^{-1},\ G_B = 4.5\ \mathrm{mmol/L},\ I_B = 15\ \mathrm{nmol/L}.$$

By performing linearization and subsequent discretization with a step size of $\delta t = 10$ min assuming the control inputs $v_1(t)$ and $v_2(t)$ are piecewise constant, we obtain the following finite-difference relations of the form (2.1) with the parameters:

$$A_d = \begin{pmatrix} 0.7557 & -34.5225 & -0.0018 \\ 0 & 0.7788 & 0.000073 \\ 0 & 0 & 0.3961 \end{pmatrix},$$

$$\mathcal{U}_d = \operatorname{conv}\left\{ \begin{pmatrix} -30.61 \\ -0.1418 \\ -0.93 \end{pmatrix}, \begin{pmatrix} -28.03 \\ -0.1418 \\ -0.93 \end{pmatrix}, \begin{pmatrix} -30.62 \\ -0.1423 \\ 2.11 \end{pmatrix}, \begin{pmatrix} -28.04 \\ -0.1423 \\ 2.11 \end{pmatrix}, \begin{pmatrix} -28.03 \\ 0.1421 \\ -0.93 \end{pmatrix}, \begin{pmatrix} 30.62 \\ 0.1421 \\ -0.93 \end{pmatrix}, \begin{pmatrix} 28.02 \\ 0.1416 \\ 2.11 \end{pmatrix}, \begin{pmatrix} 30.61 \\ 0.1416 \\ 2.11 \end{pmatrix} \right\}.$$

Note that for the system $(A_d, \mathcal{U}_d)$ the condition $0 \in \operatorname{int} \mathcal{U}_d$ is not satisfied because $\dim \mathcal{U}_d = 2$. For this reason, we will double the quantization step and transite to an equivalent system in terms of constructing the limit reachable sets with the following parameters:

If we linearize (5.10) and put $v_1(t) = v_1 k, v_2(t) = v_2 k$ for $t = [k\delta; (k+1)\delta]$ for some $\delta > 0$, we can solve the resulting system of linear differential equations explicitly. Then for $x(k) = (G(k\delta), X(k\delta), I(k\delta))^{\mathrm{T}}$ the finite-difference system relations will be correct when selecting:

$$A = A_d^2 = \begin{pmatrix} 0.5712 & -52.9824 & -0.0046 \\ 0 & 0.6065 & 0.0001 \\ 0 & 0 & 0.1569 \end{pmatrix}, \ \mathcal{U} = \mathcal{U} + A_d \mathcal{U}.$$

The eigenvalues of the matrix A are $\lambda_1 = 0.5712$, $\lambda_2 = 0.6065$ and $\lambda_3 = 0.1569$, which satisfy the conditions of Corollary 4.1. We will use this result to construct an external polyhedral estimate of the set $\mathcal{Y}_\infty$.

We will compute the vectors p^1, p^2, p^3 from Corollary 4.1:

$$p^1 = \big((1,0,0)S^{-1}\big)^{\mathrm{T}} = (1, 1500, 0.3)^{\mathrm{T}},$$

$$p^2 = \big((0,1,0)S^{-1}\big)^{\mathrm{T}} = (0, 1500, 0.28)^{\mathrm{T}},$$

$$p^3 = \big((0,0,1)S^{-1}\big)^{\mathrm{T}} = (0, 0, 1)^{\mathrm{T}}.$$

Additionally, the following equalities hold

$$\min_{v \in S^{-1}\mathcal{U}} v_1 = -997.88, \quad \min_{v \in S^{-1}\mathcal{U}} v_2 = -963.25, \quad \min_{v \in S^{-1}\mathcal{U}} v_3 = -1.54,$$

$$\max_{v \in S^{-1}\mathcal{U}} v_1 = 998.18, \quad \max_{v \in S^{-1}\mathcal{U}} v_2 = 963.53, \quad \max_{v \in S^{-1}\mathcal{U}} v_3 = 3.506.$$

Ultimately, we obtain the following estimate for the limit reachable set:

$$\mathcal{Y}_\infty \subset \hat{\mathcal{Y}}_\infty = \{x \in \mathbb{R}^3 \colon -997.88 \leqslant (p^1, x) \leqslant 998.18\} \cap$$

$$\cap \{x \in \mathbb{R}^3 \colon -963.25 \leqslant (p^2, x) \leqslant 963.53\} \cap \{x \in \mathbb{R}^3 \colon -1.54 \leqslant (p^3, x) \leqslant 3.506\}.$$

The set $\hat{\mathcal{Y}}_\infty$ is graphically represented in Figure 5.1.

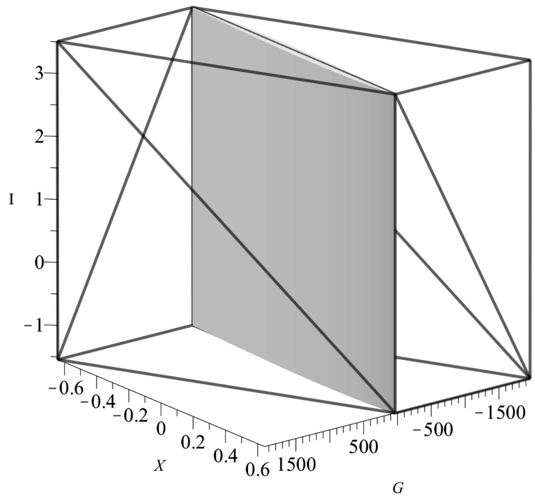

Fig. 5.1. External estimate $\mathcal{Y}_\infty$ (by lines) and reachable set $\mathcal{Y}(11)$ (by surface) for discretized system (5.10).

CONCLUSION

The article investigates various properties of the limit reachable sets for linear stationary discrete-time systems with geometric constraints on the control. Specifically, it is proven that these sets are cylinders oriented along the elements of the real Jordan basis of the system matrix, corresponding to eigenvalues with magnitudes not less than 1. The cross-section of each such cylinder is convex and can be estimated from above using appropriate supporting half-spaces, the structure of which is also described in the article. Examples demonstrate that the limit reachable sets do not necessarily have to be either open or closed, unlike similar limit null-controllable sets.

For future research, one could consider generalizing the obtained results to a class of systems where the set of admissible control values $\mathcal{U}$ has a lower dimension than the state space, and 0 is assumed to be only a relative interior point of the set $\mathcal{U}$. Additionally, it is equally important to develop numerical methods that allow the computation of the support function value in Theorem 4.1 with any predetermined accuracy.

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